The p-curl system in extension domains

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Abstract

In this paper we prove well-posedness result for a parabolic p-curl system on a three-dimensional bounded extension domains. In view of the numerical approximation, we investigate the asymptotic behavior of the solutions to suitable approximating problems. Crucial tools are a Stokes formula and a Gaffney inequality for extensions domains.

Keywords: Maxwell equations, Stokes formula, irregular domains, nonlinear semigroups, Gaffney inequality, Mosco convergence.

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Introduction

The aim of this work is to investigate the magnetic properties of irregular structures. This is a rather recent and challenging research field, where the mathematical literature is not so huge.
Irregular structures occur in many natural phenomena, thus fractals turn out to be a good model to describe such geometries. Hence, they can be used in some industrial applications.
Due to this fact, many papers appeared in the literature dealing with scalar BVPs, for instance modeling heat transfer, on domains with irregular boundaries or interfaces. Among the others, we refer to [11, 15–17, 29], [10, 35, 36] and the references listed in.
To the authors' knowledge, vector BVPs in domains with irregular boundary have been firstly studied in [30], [12] and [13], while the study of linear magnetic operators in fractal sets has been developed in [22], [20], [21] and [23]. As to the case of nonlinear vector BVPs, the literature on the so-called p-curl systems arising from electromagnetism in

the case of smooth domains goes back to the last 15-20 years: we refer to [32, 42–44] and the references listed in.

In this paper, we consider a time-dependent quasi-linear vector boundary value problem for the p-curl operator in three-dimensional irregular domains, possibly with fractal boundary. In view of its numerical approximation, we investigate the asymptotic behavior of the solutions to suitable approximating problems.
More precisely, we consider the following parabolic nonlinear vector problem in a 3D (ε, δ) domain Q with boundary a d-set (see Section 1.1 for the definitions) formally stated as

$$(\hat{P}_{3D}) \begin{cases} \frac{\partial \mathbf{u}(t,x)}{\partial t} + \operatorname{curl}\left(|\operatorname{curl}\mathbf{u}(t,x)|^{p-2}\operatorname{curl}\mathbf{u}(t,x)\right) = \mathbf{F}(t,x) & \text{in } [0,T]\times Q,\\ \operatorname{div}\mathbf{u}(t,x) = 0 & \text{in } [0,T]\times Q,\\ \boldsymbol{\nu}\times\mathbf{u} = \mathbf{0} & \text{on } [0,T]\times\partial Q,\\ \mathbf{u}(0,x) = \mathbf{u}_0(x) & \text{on } Q, \end{cases}$$

where $\mathbf{F}$ and $\mathbf{u}_0$ are given data and the "tangential trace" $\boldsymbol{\nu}\times\mathbf{u}$ has to be suitably defined on such an irregular set.

This problem can be deduced from the generalized Maxwell equations in the time-dependent 3D case:

$$(M) \begin{cases} \operatorname{curl}\mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t},\\ \dfrac{\partial \mathbf{B}}{\partial t} + \operatorname{curl}\mathbf{E} = \mathbf{F},\\ \operatorname{div}\mathbf{D} = \rho,\\ \operatorname{div}\mathbf{B} = 0, \end{cases}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, $\mathbf{D} = \epsilon\mathbf{E}$ is the electric flux density, $\epsilon > 0$ is the permittivity, $\mathbf{B} = \mu\mathbf{H}$ is the magnetic flux density, $\mu > 0$ is the permeability, $\mathbf{J}$ is the total current density, $\mathbf{F}$ is a given internal magnetic current and $\rho \geq 0$ is the charge density.
We make the following assumptions. We suppose that Q is a highly conductive medium. In this setting, $\mathbf{D}$ is very small in comparison with the eddy currents $\mathbf{J}$ (see [31]), hence it is negligible.
Moreover, we assume that the following nonlinear extension of Ohm's law holds:

$$|\mathbf{J}|^{p-2}\mathbf{J} = \sigma\mathbf{E},$$

where $\sigma > 0$ is the electric conductivity. This assumption is usually made by physicists in order to simplify the numerical discretization or to account for the thermally activated creep of the magnetic flux; see [6] and the references listed in.
We remark that, under the above hypotheses, problem $(\hat{P}_{3D})$ gives a good approximation of Bean's critical-state model for type-II superconductors [7, 18]. Without loss of generality, from now on we suppose $\mu = \sigma = 1$.

The interest in studying the motion of the magnetic field in highly conductive media with possibly fractal structure is due to the fact that many experimental results show that in superconductors oxygen crystal defects form fractal structures that seem to promote high–temperature superconductivity (see [45] and the references listed in).

We attack problem ($\hat{P}_{3D}$) via a nonlinear semigroup approach and we prove existence and uniqueness results of a "strong" solution (in the sense of Definition 2.2) under suitable assumption on the data. A key tool for the proof of this result is a nonlinear version of Stokes formula (see Theorem 1.7), whose proof relies on suitable limit arguments and trace and extension theorems. Moreover, Stokes formula (1.6) allows us to give a rigorous interpretation of the boundary condition of problem ($\hat{P}_{3D}$). These results generalize to the parabolic quasi-linear framework the results obtained in [13] in the stationary linear fractal case.

In order to consider the asymptotic analysis, we first consider an axial-symmetric case, where problem ($\hat{P}_{3D}$) reduces to a scalar parabolic problem for the p-Laplace operator with homogeneous Dirichlet boundary conditions in a two-dimensional (ε, δ) domain Ω with boundary a d-set. We also study the corresponding problems in the Lipschitz domains Ω_n which approximate Ω in the sense of Lemma 1.5. In Theorem 3.1 we prove the strong convergence of the approximating solutions to the limit irregular one.

To tackle the full 3D problem, a key tool is the use of a Gaffney inequality proved in [14] for irregular domains. By proceeding as in the axial-symmetric case, we construct a sequence of approximating problems ($\hat{P}_{3D,n}$) and we prove the Mosco convergence (or M-convergence) of the associated functionals $\tilde{\Phi}_p^{(n)}$, defined in (3.5) (see Theorem 3.6). This, in turn, allows to deduce the G-convergence of the associated subdifferentials (see Theorem 3.7), and hence to prove the convergence of the approximating solutions.

In a forthcoming paper, we will focus on the numerical approximation of problem ($\hat{P}_{3D}$) by a finite element scheme which will deeply relies on regularity results for the weak solution of the problem.

The paper is organized as follows.

In Section 1 we introduce the geometry of the problem and the functional setting and we recall some inequalities and trace results.

In Section 2 we give a strong formulation to problem ($\hat{P}_{3D}$) via nonlinear semigroup theory and we prove that it admits a unique "strong" solution in the sense of Definition 2.2.

In Section 3 we consider the asymptotic behavior of the solutions to suitable approximating problems ($\hat{P}_{3D,n}$). In Section 3.1 we first consider the axial-symmetric case in which $Q = \Omega \times I$, where $\Omega \subset \mathbb{R}^2$ is a (ε, δ) domain with boundary a d-set, we consider scalar corresponding problems in Ω and in the approximating Lipschitz domains Ω_n, for $n \in \mathbb{N}$, and we prove that the approximating solutions converge to the irregular one. Then, in Section 3.2 we analyze the asymptotic behavior of the solutions for the general 3D case under the assumptions on Q of Section 1. A key tool is a suitable Gaffney inequality for irregular domains, see Theorem 1.8.

1 Preliminaries

1.1 (ε, δ) domains and functional spaces

Throughout the paper, C denotes possibly different constants. We give the dependence of constants on some parameters in parentheses.
Let $\mathcal{G}$ (resp. $\mathcal{S}$) be an open (resp. closed) set of $\mathbb{R}^N$. By $L^p(\mathcal{G})$, for $p \geq 1$, we denote the Lebesgue space with respect to the Lebesgue measure $\mathrm{d}\mathcal{L}_N$, which will be left to the context whenever that does not create ambiguity. By $L^p(\partial\mathcal{G})$ we denote the Lebesgue space on $\partial\mathcal{G}$ with respect to a positive Borel measure μ supported on $\partial\mathcal{G}$. By $\mathcal{D}(\mathcal{G})$ we denote the space of infinitely differentiable functions with compact support in $\mathcal{G}$. By $W^{s,p}(\mathcal{G})$, where $s \in \mathbb{R}^+$, we denote the usual (possibly fractional) Sobolev spaces (see [1], [37]); $W^{s,p}_0(\mathcal{G})$ is the closure of $D(\mathcal{G})$ with respect to the $\|\cdot\|_{W^{s,p}}$-norm. $W^{-s,p'}(\mathcal{G})$ denotes the dual space of $W^{s,p}_0(\mathcal{G})$. By $C(\mathcal{S})$ we denote the space of continuous functions on $\mathcal{S}$. We write $B(P,r) = \{P' \in \mathbb{R}^N : |P'-P| < r\}$, $P \in \mathbb{R}^N, r > 0$, for the euclidean ball of radius r centered at P, and we denote by $|A|$ the Lebesgue measure of a subset $A \subset \mathbb{R}^N$.

In the following, we consider a wide class of possibly very irregular domains, the so-called (ε, δ) domains. These domains have been introduced by Jones in [24]. For the sake of clarity, we recall the definition.

Definition 1.1. *Let $Q \subset \mathbb{R}^N$ be open and connected and let $\varepsilon, \delta > 0$. For $x \in Q$, let $d(x) := \inf\limits_{y \in Q^c} |x-y|$. We say that Q is an (ε, δ) domain if, whenever $x, y \in Q$ with $|x-y| < \delta$, there exists a rectifiable arc $\gamma \in Q$ joining x to y such that*

$$\ell(\gamma) \leq \frac{1}{\varepsilon}|x-y| \quad \text{and} \quad d(z) \geq \frac{\varepsilon|x-z||y-z|}{|x-y|} \text{ for every } z \in \gamma.$$

Examples of (ε, δ) domains are Lipschitz domains and the Koch snowflake domain. As pointed out by Jones [24], (ε, δ) domains can have a highly non-rectifiable boundary. In particular, we will consider bounded (ε, δ) domains having as boundary a d-set.

Definition 1.2. *A closed nonempty set $\mathcal{S} \subset \mathbb{R}^N$ is a d-set (for $0 < d \leq N$) if there exist a Borel measure μ with $\operatorname{supp}\mu = \mathcal{S}$ and two positive constants c_1 and c_2 such that*

$$c_1 r^d \leq \mu(\overline{B(x,r)} \cap \mathcal{S}) \leq c_2 r^d \quad \forall\, x \in \mathcal{S}, \quad 0 < r \leq 1. \tag{1.1}$$

The measure μ is called d-measure.

We now recall the definition of Besov space specialized to our case. For generalities on Besov spaces, we refer to [26].

Definition 1.3. *Let $\mathcal{S}$ be a d-set, $0 < \alpha < 1$ and $1 \leq p \leq \infty$. $B^{p,p}_\alpha(\mathcal{S})$ is the space of functions for which the following norm is finite:*

$$\|u\|^2_{B^{p,p}_\alpha(\mathcal{S})} = \|u\|^p_{L^p(\mathcal{S})} + \iint\limits_{|x-y|<1} \frac{|u(x)-u(y)|^p}{|x-y|^{d+p\alpha}} \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).$$

In the following, we will denote the dual of the Besov space $B^{p,p}_{\alpha}(\mathcal{S})$ with $(B^{p,p}_{\alpha}(\mathcal{S}))'$; we point out that this space coincides with the space $B^{p',p'}_{-\alpha}(\mathcal{S})$, where p' is the Hölder conjugate exponent of p (see [27]).

We now state the trace theorem for functions in $W^{1,p}(Q)$ specialized to our case, where Q is a bounded (ε,δ) domain with boundary ∂Q a d-set.
From now on, we assume that

$$\frac{N-d}{p} < 1$$

and we set

$$\alpha := 1 - \frac{N-d}{p}. \tag{1.2}$$

Proposition 1.4. *Let $N-1 \le d < N$, $1 < p < \infty$ and α be as in (1.2). $B^{p,p}_{\alpha}(\partial Q)$ is the trace space of $W^{1,p}(Q)$ in the following sense:*

(i) there exists a continuous and linear operator γ_0 from $W^{1,p}(Q)$ to $B^{p,p}_{\alpha}(\partial Q)$;

(ii) there exists a continuous and linear operator Ext *from $B^{p,p}_{\alpha}(\partial Q)$ to $W^{1,p}(Q)$ such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator in $B^{p,p}_{\alpha}(\partial Q)$.*

For the proof, we refer to [26, Theorem 1, Chapter VII], see also [41, Theorem 1].

We point out that, if Q is a Lipschitz domain, its boundary ∂Q is a $(N-1)$-set. Hence, the trace space of $W^{1,p}(Q)$ is $B^{p,p}_{1-\frac{1}{p}}(\partial Q)$, and the latter space coincides with $W^{1-\frac{1}{p},p}(\partial Q)$.

We pass to vector valued functions. If $\mathcal{G} \subset \mathbb{R}^N$ is open, we denote by $L^p(\mathcal{G})^N$ the space of vector-valued functions $\mathbf{u} = (u_1, u_2, \dots, u_N)$ such that $u_i \in L^p(\mathcal{G})$ for every $i = 1, \dots, N$. If we endow it with the following norm

$$\|\mathbf{u}\|_{L^p(\mathcal{G})^N} := \left(\sum_{i=1}^{N} \|u_i\|^p_{L^p(\mathcal{G})} \right)^{\frac{1}{p}}, \tag{1.3}$$

it becomes a Banach space.

We define the following spaces:

$$W^p(\mathrm{div},\mathcal{G}) := \left\{ \mathbf{u} \in L^p(\mathcal{G})^N \,:\, \mathrm{div}\,\mathbf{u} \in L^p(\mathcal{G}) \right\},$$
$$W^p(\mathrm{curl},\mathcal{G}) := \left\{ \mathbf{u} \in L^p(\mathcal{G})^N \,:\, \mathrm{curl}\,\mathbf{u} \in L^p(\mathcal{G})^N \right\}.$$

We endow these spaces with the following norms:

$$\|\mathbf{u}\|^p_{\mathrm{div},\mathcal{G}} := \|\mathbf{u}\|^p_{L^p(\mathcal{G})^N} + \|\,\mathrm{div}\,\mathbf{u}\|^p_{L^p(\mathcal{G})},$$
$$\|\mathbf{u}\|^p_{\mathrm{curl},\mathcal{G}} := \|\mathbf{u}\|^p_{L^p(\mathcal{G})^N} + \|\,\mathrm{curl}\,\mathbf{u}\|^p_{L^p(\mathcal{G})^N}.$$

1.2 Integral theorems

In this section we state some important results and integral theorems.

First, we recall the following important approximation result for irregular domains by Sohr. For arbitrary $A, B \subseteq \mathbb{R}^N$, we define

$$\operatorname{dist}(A, B) := \inf_{x \in A, y \in B} |x - y|.$$

Lemma 1.5. *Let $Q \subseteq \mathbb{R}^N$, with $N \geq 2$, be an arbitrary domain. Then there exist a sequence $\{Q_n\}_{n=0}^{\infty}$ of bounded Lipschitz subdomains of Q and a sequence $\{\epsilon_n\}_{n=0}^{\infty}$ of positive numbers such that:*

a) $\overline{Q_n} \subseteq Q_{n+1} \quad \forall\, n \in \mathbb{N}$;

b) $\operatorname{dist}(\partial Q_{n+1}, Q_n) \geq \epsilon_{n+1} \quad \forall\, n \in \mathbb{N}$;

c) $\lim_{n \to +\infty} \epsilon_n = 0$;

d) $Q = \bigcup_{n=0}^{\infty} Q_n$.

See [39, Lemma 1.4.1, Chapter II.1.4]. Concrete examples of irregular domains enjoying the properties above are Koch-type fractal domains, where the approximating sequence $\{Q_n\}$ is given by the so-called *pre-fractal domains*, see e.g. [13, 29].

From now on, Q will denote a bounded simply connected three-dimensional (ε, δ) domain with boundary ∂Q a d-set. We remark that all the results of this paper can be extended to the case in which ∂Q is an "arbitrary closed set" in the sense of [25]. We now state Stokes and Green formulae. Let $\mathbf{u} \in W^p(\operatorname{div}, Q)$ and $v \in W^{1,p'}(Q)$. We define the following quantity:

$$l_\nu(\mathbf{u})[\gamma_0 v] := \int_Q \mathbf{u} \cdot \nabla v \,\mathrm{d}\mathcal{L}_3 + \int_Q v \operatorname{div} \mathbf{u} \,\mathrm{d}\mathcal{L}_3.$$

Theorem 1.6 (Green formula)**.** *Let $\mathbf{u} \in W^p(\operatorname{div}, Q)$, $2 \leq d < 3$, $1 < p < \infty$ and α be as in (1.2). Then $l_\nu(\mathbf{u})$ is a linear and continuous operator from $W^p(\operatorname{div}, Q)$ to $B_\alpha^{p',p'}(\partial Q)'$.*
By setting $\mathbf{u} \cdot \boldsymbol{\nu} := l_\nu(\mathbf{u})$, the following generalized Green formula holds for every $v \in W^{1,p'}(Q)$:

$$\langle \mathbf{u} \cdot \boldsymbol{\nu}, \gamma_0 v \rangle_{B_\alpha^{p',p'}(\partial Q)', B_\alpha^{p',p'}(\partial Q)} = \int_Q \mathbf{u} \cdot \nabla v \,\mathrm{d}\mathcal{L}_3 + \int_Q v \operatorname{div} \mathbf{u} \,\mathrm{d}\mathcal{L}_3. \tag{1.4}$$

The proof can be achieved by proceeding as in [28, Theorem 3.7].

Let now $\mathbf{u} \in W^p(\mathrm{curl}, Q)$ and $\mathbf{v} \in W^{1,p'}(Q)^3$. We define the following quantity:

$$l_\tau(\mathbf{u})[\gamma_0 \mathbf{v}] := \int_Q \mathbf{u} \cdot \mathrm{curl}\, \mathbf{v} \, \mathrm{d}\mathcal{L}_3 - \int_Q \mathbf{v} \cdot \mathrm{curl}\, \mathbf{u} \, \mathrm{d}\mathcal{L}_3. \tag{1.5}$$

Theorem 1.7 (Stokes formula)**.** *Let $\mathbf{u} \in W^p(\mathrm{curl}, Q)$, $2 \leq d < 3$, $1 < p < \infty$ and α be as in (1.2). Then $l_\tau(\mathbf{u})$ is a linear and continuous operator from $W^p(\mathrm{curl}, Q)$ to $(B^{p',p'}_\alpha(\partial Q)')^3$.*
By setting $\boldsymbol{\nu} \times \mathbf{u} := l_\tau(\mathbf{u})$, the following generalized Stokes formula holds for every $\mathbf{v} \in W^{1,p'}(Q)^3$:

$$\langle \boldsymbol{\nu} \times \mathbf{u}, \gamma_0 \mathbf{v} \rangle_{(B^{p',p'}_\alpha(\partial Q)')^3, B^{p',p'}_\alpha(\partial Q)^3} = \int_Q \mathbf{v} \cdot \mathrm{curl}\, \mathbf{u} \, \mathrm{d}\mathcal{L}_3 - \int_Q \mathbf{u} \cdot \mathrm{curl}\, \mathbf{v} \, \mathrm{d}\mathcal{L}_3. \tag{1.6}$$

Proof. We adapt the proof of Theorem 15.5 in [13] to this more general setting.
Let $l_\tau(\mathbf{u})$ be as defined in (1.5). Then Hölder inequality and Proposition 1.4 lead us to

$$\begin{aligned} |l_\tau(\mathbf{u})[\gamma_0 \mathbf{v}]| &= \left| \int_Q \mathbf{u} \cdot \mathrm{curl}\, \mathbf{v} \, \mathrm{d}\mathcal{L}_3 - \int_Q \mathbf{v} \cdot \mathrm{curl}\, \mathbf{u} \, \mathrm{d}\mathcal{L}_3 \right| \\ &\leq \|\mathbf{u}\|_{L^p(Q)^3} \|\,\mathrm{curl}\, \mathbf{v}\|_{L^{p'}(Q)^3} + \|\mathbf{v}\|_{L^{p'}(Q)^3} \|\mathrm{curl}\, \mathbf{u}\|_{L^p(Q)^3} \\ &\leq C\, \|\mathbf{v}\|_{W^{1,p'}(Q)^3} \|\mathbf{u}\|_{\mathrm{curl},Q} \leq C\, \|\gamma_0 \mathbf{v}\|_{B^{p',p'}_\alpha(\partial Q)^3} \|\mathbf{u}\|_{\mathrm{curl},Q}. \end{aligned} \tag{1.7}$$

This shows that $l_\tau(\mathbf{u})$ is an element of $(B^{p',p'}_\alpha(\partial Q)')^3$. We now prove that the operator $l_\tau(\mathbf{u})$ is independent from the choice of $\mathbf{v}$. From trace theorem 1.4, for every $\mathbf{v} \in B^{p',p'}_\alpha(\partial Q)^3$ there exists a function $\tilde{\mathbf{w}} := \mathrm{Ext}\, \mathbf{v} \in W^{1,p'}(Q)^3$ such that

$$\|\tilde{\mathbf{w}}\|_{W^{1,p'}(Q)} \leq C \|\mathbf{v}\|_{B^{p',p'}_\alpha(\partial Q)^3} \tag{1.8}$$

and $\gamma_0 \tilde{\mathbf{w}} = \mathbf{v}$ μ-almost everywhere. Thus we have that

$$\langle l_\tau(\mathbf{u}), \mathbf{v} \rangle_{(B^{p',p'}_\alpha(\partial Q)^3)', B^{p',p'}_\alpha(\partial Q)^3} = \langle l_\tau(\mathbf{u}), \gamma_0 \tilde{\mathbf{w}} \rangle_{(B^{p',p'}_\alpha(\partial Q)^3)', B^{p',p'}_\alpha(\partial Q)^3}.$$

The independence follows from (1.7) and (1.8).

We now consider the sequence of approximationg domains Q_n given by Lemma 1.5, which in particular are bounded Lipschitz domains such that $\overline{Q_n} \subset Q_{n+1}$ and $Q = \bigcup_{n=1}^\infty Q_n$.
By the vector Stokes formula for Lipschitz domains (see e.g. [19, §2, Theorem 2.11]), together with the dominated convergence theorem, we have

$$\lim_{n \to +\infty} \langle \boldsymbol{\nu} \times \mathbf{u}, \gamma_0 \mathbf{v} \rangle_{W^{-(1-\frac{1}{p'}),p}(\partial Q_n)^3, W^{1-\frac{1}{p'},p'}(\partial Q_n)^3} = \lim_{n \to +\infty} \left(\int_{Q_n} \mathbf{v} \cdot \mathrm{curl}\, \mathbf{u} \, \mathrm{d}\mathcal{L}_3 - \int_{Q_n} \mathbf{u} \cdot \mathrm{curl}\, \mathbf{v} \, \mathrm{d}\mathcal{L}_3 \right)$$

$$= \int_Q \mathbf{v} \cdot \mathrm{curl}\, \mathbf{u} \, \mathrm{d}\mathcal{L}_3 - \int_Q \mathbf{u} \cdot \mathrm{curl}\, \mathbf{v} \, \mathrm{d}\mathcal{L}_3$$

for every $\mathbf{u} \in W^p(\mathrm{curl}, Q)$ and $\mathbf{v} \in W^{1,p'}(Q)^3$. Thus the previous considerations allow us to define the generalized "tangential trace" as

$$\langle \boldsymbol{\nu} \times \mathbf{u}, \gamma_0 \mathbf{v} \rangle_{(B^{p',p'}_\alpha(\partial Q)')^3, B^{p',p'}_\alpha(\partial Q)^3} := \int_Q \mathbf{v} \cdot \mathrm{curl}\, \mathbf{u} \, \mathrm{d}\mathcal{L}_3 - \int_Q \mathbf{u} \cdot \mathrm{curl}\, \mathbf{v} \, \mathrm{d}\mathcal{L}_3.$$

$\square$

We point out that, if we set for $p \geq 2$

$$\mathbf{u} \in \mathbb{V}_p := \{\mathbf{u} \in W^p(\mathrm{curl}, Q) \, : \, \mathrm{curl} \left(|\, \mathrm{curl}\, \mathbf{u}|^{p-2} \, \mathrm{curl}\, \mathbf{u} \right) \in L^{p'}(Q) \text{ in the sense of distributions}\},$$

then for every $\mathbf{u} \in \mathbb{V}_p$ and $\mathbf{v} \in W^{1,p}(Q)$ Stokes formula (1.6) takes the following form:

$$\begin{aligned} \left\langle \boldsymbol{\nu} \times \mathbf{v}, |\, \mathrm{curl}\, \mathbf{u}|^{p-2} \, \mathrm{curl}\, \mathbf{u} \right\rangle_{(B^{p',p'}_\alpha(\partial Q)')^3, B^{p',p'}_\alpha(\partial Q)^3} &= \int_Q |\, \mathrm{curl}\, \mathbf{u}|^{p-2} \, \mathrm{curl}\, \mathbf{u} \cdot \mathrm{curl}\, \mathbf{v} \, \mathrm{d}\mathcal{L}_3 \\ &- \int_Q \mathrm{curl} \left(|\, \mathrm{curl}\, \mathbf{u}|^{p-2} \, \mathrm{curl}\, \mathbf{u} \right) \cdot \mathbf{v} \, \mathrm{d}\mathcal{L}_3. \end{aligned} \tag{1.9}$$

We now introduce the following Banach spaces:

$$W^p_0(\mathrm{div}, Q) := \{\mathbf{u} \in W^p(\mathrm{div}, Q) \, : \, \boldsymbol{\nu} \cdot \mathbf{u} = 0 \text{ on } \partial Q\},$$

$$W^p_0(\mathrm{curl}, Q) := \{\mathbf{u} \in W^p(\mathrm{curl}, Q) \, : \, \boldsymbol{\nu} \times \mathbf{u} = \mathbf{0} \text{ on } \partial Q\},$$

where the boundary conditions have to be interpreted in the sense of the above Theorems 1.6 and 1.7 respectively. We refer [40] for the smooth case and to [19] for the Lipschitz case. We stress the fact that $W^p_0(\mathrm{div}, Q) \subset W^p(\mathrm{div}, Q)$ and $W^p_0(\mathrm{curl}, Q) \subset W^p(\mathrm{curl}, Q)$ and we endow $W^p_0(\mathrm{div}, Q)$ and $W^p_0(\mathrm{curl}, Q)$ with the natural norms.

We conclude by recalling Gaffney inequality for irregular domains. We refer to Theorem 3.10 in [14].

Theorem 1.8 (Gaffney inequality)**.** *Let $Q \subset \mathbb{R}^3$ be a bounded simply connected (ε, δ) domain with ∂Q a d-set, with $0 < d \leq 3$. Let $\mathbf{v} \in W^{1,p}(Q)^3$ be such that $\mathbf{v} \in W^p(\mathrm{div}, Q) \cap W^p_0(\mathrm{curl}, Q)$. Then there exists $C = C(p, Q) > 0$ such that*

$$\|\mathbf{v}\|_{W^{1,p}(Q)^3} \leq C \left(\|\, \mathrm{curl}\, \mathbf{v}\|_{L^p(Q)^3} + \|\, \mathrm{div}\, \mathbf{v}\|_{L^p(Q)} \right). \tag{1.10}$$

2 Well-posedness of the 3D problem

In this section we provide existence and uniqueness results for the weak solution to problem $(\hat{P}_{3D})$ stated in the Introduction. From now on, we set $p \geq 2$ and $H := L^2(Q)^3$.

We introduce the following energy functional on H:

$$\Phi_p[\mathbf{u}] := \begin{cases} \dfrac{1}{p} \displaystyle\int_Q |\operatorname{curl} \mathbf{u}|^p \, d\mathcal{L}_3 & \text{if } \mathbf{u} \in D(\Phi_p), \\ +\infty & \text{if } \mathbf{u} \in H \setminus D(\Phi_p), \end{cases} \tag{2.1}$$

where the effective domain is $D(\Phi_p) := \{\mathbf{u} \in W_0^p(\operatorname{curl}, Q) \,:\, \operatorname{div} \mathbf{u} = 0 \text{ in } Q\}$.

We point out the boundary condition encoded in $D(\Phi_p)$ ($\boldsymbol{\nu} \times \mathbf{u} = \mathbf{0}$ on ∂Q) has to be interpreted in a weak sense, i.e. as an identity in $(B_\alpha^{p',p'}(\partial Q)')^3$ as defined in Theorem 1.7.

Proposition 2.1. *Φ_p is a weakly lower semicontinuous, proper and convex functional in H. Moreover, its subdifferential $\partial\Phi_p$ is single-valued.*

Proof. The functional Φ_p is clearly proper and convex, and the weak lower semicontinuity follows from the properties of the norms. Finally, from Proposition 2.40 in [5], $\partial\Phi_p$ is single-valued. □

We point out that Proposition 2.1 can be proved also for $1 < p < 2$.

Let T be a fixed positive number. We now consider the abstract Cauchy problem

$$(P_{3D}) \begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \partial\Phi_p[\mathbf{u}] = \mathbf{F}, & t \in [0,T] \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases}$$

where $\partial\Phi_p$ is the subdifferential of Φ_p and $\mathbf{F}$ and $\mathbf{u}_0$ are given functions.

According to [4, Section 2.1, Chapter III], we give the following definition.

Definition 2.2. *A function $\mathbf{u} : [0,T] \to H$ is a strong solution of problem (P_{3D}) if $\mathbf{u} \in C([0,T];H)$, $\mathbf{u}$ is differentiable a.e. in $(0,T)$, $\mathbf{u}(t) \in D(-\partial\Phi_p)$ a.e. and $\frac{\partial \mathbf{u}}{\partial t} + \partial\Phi_p[\mathbf{u}] = \mathbf{F}$ for a.e. $t \in [0,T]$.*

From [4, Theorem 2.1, chapter IV] the following existence and uniqueness result for the strong solution of problem (P_{3D}) holds.

Theorem 2.3. *If $\mathbf{u}_0 \in \overline{D(-\partial\Phi_p)}$ and $\mathbf{F} \in L^2([0,T];H)$, then problem (P_{3D}) has a unique strong solution $\mathbf{u} \in C([0,T];H)$ such that $\mathbf{u} \in W^{1,2}((\delta,T);H)$ for every $\delta \in (0,T)$. Moreover $\mathbf{u}(t) \in D(-\partial\Phi_p)$ for a.e. $t \in (0,T)$, $\sqrt{t}\frac{\partial \mathbf{u}}{\partial t} \in L^2(0,T;H)$ and $\Phi_p[\mathbf{u}] \in L^1(0,T)$.*

Moreover, from Theorem 1 and Remark 2 in [8] (see also [4]) we have the following result which will be crucial in order to prove the convergence results (see Section 3.2).

Theorem 2.4. *Let $\varphi : H \to (-\infty, +\infty]$ be a proper, convex, lower semicontinuous functional on a real Hilbert space H, with effective domain $D(\varphi)$. Then the subdifferential $\partial\varphi$ is a maximal monotone m-accretive operator. Moreover, $\overline{D(\varphi)} = \overline{D(\partial\varphi)}$ and*

$-\partial\varphi$ generates a nonlinear C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\overline{D(\varphi)}$ in the following sense: for each $u_0 \in \overline{D(\varphi)}$, the function $u := T(\cdot)u_0$ is the unique strong solution of the problem

$$\begin{cases} u \in C(\mathbb{R}_+; H) \cap W^{1,\infty}_{loc}((0,\infty); H) \ \ and \ \ u(t) \in D(\varphi) \ \ a.e., \\ \dfrac{\partial u}{\partial t} + \partial\varphi(u) \ni 0 \quad a.e. \ on \ \mathbb{R}_+, \\ u(0,x) = u_0(x). \end{cases}$$

In addition, $-\partial\varphi$ generates a nonlinear semigroup $\{\hat{T}(t)\}_{t\geq 0}$ on H where, for every $t \geq 0$, $\hat{T}(t)$ is the composition of the semigroup $T(t)$ on $\overline{D(\varphi)}$ with the projection on the convex set $\overline{D(\varphi)}$.

From Proposition 2.1 and Theorem 2.4, we have that the subdifferential $\partial\Phi_p$ is maximal, monotone and m-accretive operator on H, with domain dense in H.

We now denote by $T_p(t)$ the nonlinear semigroup generated by $-\partial\Phi_p$. From Proposition 3.2, page 176 in [38] the following result holds.

Proposition 2.5. *$T_p(t)$ is a strongly continuous and contractive semigroup on H.*

We now prove that the strong solution of problem (P_{3D}) actually solves problem $(\hat{P}_{3D})$. We first need a characterization of the subdifferential of Φ_p.

Theorem 2.6. *Let $\mathbf{u}(t) \in D(\Phi_p)$ and let $\mathbf{F}(t) \in H$ for a.e. $t \in (0,T]$. Then $\mathbf{F} \in \partial\Phi_p[\mathbf{u}]$ if and only if $\mathbf{u}$ solves the following problem:*

$$(\bar{P}_{3D}) \begin{cases} \operatorname{curl}(|\operatorname{curl}\mathbf{u}|^{p-2}\operatorname{curl}\mathbf{u}) = \mathbf{F} & in \ L^{p'}(Q), \\ \operatorname{div}\mathbf{u} = 0 & a.e. \ in \ Q, \\ \langle \boldsymbol{\nu} \times \mathbf{u}, \mathbf{v} \rangle_{(B^{p,p}_\alpha(\partial Q)')^3, B^{p,p}_\alpha(\partial Q)^3} = \mathbf{0} & \forall\, \mathbf{v} \in B^{p,p}_\alpha(\partial Q)^3. \end{cases}$$

Proof. Let $\mathbf{F} \in \partial\Phi_p[\mathbf{u}]$, i.e.

$$\Phi_p[\mathbf{v}] - \Phi_p[\mathbf{u}] \geq (\mathbf{F}, \mathbf{v} - \mathbf{u})_H \quad \text{for every } \mathbf{v} \in D(\Phi_p). \tag{2.2}$$

We choose $\mathbf{v} = \mathbf{u} + t\mathbf{w}$, with $\mathbf{w} \in D(\Phi_p)$ and $0 < t \leq 1$ in (2.2) and we obtain

$$t\int_Q \mathbf{F} \cdot \mathbf{w} \, \mathrm{d}\mathcal{L}_3 \leq \frac{1}{p}\int_Q |\operatorname{curl}(\mathbf{u} + t\mathbf{w})|^p \mathrm{d}\mathcal{L}_3 - \frac{1}{p}\int_Q |\operatorname{curl}\mathbf{u}|^p \mathrm{d}\mathcal{L}_3 \tag{2.3}$$

By dividing by t and passing to the limit for $t \to 0^+$ in (2.3), we obtain

$$\int_Q \mathbf{F} \cdot \mathbf{w} \, \mathrm{d}\mathcal{L}_3 \leq \int_Q |\operatorname{curl}\mathbf{u}|^{p-2}\operatorname{curl}\mathbf{u} \cdot \operatorname{curl}\mathbf{w} \, \mathrm{d}\mathcal{L}_3.$$

By taking $-\mathbf{w}$ in (2.3) we obtain the opposite inequality, and hence we get

$$\int_Q \mathbf{F} \cdot \mathbf{w} \, \mathrm{d}\mathcal{L}_3 = \int_Q |\operatorname{curl}\mathbf{u}|^{p-2}\operatorname{curl}\mathbf{u} \cdot \operatorname{curl}\mathbf{w} \, \mathrm{d}\mathcal{L}_3. \tag{2.4}$$

We first take $\mathbf{w} \in \mathcal{D}(Q)^3$. We point out that, since $p' \leq 2$, in particular $\mathbf{F} \in L^{p'}(Q)^3$. Then, from Stokes formula (1.6) it follows that

$$\operatorname{curl}\left(|\operatorname{curl}\mathbf{u}|^{p-2}\operatorname{curl}\mathbf{u}\right) = \mathbf{F} \quad \text{in } L^{p'}(Q) \tag{2.5}$$

and in particular in $L^2(Q)$.
Moreover, Stokes theorem for irregular domains (Theorem 1.7) yields that the boundary condition in $(\bar{P})$ holds in the sense of the dual space of $B^{p,p}_{\alpha}(\partial Q)^3$.

In order to prove the converse, let $\mathbf{u} \in D(\Phi_p)$ be the weak solution of problem $(\bar{P}_{3D})$. We have to prove that $\Phi_p[\mathbf{v}] - \Phi_p[\mathbf{u}] \geq (\mathbf{F}, \mathbf{v}-\mathbf{u})_H$ for every $\mathbf{v} \in D(\Phi_p)$. By using the inequality

$$\frac{1}{p}(|a|^p - |b|^p) \geq |b|^{p-2}b(a-b)$$

and the hypothesis that $\mathbf{u}$ is the weak solution of $(\bar{P}_{3D})$, the thesis follows (see e.g. [15, Theorem 3.6]). $\square$

Theorem 2.6 implies that the unique strong solution $\mathbf{u}$ of the abstract Cauchy problem (P_{3D}) solves the following problem $(\tilde{P}_{3D})$ on Q for a.e. $t \in (0,T]$ in the following weak sense:

$$(\tilde{P}_{3D}) \begin{cases} \frac{\partial \mathbf{u}}{\partial t}(t,x) + \operatorname{curl}\left(|\operatorname{curl}\mathbf{u}(t,x)|^{p-2}\operatorname{curl}\mathbf{u}(t,x)\right) = \mathbf{F}(t,x) & \text{in } L^{p'}(Q), \\ \operatorname{div}\mathbf{u} = 0 & \text{a.e. in } Q, \\ \langle \boldsymbol{\nu} \times \mathbf{u}, \mathbf{v} \rangle_{(B^{p,p}_{\alpha}(\partial Q)')^3, B^{p,p}_{\alpha}(\partial Q)^3} = \mathbf{0} & \forall\, \mathbf{v} \in B^{p,p}_{\alpha}(\partial Q)^3, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) & \text{in } H. \end{cases}$$

Hence, the above problem $(\tilde{P}_{3D})$ is the strong interpration of (P_{3D}).

3 The asymptotic behavior

3.1 The axial-symmetric case

We now consider the case of a axial-symmetric domain. We suppose in this section that $Q = \Omega \times I$, where $\Omega \subset \mathbb{R}^2$ is a 2D (ε, δ) domain with boundary a d-set and $I = [a,b] \subset \mathbb{R}$. If $\mathbf{F}(t,x) = (0,0,f(t,x_1,x_2))$, then we assume that $\mathbf{u} = (0,0,u(x_1,x_2))$. Problem $(\tilde{P}_{3D})$ then reduces to finding a function $u = u(x_1,x_2)$ on Ω such that

$$(\tilde{P}_{2D}) \begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f & \text{in } [0,T] \times \Omega, \\ u = 0 & \text{on } [0,T] \times \partial\Omega, \\ u(0,P) = u_0(P) & \text{in } \overline{\Omega}. \end{cases} \tag{3.1}$$

The domain $\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : (x_1,x_2,0) \in Q\}$ is a cross section of Q, i.e. $\Omega \times \{0\} = Q \cap \{x \in \mathbb{R}^3 : x_3 = 0\}$.

The energy functional associated with $(\tilde{P}_{2D})$ is

$$\Phi_p[u] := \begin{cases} \dfrac{1}{p} \displaystyle\int_\Omega |\nabla u|^p \, \mathrm{d}\mathcal{L}_2 & \text{if } u \in D(\Phi_p), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus D(\Phi_p), \end{cases} \tag{3.2}$$

where $D(\Phi_p) = W^{1,p}_0(\Omega)$.

For $f \in L^2([0,T]; L^2(\Omega))$ and $u_0 \in \overline{D(-\partial\Phi_p)}$, existence and uniqueness of the weak solution of problem $(\tilde{P}_{2D})$ follows by nonlinear semigroup theory as in Theorem 2.3 by following the patterns of Section 2.

We now consider approximating problems on the Lipschitz domains Ω_n which approximate the (ε, δ) domain Ω in the sense of Lemma 1.5.

We now come to the problems on the approximating Lipschitz domains. For every fixed $n \in \mathbb{N}$, we consider the following problems $(\tilde{P}_{2D,n})$:

$$(\tilde{P}_{2D,n}) \begin{cases} \frac{\partial u_n}{\partial t} - \Delta_p u_n = f & \text{in } [0,T] \times \Omega_n, \\ u_n = 0 & \text{on } [0,T] \times \partial\Omega_n, \\ u_n(0,P) = u_n^0(P) & \text{in } \overline{\Omega}_n. \end{cases} \tag{3.3}$$

We set

$$W^{1,p}_0(\Omega_n) := \overline{\{u \in C^1_0(\Omega) : \operatorname{supp} u \subset \Omega_n\}}^{W^{1,p}(\Omega)}.$$

Again from nonlinear semigroup theory, for every $n \in \mathbb{N}$, given $f \in L^2([0,T]; L^2(\Omega))$ and $u_n^0 \in \overline{D(-\partial\Phi_p^{(n)})}$ there exists a unique weak solution $u_n \in W^{1,p}_0(\Omega_n)$ of problem $(\tilde{P}_{2D,n})$.

The following result states the convergence of the pre-fractal solutions u_n to the solution u of problem $(\tilde{P}_{2D})$ in a suitable sense.

Theorem 3.1. *Let u and u_n be the solutions of the homogeneous problems associated to $(\tilde{P}_{2D})$ and $(\tilde{P}_{2D,n})$ respectively. Then u_n strongly converges to u in $W^{1,p}_0(\Omega)$ as $n \to +\infty$ for every $t \in [0,T]$.*

Proof. The result follows from [33, Corollary 2, p. 557] since Ω_n is an increasing sequence of sets invading Ω and $\mathrm{cap}_{p,\Omega}(E \setminus \Omega_n) \to 0$ when $n \to +\infty$ for any compact subset E of Ω, where, for any compact subset $E \subset \Omega$, its relative p-capacity with respect to Ω is defined by

$$\mathrm{cap}_{p,\Omega}(E) = \inf \left\{ \|\varphi\|^2_{W^{1,p}(\Omega)} : \varphi \in \mathcal{D}(\Omega) \text{ and } \varphi \geq 1 \text{ on } E \right\},$$

see [33, p. 531]. □

3.2 The general 3D case

We now investigate the approximation of the solution of the homogeneous 3D problem associated to $(\tilde{P}_{3D})$ in terms of smoother solutions, as in the axial-symmetric case in Section 3.1. A crucial tool in this asymptotic study will be Gaffney inequality (see Theorem 1.8).

We introduce the new energy functional on $H = L^2(Q)^3$:

$$\tilde{\Phi}_p[\mathbf{u}] := \begin{cases} \dfrac{1}{p} \displaystyle\int_Q |\operatorname{curl} \mathbf{u}|^p \, \mathrm{d}\mathcal{L}_3 & \text{if } \mathbf{u} \in D(\tilde{\Phi}_p), \\ +\infty & \text{if } \mathbf{u} \in H \setminus D(\tilde{\Phi}_p), \end{cases} \tag{3.4}$$

where in this case the effective domain is $D(\tilde{\Phi}_p) := D(\Phi_p) \cap W^{1,p}(Q)$. We point out that, for irregular domains, in general $D(\Phi_p)$ is not a subspace of $W^{1,p}(Q)$; for a counterexample in the Lipschitz case, see [2, page 832].

The natural norm on $D(\tilde{\Phi}_p)$ is the following:

$$\|\mathbf{u}\|^p_{D(\tilde{\Phi}_p)} := \|\mathbf{u}\|^p_{W^{1,p}(Q)^3} + \|\operatorname{curl} \mathbf{u}\|^p_{L^p(Q)^3},$$

which is equivalent just to $\|\operatorname{curl} \mathbf{u}\|^p_{L^p(Q)^3}$, thanks to Gaffney inequality (1.10).

Proceeding as in Section 2, the following result holds.

Proposition 3.2. *$\tilde{\Phi}_p$ is a weakly lower semicontinuous, proper and convex functional in H. Moreover, its subdifferential $\partial\tilde{\Phi}_p$ is single-valued.*

From nonlinear semigroup theory, by using the techniques of Section 2, it follows that the following problem admits a unique weak solution:

$$(\tilde{P}) \begin{cases} \frac{\partial \mathbf{u}}{\partial t}(t,x) + \operatorname{curl}\left(|\operatorname{curl} \mathbf{u}(t,x)|^{p-2} \operatorname{curl} \mathbf{u}(t,x)\right) = \mathbf{0} & \text{in } L^{p'}(Q), \\ \operatorname{div} \mathbf{u} = 0 & \text{a.e. in } Q, \\ \langle \boldsymbol{\nu} \times \mathbf{u}, \mathbf{v} \rangle_{(B^{p,p}_\alpha(\partial Q)')^3, B^{p,p}_\alpha(\partial Q)^3} = \mathbf{0} & \forall\, \mathbf{v} \in B^{p,p}_\alpha(\partial Q)^3, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) & \text{in } H. \end{cases}$$

We now consider corresponding problems on the approximating Lipschitz domains Q_n, for $n \in \mathbb{N}$, given by Lemma 1.5. We point out that, if $u \in L^2(Q)$, then $u \in L^2(Q_n)$ for every $n \in \mathbb{N}$. We denote by $\boldsymbol{\nu}_n$ the normal outward unit vector to Q_n. Moreover, we recall that, since Q_n is Lipschitz for every $n \in \mathbb{N}$, the trace space of $W^{1,p}(Q_n)$ is $W^{1-\frac{1}{p},p}(Q_n)$.

Let $\mathbb{W}$ be the space of restrictions to Q_n of functions $\mathbf{u}$ defined on Q for which the following norm is finite:

$$\|\mathbf{u}\|^p_{\mathbb{W}} := \|\mathbf{u}\|^p_{W^{1,p}(Q_n)^3} + \|\operatorname{curl} \mathbf{u}\|^p_{L^p(Q_n)^3}.$$

We introduce the energy functionals on Q_n defined on $H = L^2(Q)^3$:

$$\tilde{\Phi}_p^{(n)}[\mathbf{u}] := \begin{cases} \frac{1}{p} \int\limits_{Q_n} |\operatorname{curl} \mathbf{u}|^p \, \mathrm{d}\mathcal{L}_3 & \text{if } \mathbf{u}|_{Q_n} \in D(\tilde{\Phi}_p^{(n)}), \\ +\infty & \text{if } \mathbf{u} \in H \setminus D(\tilde{\Phi}_p^{(n)}), \end{cases} \tag{3.5}$$

where in this case the effective domain is

$$D(\tilde{\Phi}_p^{(n)}) := \{\mathbf{u} \in \mathbb{W} \, : \, \operatorname{div} \mathbf{u} = 0 \text{ in } Q_n \text{ and } \boldsymbol{\nu}_n \times \mathbf{u} = \mathbf{0} \text{ on } \partial Q_n\}.$$

As in the previous case, the norm of $D(\tilde{\Phi}_p^{(n)})$ is equivalent to $\| \operatorname{curl} \mathbf{u}\|^p_{L^p(Q_n)^3}$.

Again using the techniques of Section 2, we prove that for every $n \in \mathbb{N}$ the following problem admits a unique weak solution $\mathbf{u}_n$:

$$(\tilde{P}_n) \begin{cases} \frac{\partial \mathbf{u}_n}{\partial t}(t,x) + \operatorname{curl}\left(|\operatorname{curl} \mathbf{u}_n(t,x)|^{p-2} \operatorname{curl} \mathbf{u}_n(t,x)\right) = \mathbf{0} & \text{in } L^{p'}(Q_n), \\ \operatorname{div} \mathbf{u}_n = 0 & \text{a.e. in } Q_n, \\ \langle \boldsymbol{\nu}_n \times \mathbf{u}_n, \mathbf{v} \rangle_{(W^{1-\frac{1}{p},p}(\partial Q_n)')^3, W^{1-\frac{1}{p},p}(\partial Q_n)^3} = \mathbf{0} & \forall\, \mathbf{v} \in W^{1-\frac{1}{p},p}(\partial Q_n)^3, \\ \mathbf{u}_n(0,x) = \mathbf{u}_0^{(n)}(x) & \text{in } H. \end{cases}$$

For the sake of completeness, we explicitly write the existence and uniqueness theorems for problems $(\tilde{P})$ and $(\tilde{P}_n)$. Let $\partial\tilde{\Phi}_p$ and $\partial\tilde{\Phi}_p^{(n)}$ denote the subdifferentials of $\tilde{\Phi}_p$ and $\tilde{\Phi}_p^{(n)}$ respectively. Let also $\tilde{T}_p(t)$ and $\tilde{T}_p^{(n)}(t)$ denote the nonlinear semigroups generated by $-\tilde{\Phi}_p$ and $-\tilde{\Phi}_p^{(n)}$ respectively.

Theorem 3.3. *If $\mathbf{u}_0 \in \overline{D(-\partial\tilde{\Phi}_p)}$, then problem $(\tilde{P})$ has a unique strong solution $\mathbf{u} \in C([0,T]; H)$ defined by $\mathbf{u} = \tilde{T}_p(\cdot)\mathbf{u}_0$ such that $\mathbf{u} \in W^{1,2}((\delta,T); H)$ for every $\delta \in (0,T)$. Moreover $\mathbf{u} \in D(-\partial\tilde{\Phi}_p)$ for a.e. $t \in (0,T)$, $\sqrt{t}\frac{\partial \mathbf{u}}{\partial t} \in L^2(0,T;H)$ and $\tilde{\Phi}_p[\mathbf{u}] \in L^1(0,T)$.*

Theorem 3.4. *For every $n \in \mathbb{N}$, if $\mathbf{u}_0^{(n)} \in \overline{D(-\partial\tilde{\Phi}_p^{(n)})}$, then problem $(\tilde{P}_n)$ has a unique strong solution $\mathbf{u}_n \in C([0,T]; H)$ defined by $\mathbf{u}_n = \tilde{T}_p^{(n)}(\cdot)\mathbf{u}_0$ such that $\mathbf{u}_n \in W^{1,2}((\delta,T); H)$ for every $\delta \in (0,T)$. Moreover $\mathbf{u}_n \in D(-\partial\tilde{\Phi}_p^{(n)})$ for a.e. $t \in (0,T)$, $\sqrt{t}\frac{\partial \mathbf{u}_n}{\partial t} \in L^2(0,T;H)$ and $\tilde{\Phi}_p^{(n)}[\mathbf{u}_n] \in L^1(0,T)$.*

We are now interested in proving a convergence result similar to the 2D axial-symmetric case given in Theorem 3.1. In order to do so, we will use the notion of *M-convergence* of energy functionals.

We recall the definition of M-convergence adapted to our case. This definition was first introduced by Mosco in [33]; here we recall the definition given in [34, Definition 2.1.1].

Definition 3.5. *A sequence of proper and convex functionals $\left\{\Phi_p^{(n)}\right\}$ defined on an Hilbert space H M-converges to a functional Φ_p in H if the following hold:*

a) *for every $\{\mathbf{v}_n\} \in H$ weakly converging to $\mathbf{u} \in H$*

$$\varliminf_{n\to\infty} \Phi_p^{(n)}[\mathbf{v}_n] \geq \Phi_p[\mathbf{u}].$$

b) *for every* $\mathbf{u} \in H$ *there exists a sequence* $\{\mathbf{w}_n\} \in H$ *strongly converging to* $\mathbf{u}$ *in* H*, such that*

$$\overline{\lim_{n\to\infty}} \Phi_p^{(n)}[\mathbf{w}_n] \leq \Phi_p[\mathbf{u}].$$

Theorem 3.6. *Let* $\tilde{\Phi}_p$ *and* $\tilde{\Phi}_p^{(n)}$ *be defined as in* (3.4) *and* (3.5) *respectively. Then* $\tilde{\Phi}_p^{(n)}$ *M-converges to the functional* $\tilde{\Phi}_p$*.*

Proof. We have to prove conditions a) and b) in Definition 3.5.
Proof of condition a). Let $\mathbf{v}_n \in H$ be a weakly converging sequence to $\mathbf{u} \in H$. We can suppose $v_n \in D(\tilde{\Phi}_p^{(n)})$ and

$$\underline{\lim}_{n\to\infty} \tilde{\Phi}_p^{(n)}[\mathbf{v}_n] < \infty,$$

otherwise the thesis follows trivially. Thus there exists a constant independent from n such that

$$\frac{1}{p}\int_{Q_n} |\operatorname{curl} \mathbf{v}_n|^p \, \mathrm{d}\mathcal{L}_3 \leq C. \tag{3.6}$$

In particular, we have that $\|\mathbf{v}_n\|_{D(\tilde{\Phi}_p^{(n)})} < C$.
We remark that, since Q_n is Lipschitz, it enjoys the extension property. We now consider the trivial extension $\tilde{\mathbf{v}}_n$ of $\mathbf{v}_n$ to Q. By direct inspection, it holds that

$$\|\tilde{\mathbf{v}}_n\|_{D(\tilde{\Phi}_p)} = \|\mathbf{v}_n\|_{D(\tilde{\Phi}_p^{(n)})} \leq C.$$

Therefore, there exists a subsequence (which we still denote by $\tilde{\mathbf{v}}_n$) weakly converging to some $\tilde{\mathbf{v}}$ in $D(\tilde{\Phi}_p)$; moreover, $\tilde{\mathbf{v}}_n$ strongly converges to $\tilde{\mathbf{v}}$ in $L^p(Q)^3$ (and hence also in $L^2(Q)^3$ since $p \geq 2$). We now prove that $\tilde{\mathbf{v}} = \mathbf{u}$ in $L^2(Q)^3$, that is

$$\int_Q (\tilde{\mathbf{v}} - \mathbf{u}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}_3 = 0$$

for every $\boldsymbol{\varphi} \in L^2(Q)^3$.
We first note that

$$\begin{aligned}\int_Q (\tilde{\mathbf{v}} - \mathbf{u}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}_3 &= \int_Q (\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_n + \tilde{\mathbf{v}}_n - \mathbf{u}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}_3 \\ &= \int_Q (\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_n) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}_3 + \int_{Q_n} (\mathbf{v}_n - \mathbf{u}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}_3 + \int_{Q\setminus Q_n} (\tilde{\mathbf{v}}_n - \mathbf{u}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathcal{L}_3.\end{aligned} \tag{3.7}$$

We claim that each term on the right-hand side of (3.7) tends to zero as $n \to +\infty$. From the strong convergence of $\tilde{\mathbf{v}}_n$ to $\tilde{\mathbf{v}}$ in $L^2(Q)^3$ and the weak convergence of $\mathbf{v}_n$ to

$\mathbf{u}$ in $L^2(Q)^3$, we deduce our claim for the first two terms. As to the third, from Hölder inequality we deduce that

$$\int_{Q\setminus Q_n} |(\tilde{\mathbf{v}}_n - \mathbf{u})\cdot\boldsymbol{\varphi}|\,\mathrm{d}\mathcal{L}_3 \le \|\boldsymbol{\varphi}\|_{L^2(Q\setminus Q_n)^3}(\|\tilde{\mathbf{v}}_n\|_{L^2(Q)^3} + \|\mathbf{u}\|_{L^2(Q)^3}) \xrightarrow[n\to+\infty]{} 0$$

since Q is bounded, $|Q\setminus Q_n| \to 0$ as $n\to+\infty$ and $\tilde{\mathbf{v}}_n$ is equibounded in $D(\tilde{\Phi}_p)$ and in $L^2(Q)^3$. Hence $\tilde{\mathbf{v}}_n \rightharpoonup \mathbf{u}$ in $D(\tilde{\Phi}_p)$ and $\tilde{\mathbf{v}}_n \to \mathbf{u}$ in $L^p(Q)^3$. The thesis then follows from the lower semicontinuity of the norm.

Proof of condition b). We prove that for every $\mathbf{u}\in H$ we can construct a sequence $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ strongly converging to $\mathbf{u}$ in H such that

$$\tilde{\Phi}_p[\mathbf{u}] \ge \overline{\lim_{n\to\infty}}\,\tilde{\Phi}_p^{(n)}[\mathbf{w}_n].$$

We suppose that $\mathbf{u}\in D(\tilde{\Phi}_p)$, otherwise $\tilde{\Phi}_p[\mathbf{u}] = +\infty$ and the thesis follows trivially. We set

$$\mathbf{w}_n := \begin{cases} \mathbf{u} & \text{in } Q_n,\\ \mathbf{0} & \text{in } Q\setminus Q_n. \end{cases}$$

We point out that $\mathbf{w}_n$ strongly converges to $\mathbf{u}$ in H. Indeed, it holds that

$$\|\mathbf{w}_n - \mathbf{u}\|^2_{L^2(Q)^3} = \|\mathbf{w}_n - \mathbf{u}\|^2_{L^2(Q_n)^3} + \|\mathbf{w}_n - \mathbf{u}\|^2_{L^2(Q\setminus Q_n)^3} = \|\mathbf{u}\|^2_{L^2(Q\setminus Q_n)^3} \xrightarrow[n\to+\infty]{} 0.$$

We now prove condition b) of Definition 3.5 for $\mathbf{w}_n$. We have that

$$\begin{aligned}\overline{\lim_{n\to\infty}}\,\tilde{\Phi}_p^{(n)}[\mathbf{w}_n] &= \overline{\lim_{n\to\infty}}\,\frac{1}{p}\int_{Q_n} |\operatorname{curl}\mathbf{w}_n|^p\,\mathrm{d}\mathcal{L}_3\\ &= \overline{\lim_{n\to\infty}}\,\frac{1}{p}\int_{Q_n} |\operatorname{curl}\mathbf{u}|^p\,\mathrm{d}\mathcal{L}_3 = \frac{1}{p}\int_{Q} |\operatorname{curl}\mathbf{u}|^p\,\mathrm{d}\mathcal{L}_3 = \tilde{\Phi}_p[\mathbf{u}],\end{aligned}$$

where the second-to-last equality follows from Lemma 1.5. □

The M-convergence of the energy functionals is equivalent to the G-convergence of the associated subdifferentials, as stated in the following result.

Theorem 3.7. *$\tilde{\Phi}_p^{(n)}$ M-converges to $\tilde{\Phi}_p$ in H if and only if $\partial\tilde{\Phi}_p^{(n)}$ G-converges to $\partial\tilde{\Phi}_p$.*

For the proof see Theorem 3.66 in [3].

Theorem 3.8. *Let $\tilde{\Phi}_p^{(n)}$ and $\tilde{\Phi}_p$ be as in Theorem 3.6. Let $\tilde{T}_p^{(n)}(t)$, $\tilde{T}_p(t)$, $\mathbf{u}_0^{(n)}$ and $\mathbf{u}_0$ be as in Theorems 3.3 and 3.4. If $\mathbf{u}_0^{(n)} \to \mathbf{u}_0$ strongly in H, then $\{\mathbf{u}_n(t)\}$ converges to $\mathbf{u}(t)$ strongly in H for every $t\in[0,T]$.*

This convergence result follows from Theorem 3.6, Theorem 3.7 and Theorems 3.16 and 4.2 in [9].

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