Transmission problems for the fractional p-Laplacian across fractal interfaces

Simone Creo, Maria Rosaria Lancia and Paola Vernole

Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sapienza Università di Roma, Via A. Scarpa 16, 00161 Roma, Italy.

Abstract

We consider a parabolic transmission problem, involving nonlinear fractional operators of different order, across a fractal interface Σ. The transmission condition is of Robin type and it involves the jump of the p-fractional normal derivatives on the irregular interface. After proving existence and uniqueness results for the weak solution of the problem at hand, via a semigroup approach, we investigate the regularity of the nonlinear fractional semigroup.

Keywords: fractional p-Laplacian, fractal domains, fractional Green formula, nonlinear semigroups, ultracontractivity.

2010 Mathematics Subject Classification: Primary: 35R11, 47H20. Secondary: 28A80, 35B65, 47J35.

Introduction

Aim of this paper is to study a parabolic nonlocal transmission problem of the form

$$
(\tilde{P}) \begin{cases}
\frac{\partial u_1}{\partial t}(t,x) + (-\Delta_p)^{\beta}_{\Omega_1} u_1(t,x) = f_1(t,x) & \text{in } (0,T] \times \Omega_1, \\
\frac{\partial u_2}{\partial t}(t,x) + (-\Delta_p)^{\alpha}_{\Omega_2} u_2(t,x) = f_2(t,x) & \text{in } (0,T] \times \Omega_2, \\
u_1 = w u_2 & \text{on } (0,T] \times \Sigma, \\
\mathcal{N} u + b|u_2|^{p-2} u_2 = 0 & \text{on } (0,T] \times \Sigma, \\
\mathcal{N}_p^{p'(1-\alpha)} u_2 = 0 & \text{on } (0,T] \times \partial\Omega, \\
u_1(0,x) = u_1^0(x) & \text{in } \Omega_1, \\
u_2(0,x) = u_2^0(x) & \text{in } \Omega_2,
\end{cases}
$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded polygonal domain and Σ is a fractal interface of Koch type which divides Ω in two subdomains Ω_1 and Ω_2 (see Figure 1). Here $p > 1$, $(-\Delta_p)^{\beta}_{\Omega_1}$ and $(-\Delta_p)^{\alpha}_{\Omega_2}$ are the regional fractional p-Laplacians on Ω_1 and Ω_2 of order β and α respectively (see Section 2) and $\alpha, \beta \in (0,1)$ such that $\alpha \geq \beta$. $\mathcal{N}_p^{p'(1-\alpha)}$ is the (α,p)-fractional normal derivative on $\partial\Omega$ and $\mathcal{N}u$ denotes the jump of the p-fractional normal derivatives, to be suitably defined (see Section 3). w, b, f_1, f_2, u_1^0 and u_2^0 are given functions.

There is a huge literature on fractional operators. This is due to the fact that they describe mathematically many physical phenomena which exhibit deviations from standard diffusion. This is the so-called *anomalous diffusion*, and it is an important topic not only in physics, but also in finance and probability (see [1, 22, 33, 35]).

This diffusion is present in several models appearing in the literature. Among the others, we mention the fractional Brownian motion, the continuous time random walk, the Lévy flight as well as random walk models based on evolution equations of single and distributed fractional order in time and/or space [13, 19, 32, 35, 37].

The study of transmission problems involving linear fractional diffusion operators has been considered for the first time in the case of a Lipschitz interface in [18] (see also [16, 17]). As to the case of irregular interfaces, the first examples in the literature of transmission problems across fractal interfaces for linear second order operators with second order transmission conditions can be found in [27, 31, 30]. From the physical point of view, these latter problems describe, in electrostatics and magnetostatics, the heat flow across highly conductive thin layers (see [34] and the references listed in). Further examples can be found in [11].

As to the case of fractional operators in irregular domains, Robin-Venttsel'-type boundary value problems for the regional fractional p-Laplacian in extension domains with highly irregular boundary have been recently investigated in [8], and their approximation in terms of smoother problems has been studied in [10] (see [9] for the linear case).

In the present paper, we study problem $(\tilde{P})$, formally stated above, by a semigroup approach. We firstly prove existence and uniqueness of the "strong" solution of the associated abstract Cauchy problem (see Theorem 3.3), then we prove that such strong solution actually solves problem $(\tilde{P})$ in a suitable weak sense (see Theorem 3.6). A key issue is to prove that the jump condition on Σ is satisfied. These results are achieved by a suitable characterization of the subdifferential of the nonlinear functional associated to the problem and by a p-fractional Green formula for irregular domains (see Theorem 2.2).

Finally, in Theorem 4.7 we prove the ultracontractivity of the associated semigroup. This result deeply relies on a fractional logarithmic Sobolev inequality adapted to the present framework (see Proposition 4.1).

It turns out that, under our hypotheses on α and β, the dominant diffusion is the one in Ω_1. The case when $\alpha \leq \beta$ is also investigated in Section 5.

The paper is organized as follows.

In Section 1 we introduce the domain Ω and the functional setting and we recall some known trace and embedding results.

In Section 2 we recall the definition of regional fractional p-Laplacian and we state a p-fractional Green formula for irregular domains.

In Section 3 we prove via semigroup theory that problem $(\tilde{P})$ admits a unique solution in a suitable weak sense.

In Section 4 we prove that the semigroup associated to our problem is ultracontractive.

In Section 5 we consider the case $\alpha \leq \beta$ and we discuss some open problems.

1 Preliminaries

1.1 Geometry and functional spaces

Given $P, P_0 \in \mathbb{R}^N$, in this paper we denote by $|P - P_0|$ the Euclidean distance in $\mathbb{R}^N$ and by $B(P_0, r) = \{P \in \mathbb{R}^N : |P - P_0| < r\}$, for $r > 0$, the Euclidean ball. We also denote by $\mathcal{L}_N$ the N-dimensional Lebesgue measure.

We denote by Σ the Koch snowflake, i.e. the union of three co-planar Koch curves K_1, K_2 and K_3 (see [14]). We assume that the junction points A_1, A_3 and A_5 are the vertices of a regular triangle with unit side length, i.e. $|A_1 - A_3| = |A_1 - A_5| = |A_3 - A_5| = 1$. For $i = 1, 2, 3$, K_i is the uniquely determined self-similar set with respect to a family Ψ^i of four suitable contractions $\psi_1^{(i)}, \dots, \psi_4^{(i)}$, with respect to the same ratio $\frac{1}{3}$ (see [15]). The Hausdorff dimension of the Koch snowflake is given by $d_f = \frac{\ln 4}{\ln 3}$. One can define, in a natural way, a finite Borel measure μ supported on K by

$$\mu_\Sigma := \mu_1 + \mu_2 + \mu_3, \tag{1.1}$$

where μ_i denotes the normalized d_f-dimensional Hausdorff measure restricted to K_i, for $i = 1, 2, 3$.
In this paper we consider a bounded open polygonal domain $\Omega \subset \mathbb{R}^2$ (for simplicity, one can take a rectangle) which is divided in two subdomains Ω_1 and Ω_2 by the Koch snowflake Σ. More precisely, $\Omega = \Omega_1 \cup \Omega_2$, $\Sigma = \overline{\Omega_1} \cap \overline{\Omega_2}$, $\partial\Omega_1 = \Sigma$ and $\partial\Omega_2 = \Gamma \cup \Sigma$ (see Figure 1).

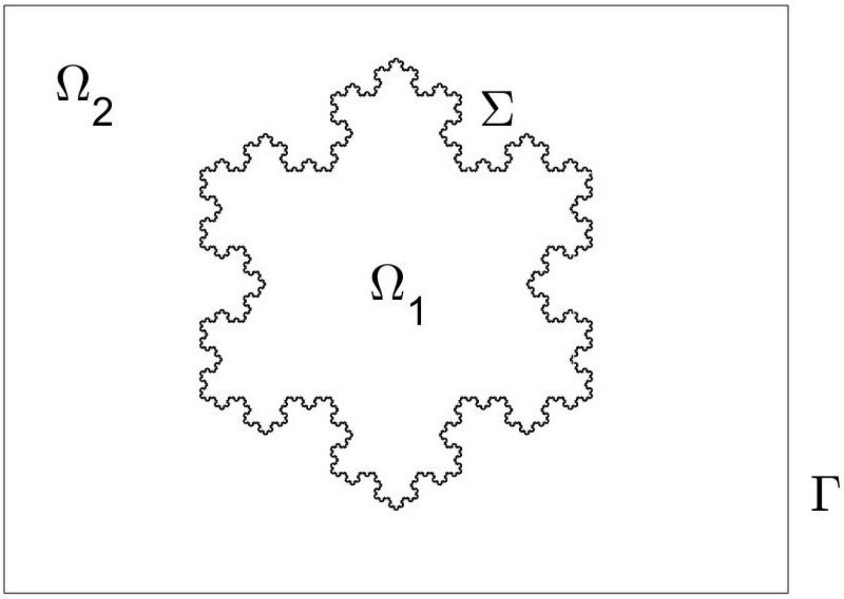

Figure 1: The domain Ω.

Let $\mathcal{G}$ (resp. $\mathcal{S}$) be an open (resp. closed) set of $\mathbb{R}^N$. By $L^p(\mathcal{G})$, for $p > 1$, we denote the Lebesgue space with respect to the Lebesgue measure $\mathcal{L}_N$, which will be left to the context whenever that does not create ambiguity. By $L^p(\partial\mathcal{G}, \mu)$ we denote the Lebesgue space on $\partial\mathcal{G}$ with respect to a Hausdorff measure μ supported on $\partial\mathcal{G}$. By $\mathcal{D}(\mathcal{G})$ we denote the space of infinitely differentiable functions with compact support in $\mathcal{G}$. By $C(\mathcal{S})$ we denote the space of continuous functions on $\mathcal{S}$.
By $W^{s,p}(\mathcal{G})$, where $0 < s < 1$, we denote the fractional Sobolev space of exponent s. We point out that it is a Banach space if we endow it with the following norm:

$$\|u\|^p_{W^{s,p}(\mathcal{G})} = \|u\|^p_{L^p(\mathcal{G})} + \iint\limits_{\mathcal{G}\times\mathcal{G}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y).$$

Moreover, we denote by $|u|_{W^{s,p}(\mathcal{G})}$ the seminorm associated to $\|u\|_{W^{s,p}(\mathcal{G})}$ and, for $u, v \in W^{s,p}(\mathcal{G})$, we set

$$(u,v)_{s,p} := \iint\limits_{\mathcal{G}\times\mathcal{G}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\, \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y).$$

In the following we will denote by $|A|$ the Lebesgue measure of a measurable subset $A \subset \mathbb{R}^N$. For f in $W^{s,p}(\mathcal{G})$, we define the trace operator γ_0 as

$$\gamma_0 f(x) := \lim_{r\to 0} \frac{1}{|B(x,r)\cap\mathcal{G}|} \int\limits_{B(x,r)\cap\mathcal{G}} f(y)\, \mathrm{d}\mathcal{L}_N(y) \tag{1.2}$$

at every point $x \in \overline{\mathcal{G}}$ where the limit exists. The limit (1.2) exists at quasi every $x \in \overline{\mathcal{G}}$ with respect to the (s,p)-capacity (see [2], Definition 2.2.4 and Theorem 6.2.1 page 159). In the sequel we will omit the trace symbol and the interpretation will be left to the context.

We recall two trace theorems, one for polygonal domains and one for irregular domains. We first state the trace theorem in the polygonal case. For the proof we refer to [6].

Proposition 1.1. *Let $\frac{1}{p} < s < 1$ and let $\mathcal{G}$ be a polygonal domain. Then $W^{s-\frac{1}{p},p}(\partial\mathcal{G})$ is the trace space to $\partial\mathcal{G}$ of $W^{s,p}(\mathcal{G})$ in the following sense:*

(i) γ_0 is a continuous and linear operator from $W^{s,p}(\mathcal{G})$ to $W^{s-\frac{1}{p},p}(\partial\mathcal{G})$;

(ii) there exists a continuous linear operator Ext from $W^{s-\frac{1}{p},p}(\partial\mathcal{G})$ to $W^{s,p}(\mathcal{G})$ such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator in $W^{s-\frac{1}{p},p}(\partial\mathcal{G})$.

We now state the trace theorem for the case of a domain with fractal boundary. We recall that the Koch snowflake is a d_f-set and the measure μ_Σ is a d_f-measure in the following sense (for more details, see [25]).

Definition 1.2. *A closed nonempty set $\mathcal{M} \subset \mathbb{R}^N$ is a d-set (for $0 < d \le N$) if there exist a Borel measure μ with $\operatorname{supp}\mu = \mathcal{M}$ and two positive constants c_1 and c_2 such that*

$$c_1 r^d \le \mu(B(x,r)\cap\mathcal{M}) \le c_2 r^d \quad \forall\, x \in \mathcal{M}. \tag{1.3}$$

The measure μ is called d-measure.

From now on we denote the d_f-measure on Σ simply by μ.

We now recall the definition of Besov space specialized to our case. For generalities on Besov spaces, we refer to [25].

Definition 1.3. *Let $\mathcal{F}$ be a d-set with respect to a d-measure μ and $\gamma = s - \frac{N-d}{p}$. $B^{p,p}_\gamma(\mathcal{F})$ is the space of functions for which the following norm is finite:*

$$\|u\|^p_{B^{p,p}_\gamma(\mathcal{F})} = \|u\|^p_{L^p(\mathcal{F})} + \iint\limits_{|x-y|<1} \frac{|u(x)-u(y)|^p}{|x-y|^{d+\gamma p}} \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).$$

Let p' be the Hölder conjugate exponent of p. In the following, we will denote the dual of the Besov space $B^{p,p}_\gamma(\mathcal{F})$ with $(B^{p,p}_\gamma(\mathcal{F}))'$; we point out that this space coincides with the space $B^{p',p'}_{-\gamma}(\mathcal{F})$ (see [26]).

We now state the trace theorem to the fractal set Σ. For the proof, we refer to [25, Theorem 1, Chapter VII].

Proposition 1.4. *Let $\mathcal{F}$ denote Ω_1 or Ω_2. Let $\frac{2-d_f}{p} < s < 1$ and $\gamma(s) := s - \frac{2-d_f}{p} > 0$. $B^{p,p}_{\gamma(s)}(\Sigma)$ is the trace space of $W^{s,p}(\mathcal{F})$ in the following sense:*

(i) γ_0 is a continuous linear operator from $W^{s,p}(\mathcal{F})$ to $B^{p,p}_{\gamma(s)}(\Sigma)$;

(ii) there exists a continuous linear operator Ext *from $B^{p,p}_{\gamma(s)}(\Sigma)$ to $W^{s,p}(\mathcal{F})$ such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator in $B^{p,p}_{\gamma(s)}(\Sigma)$.*

From now on we set

$$\gamma(s) := s - \frac{2-d_f}{p} > 0. \tag{1.4}$$

If u is a suitable function defined on the whole Ω, from now on we set $u_i := u|_{\Omega_i}$ for $i = 1, 2$.

We point out that, if $u \in L^p(\Omega)$, then it follows that $u_1 \in L^q(\Omega_1)$ and $u_2 \in L^q(\Omega_2)$. Hence, for $1 \le q < \infty$, we have that

$$\|u\|^q_q := \|u\|^q_{L^q(\Omega)} = \|u_1\|^q_{L^q(\Omega_1)} + \|u_2\|^q_{L^q(\Omega_2)} \tag{1.5}$$

If $q = \infty$, it holds that

$$\|u\|_\infty := \|u\|_{L^\infty(\Omega)} = \max\left\{\|u_1\|_{L^\infty(\Omega_1)}, \|u_2\|_{L^\infty(\Omega_2)}\right\}.$$

We now introduce the following Sobolev-type space:

$$\mathbb{W}^{\alpha,\beta}_p(\Omega) := \{u \in L^p(\Omega) \,:\, u_1 \in W^{\beta,p}(\Omega_1),\, u_2 \in W^{\alpha,p}(\Omega_2) \text{ and } u_1 = w u_2 \text{ on } \Sigma\}, \tag{1.6}$$

where w is a suitable smooth function defined on Σ. We endow this space with the following norm:

$$\|u\|^p_{\mathbb{W}^{\alpha,\beta}_p(\Omega)} := \|u\|^p_{L^p(\Omega)} + |u_1|^p_{W^{\beta,p}(\Omega_1)} + |u_2|^p_{W^{\alpha,p}(\Omega_2)}. \tag{1.7}$$

Finally, we recall that the set Ω_1 belongs to the more general class of the so-called (ϵ, δ) *domains* having as boundary a d-set (for all the details we refer to the seminal paper of Jones [23]). Ω_2 is a bounded (ϵ, δ) domain with boundary an arbitrary closed set in the sense of [24]. Such domains, even though they can be very irregular, enjoy the following important extension property; for details, we refer to Theorem 1, page 103 and Theorem 3, page 155 in [25].

Theorem 1.5. *Let $0 < s < 1$. There exists a linear extension operator $\mathcal{E}\mathrm{xt}\colon W^{s,p}(\Omega_1) \to W^{s,p}(\mathbb{R}^2)$ such that*

$$\|\mathcal{E}\mathrm{xt}\, w\|^p_{W^{s,p}(\mathbb{R}^2)} \le \bar{C}_s \|w\|^p_{W^{s,p}(\Omega_1)}, \tag{1.8}$$

with $\bar{C}_s$ depending on s.

The domain Ω_2 satisfies an analogous extension property, we refer to Theorem 1 in [24] for the details.

Domains satisfying property (1.8) are the so-called $W^{s,p}$-extension domain.

1.2 Sobolev embeddings

We now recall some important Sobolev-type embeddings for fractional Sobolev spaces on $W^{s,p}$-extension domains, see [12, Theorem 6.7] and [25, Lemma 1, p. 214] respectively.

From now on, we denote the Hausdorff dimension of Σ simply by d. We set

$$p^*(s) := \frac{2p}{2 - sp}.$$

Theorem 1.6. *Let $s \in (0,1)$ and $p \ge 1$ be such that $sp < 2$. Let $\Omega \subseteq \mathbb{R}^2$ be a $W^{s,p}$-extension domain. Then $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for every $q \in [1, p^*(s)]$, i.e. there exists a positive constant $C = C(s, p, \Omega)$ such that, for every $u \in W^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \le C \|u\|_{W^{s,p}(\Omega)}. \tag{1.9}$$

We point out that for every $0 < s < 1$ such that $sp < 2$ it holds $p^*(s) \ge p$.

From now on, let $\alpha, \beta \in (\frac{2-d}{p}, 1)$ be two real numbers such that $\alpha \ge \beta$. We recall that if $u \in \mathbb{W}^{\alpha,\beta}_p(\Omega)$, then $u_1 \in W^{\beta,p}(\Omega_1)$ and $u_2 \in W^{\alpha,p}(\Omega_2)$; from Theorem 1.6, these spaces are continuously embedded in $L^{p^*(\beta)}(\Omega_1)$ and in $L^{p^*(\alpha)}(\Omega_2)$ respectively. Moreover, it holds that $p^*(\alpha) \ge p^*(\beta)$, hence $L^{p^*(\alpha)}(\Omega_2) \hookrightarrow L^{p^*(\beta)}(\Omega_2)$; hence, we have the following continuous embedding:

$$\mathbb{W}^{\alpha,\beta}_p(\Omega) \hookrightarrow L^{p^*(\beta)}(\Omega). \tag{1.10}$$

We now recall the definition of $\gamma(s)$ given in (1.4). If $u \in \mathbb{W}_p^{\alpha,\beta}(\Omega)$, from Proposition 1.4, we have that

$$\gamma_0 u_1 \in B^{p,p}_{\gamma(\beta)}(\Sigma) \tag{1.11}$$

and

$$\gamma_0 u_2 \in B^{p,p}_{\gamma(\alpha)}(\Sigma). \tag{1.12}$$

We point out that, since $\partial\Omega_2 = \Gamma \cup \Sigma$ and Γ is Lipschitz, from Proposition 1.1 we have that $u_2|_\Gamma \in \mathbb{W}^{\alpha-\frac{1}{p},p}(\Gamma)$.

2 The regional fractional p-Laplacian and the Green formula

We recall the definition of the regional fractional p-Laplacian. We refer to [40] and the references listed in.

Let $s \in (0,1)$ and $p > 1$. For $\mathcal{G} \subseteq \mathbb{R}^N$, we define the space

$$\mathcal{L}^{p-1}_s(\mathcal{G}) := \left\{ u\colon \mathcal{G} \to \mathbb{R} \text{ measurable } : \int\limits_{\mathcal{G}} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+sp}} \, \mathrm{d}\mathcal{L}_N(x) < \infty \right\}.$$

The regional fractional Laplacian $(-\Delta_p)^s_{\mathcal{G}}$ is defined as follows, for $x \in \mathcal{G}$:

$$\begin{aligned} (-\Delta_p)^s_{\mathcal{G}} u(x) &= C_{N,p,s} \mathrm{P.V.} \int\limits_{\mathcal{G}} |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+sp}} \, \mathrm{d}\mathcal{L}_N(y) \\ &= C_{N,p,s} \lim_{\varepsilon \to 0^+} \int\limits_{\{y \in \mathcal{G} \,:\, |x-y|>\varepsilon\}} |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+sp}} \, \mathrm{d}\mathcal{L}_N(y), \end{aligned} \tag{2.1}$$

provided that the limit exists, for every function $u \in \mathcal{L}^{p-1}_s(\mathcal{G})$. The positive constant $C_{N,p,s}$ is defined as follows:

$$C_{N,p,s} = \frac{s 2^{2s} \Gamma(\frac{ps+p+N-2}{2})}{\pi^{\frac{N}{2}} \Gamma(1-s)},$$

where Γ is the Euler function.

We now introduce the p-fractional normal derivative on irregular set. We remark that the fractional normal derivative for smooth domains has been introduced in [20, 21] for the case $p = 2$ and it has then been extended to the case $p \geq 2$ in [40].

We recall a p-fractional Green formula for domains with fractal boundary, which, in turn, allows us to define the p-fractional normal derivative on non-smooth domains, see [8, Section 2]. A key tool is the use of a limit argument since the fractal domain Ω_1 (resp. Ω_2) can be approximated by an increasing (resp. decreasing) sequence of pre-fractal domains Ω_1^n (resp. Ω_2^n). We point out that the sequence of pre-fractal domains Ω_i^n, for $i=1,2$, consists of polygonal non-convex domains.

For $i=1,2$, we define the space

$$V((-\Delta_p)^s_{\Omega_i},\Omega_i):=\{u\in W^{s,p}(\Omega_i)\,:\,(-\Delta_p)^s_{\Omega_i}u\in L^{p'}(\Omega_i)\text{ in the sense of distributions}\},$$

which is a Banach space equipped with the norm

$$\|u\|_{V((-\Delta_p)^s_{\Omega_i},\Omega_i)}:=\|u\|_{W^{s,p}(\Omega_i)}+\|(-\Delta_p)^s_{\Omega_i}u\|_{L^{p'}(\Omega_i)}.$$

Analogously, for every $n\in\mathbb{N}$, we define the space $V((-\Delta_p)^s_{\Omega_i^n},\Omega_i^n)$ on Ω_i^n as follows:

$$V((-\Delta_p)^s_{\Omega_i^n},\Omega_i^n):=\{u\in W^{s,p}(\Omega_i)\,:\,(-\Delta_p)^s_{\Omega_i^n}u\in L^{p'}(\Omega_i^n)\text{ in the sense of distributions}\}.$$

We now give a notion of p-fractional normal derivative on the boundary of the pre-fractal domains Ω_i^n.

Definition 2.1. *Let $n\in\mathbb{N}$ and $u\in V((-\Delta_p)^s_{\Omega_i^n},\Omega_i^n)$ for either $i=1$ or $i=2$. We say that u has a weak p-fractional normal derivative in $(W^{s-\frac{1}{p},p}(\partial\Omega_i^n))'$ if there exists $g\in(W^{s-\frac{1}{p},p}(\partial\Omega_i^n))'$ such that*

$$\langle g,v\rangle_{(W^{s-\frac{1}{p},p}(\partial\Omega_i^n))',W^{s-\frac{1}{p},p}(\partial\Omega_i^n)}=-\int\limits_{\Omega_i^n}(-\Delta_p)^s_{\Omega_i^n}u\,v\,\mathrm{d}\mathcal{L}_2 \tag{2.2}$$

$$+\frac{C_{2,p,s}}{2}\iint\limits_{\Omega_i^n\times\Omega_i^n}|u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)$$

for every $v\in W^{s,p}(\Omega_i^n)$. In this case, g is uniquely determined and we call $C_{p,s}\mathcal{N}_p^{p'(1-s)}u:=g$ the weak p-fractional normal derivative of u, where

$$C_{p,s}:=\frac{(p-1)C_{1,p,s}}{(sp-(p-2))(sp-(p-2)-1)}\int\limits_0^\infty\frac{|t-1|^{(p-2)+1-sp}-(t\vee 1)^{p-sp-1}}{t^{p-sp}}\,\mathrm{d}t.$$

We point out that, when $s\to 1^-$ in (2.2), we recover the quasi-linear Green formula for Lipschitz domains [6].

By proceeding as in the proof of Theorem 2.2 in [8] (see also Theorem 3.2 in [10]), we can prove the following "fractional Green formula" for the fractal domain Ω_1. We can proceed analogously for the fractal domain Ω_2.

Theorem 2.2 (Fractional Green formula)**.** *Let $\gamma(s)$ be as defined in (1.4). There exists a bounded linear operator $\mathcal{N}_p^{p'(1-s)}$ from $V((-\Delta_p)^s_{\Omega_1},\Omega_1)$ to $(B^{p,p}_{\gamma(s)}(\Sigma))'$.*

The following generalized Green formula holds for all $u\in V((-\Delta_p)^s_{\Omega_1},\Omega_1)$ and $v\in W^{s,p}(\Omega_1)$:

$$\begin{aligned} C_{p,s}\left\langle \mathcal{N}_p^{p'(1-s)}u,v\right\rangle_{(B^{p,p}_{\gamma(s)}(\Sigma))',B^{p,p}_{\gamma(s)}(\Sigma)} &= -\int_{\Omega_1}(-\Delta_p)^s_{\Omega_1}u\,v\,\mathrm{d}\mathcal{L}_2 \\ &+\frac{C_{2,p,s}}{2}\iint_{\Omega_1\times\Omega_1}|u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y). \end{aligned} \tag{2.3}$$

We remark that, when $s\to 1^-$ in (2.3), we recover the Green formula stated in [31] for fractal domains. We refer the reader to [10, Remark 3.3] for the detailed proof.

3 Existence and uniqueness results

3.1 The energy functional

From now on, let $p\geq 2$ and let $b\in L^\infty(\partial\Omega)$ be a strictly positive bounded function on $\partial\Omega$. We define the space

$$H:=L^2(\Omega).$$

We point out that H is a Hilbert space with the scalar product $(u,v)_H:=\int_{\Omega_1}u_1v_1\,\mathrm{d}\mathcal{L}_2+\int_{\Omega_2}u_2v_2\,\mathrm{d}\mathcal{L}_2$.

We recall the fractional transmission problem formally stated in the Introduction:

$$(\tilde{P})\begin{cases} \frac{\partial u_1}{\partial t}(t,x)+(-\Delta_p)^\beta_{\Omega_1}u_1(t,x)=f_1(t,x) & \text{in } (0,T]\times\Omega_1,\\ \frac{\partial u_2}{\partial t}(t,x)+(-\Delta_p)^\alpha_{\Omega_2}u_2(t,x)=f_2(t,x) & \text{in } (0,T]\times\Omega_2,\\ u_1=wu_2 & \text{on } (0,T]\times\Sigma,\\ \mathcal{N}u+b|u_2|^{p-2}u_2=0 & \text{on } (0,T]\times\Sigma,\\ \mathcal{N}_p^{p'(1-\alpha)}u_2=0 & \text{on } (0,T]\times\Gamma,\\ u_1(0,x)=u_1^0(x) & \text{in } \Omega_1,\\ u_2(0,x)=u_2^0(x) & \text{in } \Omega_2, \end{cases}$$

where $u\in\mathbb{W}_p^{\alpha,\beta}(\Omega)$, $w\in B^{p,p}_\theta(\Sigma)$ for $\theta\geq\gamma(\beta)$ such that $\theta+\alpha-\beta>\frac{d}{p}$, $\mathcal{N}_p^{p'(1-\alpha)}$ is the (α,p)-normal derivative defined by the Green formula (2.3), $u_i:=u|_{\Omega_i}$ for $i=1,2$

(analogously for f_i) and f_i and u_i^0 are given functions for $i = 1, 2$.
We remark that, under the hypotheses on θ, we have that $wu_2 \in B^{p,p}_{\gamma(\beta)}(\Sigma)$, see [42].

As stated in the Introduction, $\mathcal{N}u$ is the jump of the nonlinear fractional normal derivative in a suitable weak sense. Let $u \in \mathbb{W}^{\alpha,\beta}(\Omega)$. Then, from Theorem 2.2, $\mathcal{N}_p^{p'(1-\beta)} u_1$ and $\mathcal{N}_p^{p'(1-\alpha)} u_2$ are defined by (2.3) as elements of the dual spaces of $B^{p,p}_{\gamma(\beta)}(\Sigma)$ and $B^{p,p}_{\gamma(\alpha)}(\Sigma)$ respectively.
Since by hypothesis $\alpha \geq \beta$, we have that $B^{p,p}_{\gamma(\alpha)}(\Sigma) \subseteq B^{p,p}_{\gamma(\beta)}(\Sigma)$. Hence, for $u, v \in \mathbb{W}^{\alpha,\beta}_p(\Omega)$ we set $\mathcal{N}u$ in the following way:

$$\langle \mathcal{N}u, v\rangle := C_{p,\beta} \langle \mathcal{N}_p^{p'(1-\beta)} u_1, v_1\rangle_{(B^{p,p}_{\gamma(\beta)}(\Sigma))', B^{p,p}_{\gamma(\beta)}(\Sigma)} - C_{p,\alpha} \langle \mathcal{N}_p^{p'(1-\alpha)} u_2, v_2\rangle_{(B^{p,p}_{\gamma(\alpha)}(\Sigma))', B^{p,p}_{\gamma(\alpha)}(\Sigma)}; \tag{3.1}$$

with this definition, $\mathcal{N}u$ is an element of $(B^{p,p}_{\gamma(\alpha)}(\Sigma))'$.

We define the following energy functional for every $u \in H$:

$$\Phi_p^{\alpha,\beta}[u] := \begin{cases} \dfrac{C_{2,p,\beta}}{2p} \displaystyle\iint\limits_{\Omega_1\times\Omega_1} \frac{|u_1(x)-u_1(y)|^p}{|x-y|^{\beta p+2}}\, \mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y) & + \dfrac{C_{2,p,\alpha}}{2p} \displaystyle\iint\limits_{\Omega_2\times\Omega_2} \frac{|u_2(x)-u_2(y)|^p}{|x-y|^{\alpha p+2}}\, \mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y) \\ + \dfrac{1}{p} \displaystyle\int\limits_{\Sigma} b|u_2|^p \,\mathrm{d}\mu & \text{if } u \in D(\Phi_p^{\alpha,\beta}), \\ +\infty & \text{if } u \in H \setminus D(\Phi_p^{\alpha,\beta}), \end{cases} \tag{3.2}$$

where the effective domain is $D(\Phi_p^{\alpha,\beta}) := \mathbb{W}^{\alpha,\beta}_p(\Omega)$.
The following result follows as in Proposition 3.1 in [8].

Proposition 3.1. *$\Phi_p^{\alpha,\beta}$ is a weakly lower semicontinuous, proper and convex functional in H. Moreover, its subdifferential $\partial\Phi_p^{\alpha,\beta}$ is single-valued.*

We point out that Proposition 3.1 can be proved also for $1 < p < 2$.

3.2 The abstract Cauchy problem

Let T be a fixed positive number. We now consider the abstract Cauchy problem

$$(P) \begin{cases} \frac{\partial u}{\partial t} + \mathcal{A}_p^{\alpha,\beta} u = f, \quad t \in [0, T] \\ u(0) = u_0, \end{cases}$$

where $\mathcal{A}_p^{\alpha,\beta}$ is the subdifferential of $\Phi_p^{\alpha,\beta}$ and f and u_0 are given functions.
According to [3, Section 2.1, chapter II], we give the following definition.

Definition 3.2. *A function* $u : [0,T] \to H$ *is a strong solution of problem* (P) *if* $u \in C([0,T];H)$, $u(t)$ *is differentiable a.e. in* $(0,T)$, $u(t) \in D(-\mathcal{A}_p^{\alpha,\beta})$ *a.e. and* $\frac{\partial u}{\partial t} + \mathcal{A}_p^{\alpha,\beta} u = f$ *for a.e.* $t \in [0,T]$.

From [3, Theorem 2.1, chapter IV] the following existence and uniqueness result for the strong solution of problem (P) holds.

Theorem 3.3. *If* $u_0 \in \overline{D(-\mathcal{A}_p^{\alpha,\beta})}$ *and* $f \in L^2([0,T];H)$, *then problem* (P) *has a unique strong solution* $u \in C([0,T];H)$ *such that* $u \in W^{1,2}((\delta,T);H)$ *for every* $\delta \in (0,T)$. *Moreover* $u \in D(-\mathcal{A}_p^{\alpha,\beta})$ *a.e. for* $t \in (0,T)$, $\sqrt{t}\frac{\partial u}{\partial t} \in L^2(0,T;H)$ *and* $\Phi_p^{\alpha,\beta}[u] \in L^1(0,T)$.

From Theorem 1 and Remark 2 in [5] (see also [3]) we have the following result.

Theorem 3.4. *Let* $\varphi : \ H \to (-\infty,+\infty]$ *be a proper, convex, lower semicontinuous functional on a real Hilbert space* H, *with effective domain* $D(\varphi)$. *Then the subdifferential* $\partial\varphi$ *is a maximal monotone* m*-accretive operator. Moreover,* $\overline{D(\varphi)} = \overline{D(\partial\varphi)}$ *and* $-\partial\varphi$ *generates a nonlinear* C_0*-semigroup* $\{T(t)\}_{t\geq 0}$ *on* $\overline{D(\varphi)}$ *in the following sense: for each* $u_0 \in \overline{D(\varphi)}$, *the function* $u := T(\cdot)u_0$ *is the unique strong solution of the problem*

$$\begin{cases} u \in C(\mathbb{R}_+;H) \cap W^{1,\infty}_{loc}((0,\infty);H) \ \text{ and } \ u(t) \in D(\varphi) \ \text{ a.e.}, \\ \dfrac{\partial u}{\partial t} + \partial\varphi(u) \ni 0 \ \text{ a.e. on } \mathbb{R}_+, \\ u(0,x) = u_0(x). \end{cases}$$

In addition, $-\partial\varphi$ *generates a nonlinear semigroup* $\{\tilde{T}(t)\}_{t\geq 0}$ *on* H *where, for every* $t \geq 0$, $\tilde{T}(t)$ *is the composition of the semigroup* $T(t)$ *on* $\overline{D(\varphi)}$ *with the projection on the convex set* $\overline{D(\varphi)}$.

From Proposition 3.1 and Theorem 3.4, we have that the subdifferential $\partial\Phi_p^{\alpha,\beta}$ is maximal, monotone and m-accretive operator on H, with domain dense in H.
We now denote by $T_p^{\alpha,\beta}(t)$ the nonlinear semigroup generated by $-\partial\Phi_p^{\alpha,\beta}$. From Proposition 3.2, page 176 in [36] the following result holds.

Proposition 3.5. $T_p^{\alpha,\beta}(t)$ *is a strongly continuous and contractive semigroup on* H.

3.3 Strong formulation

We now prove that the strong solution of problem (P) actually solves problem $(\tilde{P})$. We first need a characterization of the subdifferential of $\Phi_p^{\alpha,\beta}$.

Theorem 3.6. *Let u belong to $\mathbb{W}_p^{\alpha,\beta}(\Omega)$ for a.e. $t \in (0,T]$, and let $f \in H$. Then $f \in \partial\Phi_p^{\alpha,\beta}[u]$ if and only if u solves the following problem:*

$$(\bar{P}) \begin{cases} (-\Delta_p)^{\beta}_{\Omega_1} u_1 = f_1 & \text{in } L^{p'}(\Omega_1), \\ (-\Delta_p)^{\alpha}_{\Omega_2} u_2 = f_2 & \text{in } L^{p'}(\Omega_2), \\ u_1 = w u_2 & \text{on } \Sigma, \\ \langle \mathcal{N} u, v\rangle + \langle b|u_2|^{p-2}u_2, v_2\rangle_{L^{p'}(\Sigma), L^p(\Sigma)} = 0 & \forall\, v \in B^{p,p}_{\gamma(\alpha)}(\Sigma), \\ \mathcal{N}_p^{p'(1-\alpha)} u_2 = 0 & \text{on } (W^{\alpha-\frac{1}{p},p}(\Gamma))'. \end{cases}$$

Proof. Let $f \in \partial\Phi_p^{\alpha,\beta}$, i.e.

$$\Phi_p^{\alpha,\beta}[v] - \Phi_p^{\alpha,\beta}[u] \geq (f, v-u)_H \quad \text{for every } v \in \mathbb{W}_p^{\alpha,\beta}(\Omega). \tag{3.3}$$

We choose $v = u + tz$, with $z \in \mathbb{W}_p^{\alpha,\beta}(\Omega)$ and $0 < t \leq 1$ in (3.3) and we obtain

$$\begin{aligned} & t\int_{\Omega_1} f_1\, z_1 \,\mathrm{d}\mathcal{L}_2 + t\int_{\Omega_2} f_2\, z_2 \,\mathrm{d}\mathcal{L}_2 \leq \\ & \frac{C_{2,p,\beta}}{2p} \iint_{\Omega_1\times\Omega_1} \frac{|(u_1+tz_1)(x) - (u_1+tz_1)(y)|^p - |u_1(x)-u_1(y)|^p}{|x-y|^{\beta p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y) \\ & + \frac{C_{2,p,\alpha}}{2p} \iint_{\Omega_2\times\Omega_2} \frac{|(u_2+tz_2)(x) - (u_2+tz_2)(y)|^p - |u_2(x)-u_2(y)|^p}{|x-y|^{\alpha p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y) \\ & + \frac{1}{p}\int_{\Sigma} b(|u_2+tz_2|^p - |u_2|^p)\,\mathrm{d}\mu. \end{aligned} \tag{3.4}$$

We first take $z \in \mathcal{D}(\Omega_1)$ and, by passing to the limit for $t \to 0^+$ in (3.4), we obtain

$$\int_{\Omega_1} f_1\, z_1\,\mathrm{d}\mathcal{L}_2 \leq \frac{C_{2,p,\beta}}{2} \iint_{\Omega_1\times\Omega_1} \frac{|u_1(x)-u_1(y)|^{p-2}(u_1(x)-u_1(y))(z_1(x)-z_1(y))}{|x-y|^{\beta p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y).$$

By taking $-z$ in (3.4) we obtain the opposite inequality, and hence we get

$$\int_{\Omega_1} f_1\, z_1\,\mathrm{d}\mathcal{L}_2 = \frac{C_{2,p,\beta}}{2} \iint_{\Omega_1\times\Omega_1} \frac{|u_1(x)-u_1(y)|^{p-2}(u_1(x)-u_1(y))(z_1(x)-z_1(y))}{|x-y|^{\beta p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y).$$

Since $z \in \mathcal{D}(\Omega_1)$ and $p' \leq 2$, it turns out that in particular $f_1 \in L^{p'}(\Omega_1)$. Hence, the p-fractional Green formula for fractal domains given by Theorem 2.2 yields that

$$(-\Delta_p)^{\beta}_{\Omega_1} u_1 = f_1 \quad \text{in } L^{p'}(\Omega_1) \tag{3.5}$$

(and in particular in $L^2(\Omega_1)$).

We remark that, if we take $z \in \mathcal{D}(\Omega_2)$ and we proceed analogously, we obtain that $(-\Delta_p)^{\alpha}_{\Omega_2} u_2 = f_2$ in $L^{p'}(\Omega_2)$ and in $L^2(\Omega_2)$.

We now go back to (3.4). Dividing by $t > 0$ and passing to the limit for $t \to 0^+$, we get

$$\int_{\Omega_1} f_1 z_1 \,\mathrm{d}\mathcal{L}_2 + \int_{\Omega_2} f_2 z_2 \,\mathrm{d}\mathcal{L}_2 \leq \frac{C_{2,p,\beta}}{2} \iint_{\Omega_1\times\Omega_1} \frac{|u_1(x)-u_1(y)|^{p-2}(u_1(x)-u_1(y))(z_1(x)-z_1(y))}{|x-y|^{\beta p+2}} \,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)$$

$$+ \frac{C_{2,p,\alpha}}{2} \iint_{\Omega_2\times\Omega_2} \frac{|u_2(x)-u_2(y)|^{p-2}(u_2(x)-u_2(y))(z_2(x)-z_2(y))}{|x-y|^{\alpha p+2}} \,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y) + \int_{\Sigma} b|u_2|^{p-2} u_2 z_2 \,\mathrm{d}\mu.$$

As above, by taking $-z$ we obtain the opposite inequality, hence we get the equality. Then, by Theorem 2.2 and (3.5) we obtain that

$$C_{p,\beta} \left\langle \mathcal{N}_p^{p'(1-\beta)} u_1, z_1 \right\rangle_{(B^{p,p}_{\gamma(\beta)}(\Sigma))', B^{p,p}_{\gamma(\beta)}(\Sigma)} - C_{p,s} \left\langle \mathcal{N}_p^{p'(1-\alpha)} u_2, z_2 \right\rangle_{(B^{p,p}_{\gamma(\alpha)}(\partial\Omega_2))', B^{p,p}_{\gamma(\alpha)}(\partial\Omega_2)} + \int_{\Sigma} b|u_2|^{p-2} u_2\, z_2 \,\mathrm{d}\mu = 0 \tag{3.6}$$

for every $z \in \mathbb{W}^{\alpha,\beta}_p(\Omega)$. Choosing suitably z_2 such that it vanishes on Σ, we obtain that $\mathcal{N}_p^{p'(1-\alpha)} u_2 = 0$ in $(W^{\alpha-\frac{1}{p},p}(\Gamma))'$, while choosing suitably z_2 vanishing on Γ, and taking into account the definition of $\mathcal{N}u$ given in (3.1), we have that

$$\langle \mathcal{N}u, z\rangle + \int_{\Sigma} b|u_2|^{p-2} u_2\, z_2 \,\mathrm{d}\mu = 0$$

holds in $(B^{p,p}_{\gamma(\alpha)}(\Sigma))'$. This proves the assertion.

In order to prove the converse, let $u \in \mathbb{W}^{\alpha,\beta}_p(\Omega)$ be the weak solution of problem $(\bar{P})$. We have to prove that $\Phi^{\alpha,\beta}_p[v] - \Phi^{\alpha,\beta}_p[u] \geq (f, v-u)_H$ for every $v \in \mathbb{W}^{\alpha,\beta}_p(\Omega)$. By using the inequality

$$\frac{1}{p}(|a|^p - |b|^p) \geq |b|^{p-2} b(a-b)$$

and the hypothesis that u is the weak solution of $(\bar{P})$, the thesis follows (see e.g. [8, Theorem 3.6]). $\square$

Theorem 3.6 implies that the unique strong solution u of the abstract Cauchy problem (P) solves the following Robin-type problem $(\tilde{P})$ on Ω for a.e. $t \in (0, T]$ in the following weak sense:

$$(\tilde{P})\begin{cases}\frac{\partial u_1}{\partial t}(t,x)+(-\Delta_p)^{\beta}_{\Omega_1}u_1(t,x)=f_1(t,x) & \text{for a.e. } x\in\Omega_1,\\[2mm] \frac{\partial u_2}{\partial t}(t,x)+(-\Delta_p)^{\alpha}_{\Omega_2}u_2(t,x)=f_2(t,x) & \text{for a.e. } x\in\Omega_2,\\[2mm] u_1=wu_2 & \text{on } \Sigma,\\[2mm] \langle\mathcal{N}u,v\rangle+\langle b|u_2|^{p-2}u_2,v_2\rangle_{L^{p'}(\Sigma),L^p(\Sigma)}=0 & \forall\, v\in B^{p,p}_{\gamma(\alpha)}(\Sigma),\\[2mm] \mathcal{N}_p^{p'(1-\alpha)}u_2=0 & \text{in } (W^{\alpha-\frac{1}{p},p}(\Gamma))',\\[2mm] u(0,x)=u^0(x) & \text{in } H,\end{cases}$$

where

$$u^0(x):=\begin{cases}u_1^0(x) & \text{on } \Omega_1,\\[2mm] u_2^0(x) & \text{on } \Omega_2.\end{cases}$$

4 Ultracontractivity results

We now focus on proving the ultracontractivity of the semigroup $T_p^{\alpha,\beta}(t)$.

We first need some preliminary results. From (1.5) and (1.10), it follows that for every $q\in[1,p^*(\beta)]$

$$\|u\|_q\leq C\left(\|u\|_p+|u_1|_{W^{\beta,p}(\Omega_1)}+|u_2|_{W^{\alpha,p}(\Omega_2)}\right). \tag{4.1}$$

Moreover, for every $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that

$$\|u\|_q\leq C_\varepsilon\left(\varepsilon\|u\|_p+|u_1|_{W^{\beta,p}(\Omega_1)}+|u_2|_{W^{\alpha,p}(\Omega_2)}\right). \tag{4.2}$$

We first prove a fractional logarithmic Sobolev inequality tailored to the problem at hand.

Proposition 4.1. *Let $p\geq 2$, $\alpha,\beta\in(\frac{2-d}{p},1)$ and $\beta p\leq\alpha p<2$. Let $u\in\mathbb{W}_p^{\alpha,\beta}(\Omega)$ be non-negative on $\overline{\Omega}$ and such that $\|u\|_p^p=\|u_1\|^p_{L^p(\Omega_1)}+\|u_2\|^p_{L^p(\Omega_2)}=1$. We set*

$$\Lambda(u):=\int\limits_{\Omega_1}u_1\,\mathrm{d}\mathcal{L}_2+\int\limits_{\Omega_2}u_2\,\mathrm{d}\mathcal{L}_2. \tag{4.3}$$

Then for every $\varepsilon>0$ there exists a positive constant $\overline{C_\varepsilon}$ depending also on α, β, p and Ω such that, for every $\bar{\varepsilon}>0$,

$$\Lambda(u^p\log u)\leq\frac{2}{\beta p^2}\left[\bar{\varepsilon}\overline{C_\varepsilon}\left(|u_1|^p_{W^{\beta,p}(\Omega_1)}+|u_2|^p_{W^{\alpha,p}(\Omega_2)}\right)-\log\bar{\varepsilon}+\bar{\varepsilon}\overline{C_\varepsilon}\varepsilon\right], \tag{4.4}$$

where $u^p\log u:=(u_1^p\log u_1,u_2^p\log u_2)$.

Proof. We adapt to our aim the proof of Lemma 3.2 in [29]. We apply Jensen's inequality with $q = p^*(\beta) - p = \frac{\beta p^2}{2-\beta p}$ and we obtain

$$\Lambda(u^p \log u) \leq \frac{1}{q} \log \Lambda(u^{p+q}) = \frac{2-\beta p}{\beta p^2} \log \|u\|_{p^*(\beta)}^{p^*(\beta)} = \frac{2}{\beta p^2} \log \|u\|_{p^*(\beta)}^{p}. \tag{4.5}$$

Moreover, from the properties of the logarithmic function, for every $\bar{\varepsilon} > 0$ we have that

$$\log \|u\|_{p^*(\beta)}^p \leq \bar{\varepsilon} \|u\|_{p^*(\beta)}^p - \log \bar{\varepsilon}. \tag{4.6}$$

We then estimate $\|u\|_{p^*(\beta)}^p$ in (4.6) using (4.2) with $q = p^*(\beta)$. Hence, since $\|u\|_p^p = 1$, we obtain that for every $\varepsilon > 0$ there exists a positive constant $\overline{C_\varepsilon}$ such that

$$\Lambda(u^p \log u) \leq \frac{2}{\beta p^2} \left[\bar{\varepsilon} \overline{C_\varepsilon} \left(|u_1|_{W^{\beta,p}(\Omega_1)}^p + |u_2|_{W^{\alpha,p}(\Omega_2)}^p + \varepsilon \right) - \log \bar{\varepsilon} \right],$$

and this concludes the proof. $\square$

We now prove some preliminary lemmas which will allow us to prove the ultracontractivity of the nonlinear semigroup $T_p^{\alpha,\beta}(t)$. We adapt to the fractional framework the results of [29, Section 3.2], see also [39, 40, 41].

We first recall some known numerical inequalities. For more details we refer to [4].

Proposition 4.2. *Let $a, b \in \mathbb{R}^N$. If $r \in (1, \infty)$, it holds that*

$$\left(|a|^{r-2}a - |b|^{r-2}b\right)(a-b) \;\geq\; (r-1)\left(|a| + |b|\right)^{r-2} |a-b|^2. \tag{4.7}$$

If $r \in [2, \infty)$, then for $c_r^* := \min\{1/(r-1), 2^{-2-r}\, 3^{-r/2}\} \in (0,1]$, it holds that

$$\left(|a|^{r-2}a - |b|^{r-2}b\right)(a-b) \;\geq\; c_r^* \, |a-b|^r. \tag{4.8}$$

We remark that (4.8) implies

$$\left(|a|^{r-2}a - |b|^{r-2}b\right) \operatorname{sgn}(a-b) \;\geq\; c_r^* \, |a-b|^{r-1}. \tag{4.9}$$

Lemma 4.3. *Let $\left\{T_p^{\alpha,\beta}(t)\right\}_{t\geq 0}$ be the Markovian semigroup on H generated by $-\partial\Phi_p^{\alpha,\beta}$. Given $t \geq 0$ and u^0, $v^0 \in L^\infty(\Omega)$, let $u(t,x) := T_p^{\alpha,\beta}(t)u^0(x)$ and $v(t,x) := T_p^{\alpha,\beta}(t)v^0(x)$ be the solutions of the homogeneous problem associated to (P) with initial data u^0 and v^0 respectively. We set $U(t,x) := u(t,x) - v(t,x)$, i.e.*

$$U(t,x) = \begin{cases} U_1(t,x) := u_1(t,x) - v_1(t,x) & \text{on } \Omega_1, \\ U_2(t,x) := u_2(t,x) - v_2(t,x) & \text{on } \Omega_2, \\ U_1 = wU_2 & \text{on } \Sigma. \end{cases}$$

Then, for every real number $r \geq 2$ and for a.e. $t \geq 0$, there exists a constant $\tilde{C} = \tilde{C}(\alpha, \beta, p)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U(t)\|_r^r \leq -r\tilde{C}\left(\iint\limits_{\Omega_1\times\Omega_1} \frac{|U_1(t,x)-U_1(t,y)|^{r+p-2}}{|x-y|^{\beta p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)\right.$$

$$\left.+\iint\limits_{\Omega_2\times\Omega_2} \frac{|U_2(t,x)-U_2(t,y)|^{r+p-2}}{|x-y|^{\alpha p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)\right) - c_p^* b_0 r\int\limits_{\Sigma} |U_2(t)|^{r+p-2}\,\mathrm{d}\mu,$$

where $c_p^* > 0$ is the constant given in Proposition 4.2 and $b_0 = \min\limits_{\overline{\Omega}} b$.

Proof. We fix $t \geq 0$. We point out that $u(t), v(t) \in \mathbb{W}_p^{\alpha,\beta}(\Omega)$, hence also $U(t) \in \mathbb{W}_p^{\alpha,\beta}(\Omega)$. For $r \geq 2$, we define the function $G_r : [0,\infty) \to [0,\infty)$ by

$$G_r(t) := \|U(t)\|_r^r = \|U_1(t)\|_{L^r(\Omega_1)}^r + \|U_2(t)\|_{L^r(\Omega_2)}^r.$$

G_r is clearly differentiable for a.e. $t \geq 0$. Taking into account the characterization of the subdifferential $\partial\Phi_p^{\alpha,\beta}$ given by Theorem 3.6, since $u(t)$ and $v(t)$ are solutions of the abstract Cauchy problem (P) with $f = 0$ and initial data u^0 and v^0 respectively, we get

$$G_r'(t) = r\int\limits_{\Omega_1} |U_1(t)|^{r-2}U_1(t)(u_1'(t)-v_1'(t))\,\mathrm{d}\mathcal{L}_2 + r\int\limits_{\Omega_2} |U_2(t)|^{r-2}U_2(t)(u_2'(t)-v_2'(t))\,\mathrm{d}\mathcal{L}_2$$

$$= -r\int\limits_{\Omega_1} |U_1(t)|^{r-2}U_1(t)\left((-\Delta_p)^{\beta}_{\Omega_1}u_1(t) - (-\Delta_p)^{\beta}_{\Omega_1}v_1(t)\right)\,\mathrm{d}\mathcal{L}_2$$

$$- r\int\limits_{\Omega_2} |U_2(t)|^{r-2}U_2(t)\left((-\Delta_p)^{\alpha}_{\Omega_2}u_2(t) - (-\Delta_p)^{\alpha}_{\Omega_2}v_2(t)\right)\,\mathrm{d}\mathcal{L}_2.$$

In order to simplify the notation, we set

$$\Psi := \begin{cases} \Psi_1 := |U_1|^{r-2}U_1 & \text{on } \Omega_1, \\ \Psi_2 := |U_2|^{r-2}U_2 & \text{on } \Omega_2. \end{cases}$$

Using the Green formula (2.3), since outward normals to Ω_1 and Ω_2 at the interface Σ have opposite sign, we get

$$
\begin{aligned}
G_r'(t) = &-r\frac{C_{2,p,\beta}}{2}\Bigg[\iint\limits_{\Omega_1\times\Omega_1}\frac{|u_1(t,x)-u_1(t,y)|^{p-2}(u_1(t,x)-u_1(t,y))(\Psi_1(t,x)-\Psi_1(t,y))}{|x-y|^{\beta p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)\\
&-\iint\limits_{\Omega_1\times\Omega_1}\frac{|v_1(t,x)-v_1(t,y)|^{p-2}(v_1(t,x)-v_1(t,y))(\Psi_1(t,x)-\Psi_1(t,y))}{|x-y|^{\beta p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)\Bigg]\\
&-r\frac{C_{2,p,\alpha}}{2}\Bigg[\iint\limits_{\Omega_2\times\Omega_2}\frac{|u_2(t,x)-u_2(t,y)|^{p-2}(u_2(t,x)-u_2(t,y))(\Psi_2(t,x)-\Psi_2(t,y))}{|x-y|^{\alpha p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)\\
&-\iint\limits_{\Omega_2\times\Omega_2}\frac{|v_2(t,x)-v_2(t,y)|^{p-2}(v_2(t,x)-v_2(t,y))(\Psi_2(t,x)-\Psi_1(t,y))}{|x-y|^{\alpha p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)\Bigg]\\
&+rC_{p,\beta}\left[\langle\mathcal{N}_p^{p'(1-\beta)}u_1,\Psi_1\rangle-\langle\mathcal{N}_p^{p'(1-\beta)}v_1,\Psi_1\rangle\right]-rC_{p,\alpha}\left[\langle\mathcal{N}_p^{p'(1-\alpha)}u_2,\Psi_2\rangle-\langle\mathcal{N}_p^{p'(1-\alpha)}v_2,\Psi_2\rangle\right].
\end{aligned}
\tag{4.10}
$$

We now apply (4.8) and, recalling the definition of $\mathcal{N}$ given in (3.1), we use the transmission condition of problem $(\tilde{P})$. Then, by using the properties of the function b and the definition of Ψ, from (4.10) we deduce that

$$
\begin{aligned}
G_r'(t) \le -rc_p^*\Bigg(&\frac{C_{2,p,\beta}}{2}\iint\limits_{\Omega_1\times\Omega_1}\frac{|U_1(t,x)-U_1(t,y)|^{r+p-2}}{|x-y|^{\beta p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)\\
&+\frac{C_{2,p,\alpha}}{2}\iint\limits_{\Omega_2\times\Omega_2}\frac{|U_2(t,x)-U_2(t,y)|^{r+p-2}}{|x-y|^{\beta p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)\Bigg)-rc_p^*b_0\int\limits_\Sigma b|U_2(t)|^{r+p-2}\,\mathrm{d}\mu.
\end{aligned}
$$

Setting $\tilde{C} := c_p^*\max\left\{\frac{C_{2,p,\beta}}{2},\frac{C_{2,p,\alpha}}{2}\right\}$, we get the thesis. $\square$

We remark that, as a consequence of Lemma 4.3, we have that $G_r(t) := \|U(t)\|_r^r = \|U_1(t)\|^r_{L^r(\Omega_1)} + \|U_2(t)\|^r_{L^r(\Omega_2)}$ is non-increasing w.r.t. t.

We now recall the following useful result. We refer to [41, Lemma 4.1].

Lemma 4.4. *Let $p, r \ge 2$ and $s \in (0,1)$. Then, for every $u, v \in W^{s,p}(\Omega)$ it holds that*

$$
C_{r,p}(|u|^{\frac{r+p-2}{p}},|u|^{\frac{r+p-2}{p}})_{s,p} \le C_{r,p}(|u|^{\frac{r-2}{p}},|u|^{\frac{r-2}{p}})_{s,p} \le (u,|u|^{r-2}u)_{s,p}, \tag{4.11}
$$

where $C_{r,p} := (r-1)\left(\dfrac{p}{r+p-2}\right)^p$.

The next two lemmas follow by adapting to the fractional setting Lemmas 3.5 and 3.6 in [29] (see also [8] and [38]).

Lemma 4.5. *Under the same notations and assumptions of Lemma 4.3, if $r:[0,\infty)\to[2,\infty)$ is an increasing differentiable function, then for a.e. $t\geq 0$ and for every $\varepsilon,\bar\varepsilon>0$ we have that*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\log\|U(t)\|_{r(t)} &\leq \frac{r'(t)}{r(t)}\Lambda\left(\frac{|U(t)|^{r(t)}}{\|U(t)\|_{r(t)}^{r(t)}}\log\frac{|U(t)|}{\|U(t)\|_{r(t)}}\right)\\
&-\frac{\tilde C(r(t)-1)}{\bar\varepsilon\overline{C_\varepsilon}}\left(\frac{p}{r(t)+p-2}\right)^p\log\bar\varepsilon\frac{\|U(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|U(t)\|_{r(t)}^{r(t)}}+\varepsilon\tilde C(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^p\frac{\|U(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|U(t)\|_{r(t)}^{r(t)}}\\
&-\frac{\tilde C(r(t)-1)}{\bar\varepsilon\overline{C_\varepsilon}}\left(\frac{p}{r(t)+p-2}\right)^{p-1}\frac{\beta p^2}{2}\frac{\|U(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|U(t)\|_{r(t)}^{r(t)}}\Lambda\left(\frac{|U(t)|^{r(t)+p-2}}{\|U(t)\|_{r(t)+p-2}^{r(t)+p-2}}\log\frac{|U(t)|}{\|U(t)\|_{r(t)+p-2}}\right),
\end{aligned}\tag{4.12}$$

where $\overline{C_\varepsilon}$ and $\tilde C$ are the constants appearing in (4.4) and (4.10) respectively and Λ is defined as in (4.3).

Proof. From the chain rule and from Lemma 4.3, we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\log\|U(t)\|_{r(t)} &= -\frac{r'(t)}{r(t)}\log\|U(t)\|_{r(t)}+\frac{1}{r(t)\|U(t)\|_{r(t)}^{r(t)}}\frac{\mathrm{d}}{\mathrm{d}t}\|U(t)\|_{r(t)}^{r(t)}\leq -\frac{r'(t)}{r(t)}\log\|U(t)\|_{r(t)}\\
&+\frac{r'(t)}{r(t)}\frac{1}{\|U(t)\|_{r(t)}^{r(t)}}\Lambda\left(|U(t)|^{r(t)}\log|U(t)|\right)-\frac{c_p^*b_0}{\|U(t)\|_{r(t)}^{r(t)}}\int_\Sigma|U_2(t)|^{r(t)+p-2}\,\mathrm{d}\mu-\frac{\tilde C}{\|U(t)\|_{r(t)}^{r(t)}}\cdot\\
&\cdot\left(\iint_{\Omega_1\times\Omega_1}\frac{|U_1(t,x)-U_1(t,y)|^{p-2}(U_1(t,x)-U_1(t,y))(|U_1(t,x)|^{r(t)}U_1(t,x)-|U_1(t,y)|^{r(t)}U_1(t,y))}{|x-y|^{\beta p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)\right.\\
&\left.+\iint_{\Omega_2\times\Omega_2}\frac{|U_2(t,x)-U_2(t,y)|^{p-2}(U_2(t,x)-U_2(t,y))(|U_2(t,x)|^{r(t)}U_2(t,x)-|U_2(t,y)|^{r(t)}U_2(t,y))}{|x-y|^{\alpha p+2}}\,\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)\right)
\end{aligned}\tag{4.13}$$

Recalling the definition of Λ, using Lemma 4.4 and estimating the term on the fractal interface Σ with zero, we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\log\|U(t)\|_{r(t)} \le \frac{r'(t)}{r(t)}\Lambda\left(\frac{|U(t)|^{r(t)}}{\|U(t)\|_{r(t)}^{r(t)}}\log\frac{|U(t)|}{\|U(t)\|_{r(t)}}\right)\\
&-\frac{\tilde{C}}{\|U(t)\|_{r(t)}^{r(t)}}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p}\left(\iint\limits_{\Omega_1\times\Omega_1}\frac{\left||U_1(t,x)|^{\frac{r(t)+p-2}{p}}-|U_1(t,y)|^{\frac{r(t)+p-2}{p}}\right|^{p}}{|x-y|^{\beta p+2}}\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)\right.\\
&\left.+\iint\limits_{\Omega_2\times\Omega_2}\frac{\left||U_2(t,x)|^{\frac{r(t)+p-2}{p}}-|U_2(t,y)|^{\frac{r(t)+p-2}{p}}\right|^{p}}{|x-y|^{\alpha p+2}}\mathrm{d}\mathcal{L}_2(x)\mathrm{d}\mathcal{L}_2(y)\right) = \frac{r'(t)}{r(t)}\Lambda\left(\frac{|U(t)|^{r(t)}}{\|U(t)\|_{r(t)}^{r(t)}}\log\frac{|U(t)|}{\|U(t)\|_{r(t)}}\right)\\
&-\tilde{C}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p}\frac{\|U(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|U(t)\|_{r(t)}^{r(t)}}\left(|F_1(t)|^p_{W^{\beta,p}(\Omega_1)}+|F_2(t)|^p_{W^{\alpha,p}(\Omega_2)}\right),
\end{aligned}
\tag{4.14}
$$

where for $i=1,2$

$$
F_i(t,x) := \frac{|U_i(t)|^{\frac{r(t)+p-2}{p}}}{\|U(t)\|_{r(t)+p-2}^{\frac{r(t)+p-2}{p}}}.
$$

If we define $F=F(t,x)$ to be equal to F_i on Ω_i for $i=1,2$, then F fulfills the hypotheses of Proposition 4.1. Thus, since it holds that

$$
\Lambda(F^p\log F) = \frac{r(t)+p-2}{p}\Lambda\left(\frac{|U(t)|^{r(t)+p-2}}{\|U(t)\|_{r(t)+p-2}^{r(t)+p-2}}\log\frac{|U(t)|}{\|U(t)\|_{r(t)+p-2}}\right),
$$

the thesis follows. □

Lemma 4.6. *Under the assumptions of Lemma 4.5, for a.e.* $t\ge 0$ *we have that*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\log\|U(t)\|_{r(t)} \le -A(t)\log\|U(t)\|_{r(t)} - B(t), \tag{4.15}
$$

where

$$
A(t) := \frac{r'(t)2(p-2)}{r(t)\beta p(r(t)+p-2)}, \tag{4.16}
$$

$$
\begin{aligned}
B(t) :=& -\frac{r'(t)(p-2)(2-\beta p)}{r(t)\beta p(r(t)+p-2)}\log\omega - \tilde{C}p\\
&+\frac{2r'(t)}{r(t)\beta p(r(t)+p-2)}\log\left[\frac{r(t)}{r'(t)}\frac{\beta p^2}{2}\hat{C}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p-1}\right],
\end{aligned}
\tag{4.17}
$$

$\hat{C}=\hat{C}(\alpha,\beta,p,\Omega)$ *is a positive constant and* $\omega=\max\{|\Omega_1|,|\Omega_2|\}$.

Proof. We choose $\bar{\varepsilon} > 0$ as follows:

$$\bar{\varepsilon} := \frac{r(t)}{r'(t)} \frac{\beta p^2}{2} \frac{\tilde{C}}{\overline{C_\varepsilon}} (r(t)-1) \left(\frac{p}{r(t)+p-2} \right)^{p-1} \frac{\|U(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|U(t)\|_{r(t)}^{r(t)}}.$$

Hence from (4.12) we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \log \|U(t)\|_{r(t)} \leq \frac{r'(t)}{r(t)} \Bigg[\Lambda \left(\frac{|U(t)|^{r(t)}}{\|U(t)\|_{r(t)}^{r(t)}} \log \frac{|U(t)|}{\|U(t)\|_{r(t)}} \right) - \Lambda \left(\frac{|U(t)|^{r(t)+p-2}}{\|U(t)\|_{r(t)+p-2}^{r(t)+p-2}} \log \frac{|U(t)|}{\|U(t)\|_{r(t)+p-2}} \right) \Bigg] \\
- \frac{r'(t)}{r(t)(r(t)+p-2)} \frac{2}{\beta p} \log \left[\frac{r(t)}{r'(t)} \frac{\beta p^2}{2} \frac{\tilde{C}}{\overline{C_\varepsilon}} (r(t)-1) \left(\frac{p}{r(t)+p-2} \right)^{p-1} \frac{\|U(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|U(t)\|_{r(t)}^{r(t)}} \right] \\
+ \varepsilon \tilde{C} (r(t)-1) \left(\frac{p}{r(t)+p-2} \right)^{p} \frac{\|U(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|U(t)\|_{r(t)}^{r(t)}}.
\end{aligned} \tag{4.18}$$

We now choose

$$\varepsilon := \frac{\|U(t)\|_{r(t)}^{r(t)}}{\|U(t)\|_{r(t)+p-2}^{r(t)+p-2}}$$

and we point out that, since $r(t) \geq 2$ and $p \geq 2$, it holds that

$$(r(t)-1) \left(\frac{p}{r(t)+p-2} \right)^{p} \leq p.$$

Hence, for a suitable positive constant $\hat{C}$ depending on α, β, p and Ω, from (4.18) we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \log \|U(t)\|_{r(t)} \leq \frac{r'(t)}{r(t)} \Bigg[\Lambda \left(\frac{|U(t)|^{r(t)}}{\|U(t)\|_{r(t)}^{r(t)}} \log \frac{|U(t)|}{\|U(t)\|_{r(t)}} \right) - \Lambda \left(\frac{|U(t)|^{r(t)+p-2}}{\|U(t)\|_{r(t)+p-2}^{r(t)+p-2}} \log \frac{|U(t)|}{\|U(t)\|_{r(t)+p-2}} \right) \Bigg] \\
+ \tilde{C} p - \frac{r'(t)}{r(t)(r(t)+p-2)} \frac{2}{\beta p} \log \left[\frac{r(t)}{r'(t)} \frac{\beta p^2}{2} \hat{C} (r(t)-1) \left(\frac{p}{r(t)+p-2} \right)^{p-1} \frac{\|U(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|U(t)\|_{r(t)}^{r(t)}} \right].
\end{aligned} \tag{4.19}$$

We now set

$$K(q,U) := \Lambda \left(\frac{|U|^q}{\|U\|_q^q} \log \frac{|U|}{\|U\|_q} \right) = \int_{\Omega_1} \frac{|U_1|^q}{\|U\|_q^q} \log \frac{|U_1|}{\|U\|_q} \, \mathrm{d}\mathcal{L}_2 + \int_{\Omega_2} \frac{|U_2|^q}{\|U\|_q^q} \log \frac{|U_2|}{\|U\|_q} \, \mathrm{d}\mathcal{L}_2.$$

The functional $K(q,U)$ satisfies the following property: for every $q_2 \geq q_1 \geq 1$ and for every $U \in L^\infty(\Omega)$

$$K(q_2,U) - K(q_1,U) \geq \log \frac{\|U\|_{q_1}}{\|U\|_{q_2}}. \tag{4.20}$$

Applying (4.20) with $q_1 = r(t)$ and $q_2 = (r(t)+p-2)$, from (4.19) and using the properties of the logarithmic function, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\|U(t)\|_{r(t)} &\leq \frac{r'(t)}{r(t)}\left(1-\frac{2}{\beta p}\right)\log\|U(t)\|_{r(t)+p-2} + \tilde{C}p - \frac{r'(t)}{r(t)}\left(1-\frac{2r(t)}{\beta p(r(t)+p-2)}\right)\log\|U(t)\|_{r(t)} \\ &- \frac{r'(t)}{r(t)(r(t)+p-2)}\frac{2}{\beta p}\log\left[\frac{r(t)}{r'(t)}\frac{\beta p^2}{2}\hat{C}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p-1}\right].\end{aligned} \tag{4.21}$$

We remark that, since $\beta p < 2$, $1-\frac{2}{\beta p} < 0$. Thus, from Hölder inequality we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\|U(t)\|_{r(t)} &\leq \frac{r'(t)}{r(t)}\frac{2(2-p)}{\beta p(r(t)+p-2)}\log\|U(t)\|_{r(t)} + \frac{r'(t)}{r(t)}\frac{2-\beta p}{\beta p}\frac{p-2}{r(t)+p-2}\log\omega \\ &+ \tilde{C}p - \frac{r'(t)}{r(t)(r(t)+p-2)}\frac{2}{\beta p}\log\left[\frac{r(t)}{r'(t)}\frac{\beta p^2}{2}\hat{C}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p-1}\right].\end{aligned} \tag{4.22}$$

Hence, taking into account the definitions of $A(t)$ and $B(t)$ in (4.16) and (4.17) respectively, estimate (4.15) follows. $\square$

We now prove the ultracontractivity of the semigroup $T_p^{\alpha,\beta}(t)$.

Theorem 4.7. *Let $p > 2$ and $\beta p \leq \alpha p < 2$. In the notations of the above lemmas, if $q \in [2,\infty]$, then there exist two positive constants C_1, C_2 depending on α, β, p, q and Ω such that*

$$\|T_p^{\alpha,\beta}(t)u^0 - T_p^{\alpha,\beta}(t)v^0\|_\infty \leq C_1(\max\{|\Omega_1|,|\Omega_2|\})^{\lambda_1(\beta)} e^{C_2 t} t^{-\lambda_2(\beta)} \|u^0 - v^0\|_q^{\lambda_3(\beta)}, \tag{4.23}$$

for every $u^0, v^0 \in L^q(\Omega)$ and for every $t > 0$, where

$$\begin{aligned}&\lambda_1(\beta) = \frac{2-\beta p}{2}\left[1-\left(\frac{q}{q+p-2}\right)^{\frac{2}{\beta p}}\right], \quad \lambda_2(\beta) = \frac{1}{p-2}\left[1-\left(\frac{q}{q+p-2}\right)^{\frac{2}{\beta p}}\right], \\ &\lambda_3(\beta) = \left(\frac{q}{q+p-2}\right)^{\frac{2}{\beta p}}.\end{aligned} \tag{4.24}$$

Proof. We first take $u_0, v_0 \in L^\infty(\Omega)$ and we use the same assumptions and notations of Lemma 4.6. In particular, we consider an increasing differentiable function $r\colon [0,\infty) \to [2,\infty)$ and we define $A(t)$ and $B(t)$ as in (4.16) and (4.17) respectively.

We set

$$y(t) := \log \|U(t)\|_{r(t)};$$

then, from (4.15), $y(t)$ satisfies the following ordinary differential inequality:

$$y'(t) + A(t)y(t) + B(t) \leq 0. \tag{4.25}$$

We now consider the following ODE:

$$\begin{cases} x'(t) + A(t)x(t) + B(t) = 0, \\ x(0) = y(0). \end{cases} \tag{4.26}$$

The unique solution $x(t)$ of (4.26) can be written in the following way:

$$x(t) = \exp\left(-\int_0^t A(\tau)\,\mathrm{d}\tau\right)\left[y(0) - \int_0^t B(\tau)\exp\left(\int_0^\tau A(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\tau\right]; \tag{4.27}$$

hence, the solution $y(t)$ of the ordinary differential inequality (4.25) is such that $y(t) \leq x(t)$ for every $t \in [0,\infty)$.

We now fix $t > 0$, for any given $q \geq 2$ and for $\tau \in [0,t)$ we set

$$r(\tau) := \frac{qt}{t-\tau}. \tag{4.28}$$

The function $r(\cdot)$ satisfies the hypotheses of Lemma 4.5, i.e. it is increasing and differentiable on $[0,t)$ and $r(\tau) \geq 2$ for every $\tau \in [0,t)$.

Using (4.28), we obtain that

$$A(\tau) = \frac{2}{\beta p}\,\frac{p-2}{t(q+p-2) - \tau(p-2)}$$

and

$$\begin{aligned} B(\tau) = &-\frac{(2-\beta p)(p-2)}{\beta p}\,\frac{1}{t(q+p-2)-\tau(p-2)}\log\omega - \tilde{C}p + \frac{2}{\beta p}\cdot \\ &\cdot \frac{1}{t(q+p-2)-\tau(p-2)}\log\left[\frac{\beta p^2}{2}\hat{C}\left(t(q-1)+\tau\right)\left(\frac{p(t-\tau)}{t(q+p-2)-\tau(p-2)}\right)^{p-1}\right], \end{aligned}$$

where $\tilde{C}$ and $\hat{C}$ are the constants in Lemma 4.3 and Lemma 4.6 respectively.

We now write $x(t)$ more explicitly. From standard calculations, we have that

$$\lim_{\tau\to t^-}\exp\left(-\int_0^\tau A(\sigma)\,\mathrm{d}\sigma\right) = \left(\frac{q}{q+p-2}\right)^{\frac{2}{\beta p}}. \tag{4.29}$$

Moreover, again from standard calculations, we can prove that

$$\lim_{\tau\to t^-}\int_0^\tau B(\sigma)\exp\left(\int_0^\sigma A(\xi)\,\mathrm{d}\xi\right)\,\mathrm{d}\sigma = -\frac{2-\beta p}{2}\log\omega\left[\left(\frac{q+p-2}{q}\right)^{\frac{2}{\beta p}}-1\right]-\check{C}t$$
$$+\frac{1}{p-2}\left[\left(\frac{q+p-2}{q}\right)^{\frac{2}{\beta p}}-1\right]\left[\log\left(\frac{\beta p^p}{2}\hat{C}\right)+\log t\right]+I^{(1)}+I^{(2)}-I^{(3)}, \tag{4.30}$$

where $\check{C}$ is a suitable positive constant depending on β, p, Ω and q and $I^{(1)}$, $I^{(2)}$ and $I^{(3)}$ are integral terms which do not depend on t and can be explicitly computed as in [7, proof of Lemma 3.9].

From (4.29) and (4.30) it follows that

$$\begin{aligned}\lim_{\tau\to t^-}x(\tau) = \left(\frac{q}{q+p-2}\right)^{\frac{2}{\beta p}}y(0)+\frac{2-\beta p}{2}\log\omega\left[1-\left(\frac{q}{q+p-2}\right)^{\frac{2}{\beta p}}\right]+C_2t\\ -\frac{1}{p-2}\left[1-\left(\frac{q}{q+p-2}\right)^{\frac{2}{\beta p}}\right]\left[\log\left(\frac{\beta p^p}{2}\hat{C}\right)+\log t\right]+C_I,\end{aligned} \tag{4.31}$$

where $C_2=\left(\frac{q}{q+p-2}\right)^{\frac{2}{\beta p}}\check{C}$ and $C_I=\left(\frac{q}{q+p-2}\right)^{\frac{2}{\beta p}}(I^{(3)}-I^{(1)}-I^{(2)})$.

We now point out that, as a consequence of Lemma 4.3, for every $0\le\tau<t$ it holds

$$\|U(t)\|_{r(\tau)}=\|u(t)-v(t)\|_{r(\tau)}\le\|u(\tau)-v(\tau)\|_{r(\tau)}=\|U(\tau)\|_{r(\tau)}=e^{y(\tau)}\le e^{x(\tau)}. \tag{4.32}$$

Since $y(0)=\log\|U(0)\|_{r(0)}=\log\|u^0-v^0\|_q$, from (4.31) and (4.32) we obtain

$$\lim_{\tau\to t^-}\|U(t)\|_{r(\tau)}\le\lim_{\tau\to t^-}e^{x(\tau)}=\|u^0-v^0\|_q^{\lambda_3(\beta)}\,\omega^{\lambda_1(\beta)}\,e^{C_2t}\,t^{-\lambda_2(\beta)}\left(\frac{\beta p^{p+1}}{2}\hat{C}\right)^{-\lambda_2(\beta)}e^{C_I}, \tag{4.33}$$

where the constants $\lambda_1(\beta)$, $\lambda_2(\beta)$ and $\lambda_3(\beta)$ are as defined in (4.24).

Finally, we remark that

$$\lim_{\tau\to t^-}r(\tau)=+\infty.$$

Therefore, from the definition of ω, there exists a suitable positive constant C_1 depending on α, β, p, Ω and q such that

$$\|U(t)\|_\infty=\|T_p^{\alpha,\beta}(t)u^0-T_p^{\alpha,\beta}(t)v^0\|_\infty\le C_1(\max\{|\Omega_1|,|\Omega_2|\})^{\lambda_1(\beta)}e^{C_2t}t^{-\lambda_2(\beta)}\|u^0-v^0\|_q^{\lambda_3(\beta)},$$

thus the thesis follows in the case $u^0,v^0\in L^\infty(\Omega)$.

The proof in the case $u^0,v^0\in L^q(\Omega)$ is then achieved by a density argument as in the proof of [38, Theorem 3.2.7]. □

We remark that also in the linear case, i.e. $p=2$, the semigroup $T_2^{\alpha,\beta}(t)$ is ultracontractive. The proof follows by adapting the techniques of [18, Theorem 2.16].

5 The case $\alpha \le \beta$ and some remarks

We briefly comment on the case when the fractional exponents are such that $\alpha \le \beta < 1$. In this case, the transmission conditions on Σ in the formal problem $(\tilde{P})$ read as follows:

$$\tilde{w} u_1 = u_2 \quad \text{on } (0,T] \times \Sigma, \quad \tilde{\mathcal{N}} u + b|u_1|^{p-2} u_1 = 0 \quad \text{on } (0,T] \times \Sigma,$$

where $\tilde{w} \in B^{p,p}_{\theta}(\Sigma)$ for $\theta \ge \gamma(\alpha)$ such that $\theta - \alpha + \beta > \frac{d}{p}$ and the operator $\tilde{\mathcal{N}}$ is a linear and continuous operator on $B^{p,p}_{\gamma(\beta)}(\Sigma)$ which is defined as in (3.1).
We introduce the Sobolev space

$$\tilde{\mathbb{W}}^{\alpha,\beta}_p(\Omega) := \{u \in L^p(\Omega) \,:\, u_1 \in W^{\beta,p}(\Omega_1),\, u_2 \in W^{\alpha,p}(\Omega_2) \text{ and } \tilde{w} u_1 = u_2 \text{ on } \Sigma\}. \quad (5.1)$$

which is endowed with the norm given by (1.7). This space is the effective domain of the following energy functional on $L^2(\Omega)$:

$$\tilde{\Phi}^{\alpha,\beta}_p[u] := \begin{cases} \dfrac{C_{2,p,\beta}}{2p} \displaystyle\iint_{\Omega_1 \times \Omega_1} \dfrac{|u_1(x) - u_1(y)|^p}{|x-y|^{\beta p + 2}} \, \mathrm{d}\mathcal{L}_2(x) \mathrm{d}\mathcal{L}_2(y) & + \dfrac{C_{2,p,\alpha}}{2p} \displaystyle\iint_{\Omega_2 \times \Omega_2} \dfrac{|u_2(x) - u_2(y)|^p}{|x-y|^{\alpha p + 2}} \, \mathrm{d}\mathcal{L}_2(x) \mathrm{d}\mathcal{L}_2(y) \\ + \dfrac{1}{p} \displaystyle\int_{\Sigma} b|u_1|^p \, \mathrm{d}\mu & \text{if } u \in \tilde{\mathbb{W}}^{\alpha,\beta}_p(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus \tilde{\mathbb{W}}^{\alpha,\beta}_p(\Omega). \end{cases} \quad (5.2)$$

Moreover, we point out that (1.11) and (1.12) hold also for the space $\tilde{\mathbb{W}}^{\alpha,\beta}_p(\Omega)$, while the continuous embedding (1.10) is replaced by the following

$$\tilde{\mathbb{W}}^{\alpha,\beta}_p(\Omega) \hookrightarrow L^{p^*(\alpha)}(\Omega). \quad (5.3)$$

The functional $\tilde{\Phi}^{\alpha,\beta}_p$ enjoys the same properties of the functional $\Phi^{\alpha,\beta}_p$, namely it is weakly lower semicontinuous, proper and convex on $L^2(\Omega)$ and its subdifferential $\partial \tilde{\Phi}^{\alpha,\beta}_p$ is single-valued.
We now study the following abstract Cauchy problem

$$(P) \begin{cases} \frac{\partial u}{\partial t} + \tilde{\mathcal{A}}^{\alpha,\beta}_p u = f, & t \in [0,T] \\ u(0) = u_0, \end{cases}$$

involving $\tilde{\mathcal{A}}^{\alpha,\beta}_p := \partial \tilde{\Phi}^{\alpha,\beta}_p$. As in Theorem 3.3, we can prove that the above abstract Cauchy problem admits a unique strong solution in the sense of Definition 3.2. In addition to that, we also have that the nonlinear semigroup $\tilde{T}^{\alpha,\beta}_p(t)$ generated by $-\tilde{\mathcal{A}}^{\alpha,\beta}_p$ is strongly continuous and contractive on $L^2(\Omega)$.
By means of a suitable characterization of $\partial \tilde{\Phi}^{\alpha,\beta}_p$ analogous to the one given in Theorem 3.6, we have that the unique strong solution of problem (P) actually solves the following

problem on Ω for a.e. $t \in (0,T]$ in the following weak sense:

$$(\tilde{P}) \begin{cases} \frac{\partial u_1}{\partial t}(t,x) + (-\Delta_p)^{\beta}_{\Omega_1} u_1(t,x) = f_1(t,x) & \text{for a.e. } x \in \Omega_1, \\ \frac{\partial u_2}{\partial t}(t,x) + (-\Delta_p)^{\alpha}_{\Omega_2} u_2(t,x) = f_2(t,x) & \text{for a.e. } x \in \Omega_2, \\ \tilde{w} u_1 = u_2 & \text{on } \Sigma, \\ \langle \tilde{\mathcal{N}} u, v\rangle + \langle b|u_1|^{p-2} u_1, v_1\rangle_{L^{p'}(\Sigma), L^p(\Sigma)} = 0 & \forall\, v \in B^{p,p}_{\gamma(\beta)}(\Sigma), \\ \mathcal{N}^{p'(1-\alpha)}_p u_2 = 0 & \text{in } (W^{\alpha-\frac{1}{p},p}(\Gamma))', \\ u(0,x) = u^0(x) & \text{in } L^2(\Omega). \end{cases}$$

Finally, one can easily adapt all the results of Section 4 and obtain the ultracontractivity of $\tilde{T}^{\alpha,\beta}_p(t)$. We state the main result for the sake of clarity.

Theorem 5.1. *Let $p > 2$ and $\alpha p \leq \beta p < 2$. If $q \in [2,\infty]$, then there exist two positive constants C_1, C_2 depending on α, β, p, q and Ω such that*

$$\|\tilde{T}^{\alpha,\beta}_p(t)u^0 - \tilde{T}^{\alpha,\beta}_p(t)v^0\|_\infty \leq C_1(\max\{|\Omega_1|,|\Omega_2|\})^{\lambda_1(\alpha)} e^{C_2 t} t^{-\lambda_2(\alpha)} \|u^0 - v^0\|_q^{\lambda_3(\alpha)}, \quad (5.4)$$

for every $u^0, v^0 \in L^q(\Omega)$ and for every $t > 0$, where

$$\lambda_1(\alpha) = \frac{2-\alpha p}{2}\left[1 - \left(\frac{q}{q+p-2}\right)^{\frac{2}{\alpha p}}\right], \quad \lambda_2(\alpha) = \frac{1}{p-2}\left[1 - \left(\frac{q}{q+p-2}\right)^{\frac{2}{\alpha p}}\right],$$

$$\lambda_3(\alpha) = \left(\frac{q}{q+p-2}\right)^{\frac{2}{\alpha p}}.$$

We conclude the paper by pointing out that the results of this paper can be adapted to more general frameworks.

First of all, one can replace the Koch snowflake with the so-called fractal mixtures (for details on such structures see e.g. [28]). Moreover, one can study fractional operators involving more general kernels, under suitable growth conditions.

Finally, by proceeding as in [8], we can consider the case of a domain $\Omega \subset \mathbb{R}^N$ for $N \geq 2$ such that $\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma$, where the domains Ω_i are (ϵ,δ) domains satisfying the hypotheses of [8, Section 1.2] and Σ is a general d-set or an arbitrary closed set in the sense of [24].

Acknowledgements. The authors have been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] S. Abe, S. Thurner, *Anomalous diffusion in view of Einstein's 1905 theory of Brownian motion*, Physica A, 356 (2005), 403–407.

[2] D. R. Adams, L. I. Hedberg, *Function Spaces and Potential Theory*, Springer-Verlag, Berlin, 1996.

[3] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Translated from the Romanian, Noordhoff International Publishing, Leiden, 1976.

[4] M. Biegert, *A priori estimate for the difference of solutions to quasi-linear elliptic equations*, Manuscripta Math., 133 (2010), 273–306.

[5] H. Brézis, *Propriétés régularisantes de certains semi-groupes non linéaires*, Israel J. Math., 9 (1971), 513–534.

[6] F. Brezzi, G. Gilardi, *Fundamentals of PDE for Numerical Analysis*, in: Finite Element Handbook (edited by H. Kardestuncer and D. H. Norrie), McGraw-Hill Book Co., New York, 1987.

[7] F. Cipriani, G. Grillo, *Uniform bounds for solutions to quasilinear parabolic equations,* J. Differential Equations, 177 (2001), 209–234.

[8] S. Creo, M. R. Lancia, *Fractional (s, p)-Robin-Venttsel' problems on extension domains*, NoDEA Nonlinear Differential Equations Appl., 28 (2021), 3, Paper No. 31, 33 pp. DOI: 10.1007/s00030-021-00692-w

[9] S. Creo, M. R. Lancia, P. Vernole, *Convergence of fractional diffusion processes in extension domains*, J. Evol. Equ., 20 (2020), 1, 109–139.

[10] S. Creo, M. R. Lancia, P. Vernole, *M-convergence of p-fractional energies in irregular domains*, J. Convex Anal., 28 (2021), 2, 509–534.

[11] R. Dautray, J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, Vol 2: Functional and Variational Methods*, Springer–Verlag, Berlin, 1988.

[12] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136 (2012), 521–573.

[13] A. A. Dubkov, B. Spagnolo, V. V. Uchaikin, *Lèvy flight superdiffusion: An introduction*, Int. J. Bifurc. Chaos, 18 (2008), 2649–2672.

[14] K. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1986.

[15] U. R. Freiberg, M. R. Lancia, *Energy form on a closed fractal curve*, Z. Anal. Anwendungen, 23 (2004), 115–137.

[16] C. G. Gal, M. Warma, *Elliptic and parabolic equations with fractional diffusion and dynamic boundary conditions*, Evol. Equat. Control Theory, 5 (2016), 61–103.

[17] C. G. Gal, M. Warma, *Reaction-diffusion equations with fractional diffusion on non-smooth domains with various boundary conditions*, Discrete Contin. Dyn. Syst., 36 (2016), 1279–1319.

[18] C. G. Gal, M. Warma, *Nonlocal transmission problems with fractional diffusion and boundary conditions on non-smooth interfaces*, Comm. Partial Differential Equations, 42 (2017), 579–625.

[19] R. Gorenflo, F. Mainardi, A. Vivoli, *Continuous-time random walk and parametric subordination in fractional diffusion*, Chaos, Solitons, Fractals, 34 (2007), 87–103.

[20] Q. Y. Guan, *Integration by parts formula for regional fractional Laplacian.* Commun. Math. Phys., 266 (2006), 289–329.

[21] Q. Y. Guan, Z. M. Ma, *Boundary problems for fractional Laplacians*, Stoch. Dyn., 5 (2005), 385–424.

[22] M. Jara, *Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps*, Comm. Pure Appl. Math., 62 (2009), 198–214.

[23] P. W. Jones, *Quasiconformal mapping and extendability of functions in Sobolev spaces*, Acta Math., 147 (1981), 71–88.

[24] A. Jonsson, *Besov spaces on closed subsets of* $\mathbb{R}^n$, Trans. Amer. Math. Soc., 341 (1994), 355–370.

[25] A. Jonsson, H. Wallin, *Function Spaces on Subsets of* $\mathbb{R}^n$, Part 1, Math. Reports, vol.2, Harwood Acad. Publ., London, 1984.

[26] A. Jonsson, H. Wallin, *The dual of Besov spaces on fractals*, Studia Math., 112 (1995), 285–300.

[27] M. R. Lancia, *A transmission problem with a fractal interface*, Z. Anal. Anwendungen, 21 (2002), 113–133.

[28] M. R. Lancia, V. Regis Durante, P. Vernole, *Asymptotics for Venttsel' problems for operators in non divergence form in irregular domains*, Discrete Contin. Dyn. Syst. Ser. S, 9 (2016), 1493–1520.

[29] M. R. Lancia, A. Vélez-Santiago, P. Vernole, *Quasi-linear Venttsel' problems with nonlocal boundary conditions on fractal domains*, Nonlinear Anal. Real World Appl., 35 (2017), 265–291.

[30] M. R. Lancia, P. Vernole, *Irregular heat flow problems*, SIAM J. Math. Anal., 42 (2010), 1539–1567.

[31] M. R. Lancia, P. Vernole, *Semilinear fractal problems: approximation and regularity results*, Nonlinear Anal., 80 (2013), 216–232.

[32] B. B. Mandelbrot, J. W. Van Ness, *Fractional Brownian motions, fractional noises and applications*, SIAM Rev., 10 (1969), 422–437.

[33] A. Mellet, S. Mischler, C. Mouhot, *Fractional diffusion limit for collisional kinetic equations*, Arch. Ration. Mech. Anal., 199 (2011), 493–525.

[34] H. Pham Huy, E. Sanchez-Palencia, *Phénomènes des transmission à travers des couches minces de conductivité élevée*, J. Math. Anal. Appl., 47 (1974), 284–309.

[35] W. R. Schneider, *Grey noise*, in: S. Albeverio, G. Casati, U. Cattaneo, D. Merlini, R. Moresi, eds. Stochastic Processes, Physics and Geometry, Teaneck, NJ, USA, World Scientific, 1990, 676–681.

[36] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Mathematical Surveys and Monographs 49, American Mathematical Society, Providence, RI, 1997.

[37] E. Valdinoci, *From the long jump random walk to the fractional Laplacian*, Bol. Soc. Esp. Mat. Apl. SeMA, 49 (2009), 33–44.

[38] A. Vélez-Santiago, *Quasi-linear variable exponent boundary value problems with Wentzell-Robin and Wentzell boundary conditions*, J. Funct. Anal., 266 (2014), 560–615.

[39] M. Warma, *The p-Laplace operator with the nonlocal Robin boundary conditions on arbitrary open sets*, Ann. Mat. Pura Appl., 193 (2014), 771–800.

[40] M. Warma, *The fractional Neumann and Robin type boundary conditions for the regional fractional p-Laplacian*, NoDEA Nonlinear Differential Equations Appl., 23 (2016), Art. 1, 46 pp.

[41] M. Warma, *On a fractional* (s,p)*-Dirichlet-to-Neumann operator on bounded Lipschitz domains*, J. Elliptic Parabol. Equ., 4 (2018), 223–269.

[42] J. L. Zolesio, *Multiplication dans les espaces de Besov*, Proc. Roy. Soc. Edinburgh Sect. A, 78 (1977/78), 1–2, 113–117.