M-Convergence of p-fractional energies in irregular domains

Simone Creo, Maria Rosaria Lancia and Paola Vernole

Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sapienza Università di Roma,
Via A. Scarpa 16, 00161 Roma, Italy.

Dedicated to Umberto Mosco, with friendship
and admiration, on his 80th birthday

Abstract

We study the asymptotic behavior of anomalous p-fractional energies in bad domains via the M-convergence. These energies arise naturally when studying Robin-Venttsel' problems for the regional fractional p-Laplacian. We provide a suitable notion of fractional normal derivative on irregular sets via a fractional Green formula as well as existence and uniqueness results for the solution of the Robin-Venttsel' problem by a semigroup approach. Markovianity properties of the associated semigroup are proved.

Keywords: fractional p-Laplacian, fractal domains, fractional Green formula, M-convergence, nonlinear semigroups, dynamical boundary conditions.

2010 Mathematics Subject Classification: Primary: 35R11, 35B40. Secondary: 28A80, 47H20, 47J35.

Introduction

Aim of this paper is to study a parabolic problem for the regional fractional p-Laplacian with Robin (Venttsel') boundary conditions in irregular domains via a constructive approach. Namely, our goal is not only to study the problem at hand but also to approximate the solution, if any, in terms of smoother solutions. This is a key point also in view of numerical approximations.

Fractional operators describe mathematically many physical phenomena exhibiting deviations from standard diffusion, the so-called anomalous diffusion. It is an important topic not only in physics, but also in finance and probability ([1, 26, 41, 43]; for a tutorial see [48]).

Several models appear in the literature to describe such diffusion, e.g. the fractional Brownian motion, the continuous time random walk, the Lévy flight as well as random walk models based on evolution equations of single and distributed fractional order in time and/or space [16, 23, 40, 43, 46].

The diffusion processes – which often take place across irregular interfaces or boundaries – are governed, in some situations, by the regional fractional Laplacian, therefore a rigorous formulation is needed.

In the literature, results for boundary value problems for the regional fractional Laplacian with Dirichlet, Neumann, Robin or Venttsel'-type boundary conditions on Lipschitz domains, can be found in [20], [21] and [22] along with the physical motivations. For the case of the fractional Laplacian with Robin boundary conditions in Lipschitz domains, the reader is referred to the recent paper of Claus and Warma [12]. The results on the regional fractional p-Laplacian in piecewise smooth domains are more recent [49, 50, 19].

As concerning Venttsel'-type boundary value problems in irregular domains, possibly of fractal type, among the others we refer the reader to [37, 39, 35, 13, 36]. The Robin-Venttsel' problem for the (linear) regional fractional Laplacian in irregular domains has been investigated recently in [14], where also a constructive approach is developed. The nonlinear case of the regional fractional p-Laplacian is completely unexplored and it will be object of our investigation.

As we will see, the *Mosco-convergence* (M-convergence) of suitable energy functionals is a cornerstone for the constructive approach.

More precisely, in this paper we consider the following evolution problems for the regional fractional p-Laplacian with dynamical Robin-Venttsel' boundary conditions.

The problems can be formally stated as:

$$(\tilde{P})\begin{cases}\frac{\partial u}{\partial t}(t,x)+(-\Delta_p)^s_Q u(t,x)=f(t,x) & \text{in } (0,T]\times Q,\\[2mm] \frac{\partial u}{\partial t}+\mathcal{N}_p^{p'(1-s)}u+b|u|^{p-2}u=f & \text{on } (0,T]\times \partial Q,\\[2mm] u(0,x)=u_0(x) & \text{in } \overline{Q},\end{cases}$$

and, for every $n\in\mathbb{N}$,

$$(\tilde{P}_n)\begin{cases}\frac{\partial u_n}{\partial t}(t,x)+(-\Delta_p)^s_{Q_n} u_n(t,x)=f(t,x) & \text{in } (0,T]\times Q_n,\\[2mm] \delta_n\frac{\partial u_n}{\partial t}+\mathcal{N}_p^{p'(1-s)}u_n+\delta_n b|u|^{p-2}u=\delta_n f & \text{on } (0,T]\times \partial Q_n,\\[2mm] u_n(0,x)=u_0^{(n)}(x) & \text{in } \overline{Q}_n,\end{cases}$$

where $Q\subset\mathbb{R}^3$ is a three-dimensional Koch-type cylinder and $\{Q_n\}$ is a sequence of suitable polyhedral domains approximating Q (see Section 1.1 for details).
Here $(-\Delta_p)^s_Q$ and $(-\Delta_p)^s_{Q_n}$ denote the regional fractional p-Laplacians (see (3.1)), $s\in(0,1)$, $p>1$, $\mathcal{N}_p^{p'(1-s)}u$ is the fractional normal derivative to be suitably defined, f, b, u_0 and $u_0^{(n)}$ are given functions, while T and δ_n are positive numbers.
Firstly, we will focus on giving a rigorous formulation of the parabolic problem for the regional fractional p-Laplacian with dynamical boundary conditions in irregular domains, and in suitable smoother approximating domains. Secondly, we will prove that the approximating solutions of problems $(\tilde{P}_n)$ converge in a suitable sense to the (limit) solution of problem $(\tilde{P})$.
To this aim, we introduce a suitable notion of p-fractional normal derivative on irregular sets, via a generalized fractional Green formula, and we prove that it is an element of the dual of a suitable Besov space defined on ∂Q (see Theorem 3.2).
We then consider the fractional energy functional $E_{p,s}$ defined in (2.1), which is proper, convex and weakly lower semicontinuous, and the corresponding associated subdifferential $\mathcal{A}_s$. In Theorem 4.6 we prove, via a semigroup approach, existence and uniqueness of a strong solution for a suitable abstract Cauchy problem (P) for the operator $\mathcal{A}_s$. We prove regularity properties of the semigroup, i.e. order-preserving and non-expansive on L^∞, in Theorem 4.5. By applying Theorem 4.8, we prove that problem $(\tilde{P})$ is the strong formulation of the abstract problem (P). Similar results for the approximating problems $(\tilde{P}_n)$ hold (see Theorem 4.9).
In order to study the asymptotic behavior of the approximating solutions, we consider the fractional functionals $E_{p,s}$ and $E_{p,s}^{(n)}$ on $L^2(Q,m)$ and $L^2(Q,m_n)$ respectively (see (1.6) and (1.7)). In Theorem 4.12 we study the asymptotic behavior of the solutions of problems (P_n); the functional setting is that of varying Hilbert spaces (see Section 1.3).

We use the notion of M-convergence [42] of the energy functionals adapted by Tölle to the nonlinear framework in varying Hilbert spaces [45]. The M-convergence of the functionals is equivalent to the G-convergence of their subdifferentials $\mathcal{A}_s^{(n)}$, which in turn is equivalent to the convergence of the nonlinear semigroup generated by $-\mathcal{A}_s^{(n)}$ (see [3] and [9] for the case of a fixed Banach space).
In Theorem 2.6 we prove the M-convergence of the functionals, which yields the convergence of the solutions in a suitable sense (see Theorem 4.12) via the convergence of the semigroups given in [45, Theorem 7.24]. The choice of the factor δ_n, which accounts for the jump of dimension between ∂Q and ∂Q_n, is crucial in the proof of the M-convergence.
We point out that our results can be extended to the more general class of Jones domains [27] as in [14]. Here, for the sake of simplicity, we confine ourselves to the model domain Q.
The plan of the paper is the following.
In Section 1 we recall some preliminary results on traces and varying Hilbert spaces.
In Section 2 we introduce the energy functionals $E_{p,s}^{(n)}$ and $E_{p,s}$ respectively and we prove the M-convergence.
In Section 3 we recall the definition of fractional regional p-Laplacian and we introduce the notion of weak fractional normal derivative by proving a generalized fractional Green formula.
In Section 4 we prove existence and uniqueness of a strong solution for the corresponding abstract Cauchy problems and we give a strong interpretation. We prove that the associated semigroups are Markovian. We also prove the convergence of the solutions u_n to u in the framework of varying Hilbert spaces.
In Section 5 we discuss some open problems and make some comparisons with the linear case.

1 Preliminaries

1.1 The fractal domain

Given $P, P_0 \in \mathbb{R}^N$, in this paper we denote by $|P - P_0|$ the Euclidean distance in $\mathbb{R}^N$ and by $B(P_0, r) = \{P \in \mathbb{R}^N : |P - P_0| < r\}$, for $r > 0$, the Euclidean ball. We also denote by $\mathcal{L}_N$ the N-dimensional Lebesgue measure.
We denote by F the Koch snowflake, i.e. the union of three co-planar Koch curves F_1, F_2 and F_3 (see [17]). We assume that the junction points A_1, A_3 and A_5 are the vertices of a regular triangle with unit side length, i.e. $|A_1 - A_3| = |A_1 - A_5| = |A_3 - A_5| = 1$. For $i = 1, 2, 3$, F_i is the uniquely determined self-similar set with respect to a family Ψ^i

of four suitable contractions $\psi_1^{(i)}, \ldots, \psi_4^{(i)}$, with respect to the same ratio $\frac{1}{3}$ (see [18]). The Hausdorff dimension of the Koch snowflake is given by $d_f = \frac{\ln 4}{\ln 3}$. One can define, in a natural way, a finite Borel measure μ supported on F by

$$\mu_F := \mu_1 + \mu_2 + \mu_3, \tag{1.1}$$

where μ_i denotes the normalized d_f-dimensional Hausdorff measure restricted to F_i, for $i = 1, 2, 3$. Moreover, in the following we denote by F_n the closed pre-fractal polygonal curve approximating F at the n-th step.
By $\Omega \subset \mathbb{R}^2$ we denote the open bounded set having as boundary the Koch snowflake F and, for every $n \in \mathbb{N}$, by $\Omega_n \subset \mathbb{R}^2$ we denote the open bounded set having as boundary F_n.
We define the "cylindrical-type" surfaces $S = F \times I$ and, for every $n \in \mathbb{N}$, $S_n = F_n \times I$, where $I = [0, 1]$. By Q we denote the open cylindrical domain having S as "lateral surface" and the sets $\Omega \times \{0\}$ and $\Omega \times \{1\}$ as bases; in the same spirit, for every $n \in \mathbb{N}$ we denote by Q_n the three-dimensional cylindrical domain having S_n as lateral surface and the sets $\Omega_n \times \{0\}$ and $\Omega_n \times \{1\}$ as bases (see Figure 1). We set $\tilde{\Omega} = (\Omega \times \{0\}) \cup (\Omega \times \{1\})$ and $\tilde{\Omega}_n = (\Omega_n \times \{0\}) \cup (\Omega_n \times \{1\})$ respectively.

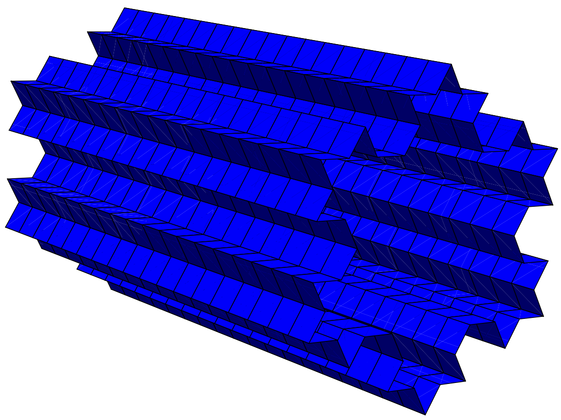

Figure 1: The fractal domain Q.

The pre-fractal domains Q_n are non-convex and polyhedral domains such that

1) Q_n is bounded and Lipschitz;

2) $Q_n \subseteq Q_{n+1}$;

3) $Q = \displaystyle\bigcup_{n=1}^{\infty} Q_n$.

1.2 Functional spaces and trace theorems

We begin by introducing suitable measure on the lateral surfaces S and S_n. By ℓ we denote the natural arc-length coordinate on each edge of the polygonal curve F_n. By $\mathrm{d}\ell$ we denote the one–dimensional measure given by the arc-length ℓ.
We introduce on S and S_n the measures

$$\mathrm{d}g = \mathrm{d}\mu_F \times \mathrm{d}\mathcal{L}_1, \tag{1.2}$$

and

$$\mathrm{d}\sigma = \mathrm{d}\ell \times \mathrm{d}\mathcal{L}_1$$

respectively, where μ_F is the d_f-normalized Hausdorff measure on F given by (1.1) and $\mathcal{L}_1$ is the one-dimensional Lebesgue measure on I.

Let $\mathcal{G}$ (resp. $\mathcal{S}$) be an open (resp. closed) set of $\mathbb{R}^N$. By $L^p(\mathcal{G})$, for $p > 1$, we denote the Lebesgue space with respect to the Lebesgue measure $\mathcal{L}_N$, which will be left to the context whenever that does not create ambiguity. By $L^p(\partial\mathcal{G}, \mu)$ we denote the Lebesgue space on $\partial\mathcal{G}$ with respect to a Hausdorff measure μ supported on $\partial\mathcal{G}$. By $D(\mathcal{G})$ we denote the space of infinitely differentiable functions with compact support in $\mathcal{G}$. By $C(\mathcal{S})$ we denote the space of continuous functions on $\mathcal{S}$.
By $W^{s,p}(\mathcal{G})$, where $0 < s < 1$, we denote the fractional Sobolev space of exponent s. We point out that it is a Banach space if we endow it with the following norm:

$$\|u\|^p_{W^{s,p}(\mathcal{G})} = \|u\|^p_{L^p(\mathcal{G})} + \iint\limits_{\mathcal{G}\times\mathcal{G}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y),$$

Moreover, for $u, v \in W^{s,p}(\mathcal{G})$ we set

$$(u, v)_{s,p} := \iint\limits_{\mathcal{G}\times\mathcal{G}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+sp}} \, \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y).$$

In the following we will denote by $|A|$ the Lebesgue measure of a measurable subset $A \subset \mathbb{R}^N$. For f in $W^{s,p}(\mathcal{G})$, we define the trace operator γ_0 as

$$\gamma_0 f(x) := \lim_{r\to 0} \frac{1}{|B(x,r)\cap\mathcal{G}|} \int\limits_{B(x,r)\cap\mathcal{G}} f(y) \, \mathrm{d}\mathcal{L}_N(y) \tag{1.3}$$

at every point $x \in \overline{\mathcal{G}}$ where the limit exists. The limit (1.3) exists at quasi every $x \in \overline{\mathcal{G}}$ with respect to the (s,p)-capacity (see [2], Definition 2.2.4 and Theorem 6.2.1 page 159). In the sequel we will omit the trace symbol and the interpretation will be left to the context.
We need two trace theorems, one for the pre-fractal and one for the fractal case. We begin with the trace theorem in the pre-fractal case. For the proof we refer to [10].

Proposition 1.1. *Let $\frac{1}{p} < s < 1$. Then $W^{s-\frac{1}{p},p}(\partial Q_n)$ is the trace space to S_n of $W^{s,p}(Q_n)$ in the following sense:*

(i) γ_0 is a continuous and linear operator from $W^{s,p}(Q_n)$ to $W^{s-\frac{1}{p},p}(\partial Q_n)$;

(ii) there exists a continuous linear operator Ext from $W^{s-\frac{1}{p},p}(\partial Q_n)$ to $W^{s,p}(Q_n)$ such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator in $W^{s-\frac{1}{p},p}(\partial Q_n)$.

As to the fractal domain Q, we note that Q is a (ε, δ) domain in the sense of Jones [27] with boundary an arbitrary closed set in the sense of Jonsson [28].
We recall the definition of Besov spaces on an arbitrary closed set $\tilde{\mathcal{F}}$ specialized to our case. For generalities on these Besov spaces, we refer to [28]. Let us suppose that there is a measure $\mu_{\tilde{\mathcal{F}}}$ on $\tilde{\mathcal{F}}$ satisfying the following condition: for $0 < d_1 \leq d_2 \leq N$, there exist two positive constants $\tilde{c}_1$ and $\tilde{c}_2$ such that

$$\tilde{c}_1\, k^{d_1} \mu_{\tilde{\mathcal{F}}}(B(x,r)) \leq \mu_{\tilde{\mathcal{F}}}(B(x,kr)) \leq \tilde{c}_2\, k^{d_2} \mu_{\tilde{\mathcal{F}}}(B(x,r)) \tag{1.4}$$

for all $x \in \tilde{\mathcal{F}}$, $r > 0$, $k \geq 1$ such that $kr \leq 1$. When $d_1 = d_2$, the set $\tilde{\mathcal{F}}$ is a d-set (see [29]). We remark that S is a $(d_f + 1)$-set, while the boundary $\partial Q = S \cup (\Omega \times \{0\}) \cup (\Omega \times \{1\})$ is neither a 2-set nor a $(d_f + 1)$-set.

Definition 1.2. *Let $\tilde{\mathcal{F}} \subset \mathbb{R}^N$ be an arbitrary closed set and $\mu_{\tilde{\mathcal{F}}}$ be a measure defined on $\tilde{\mathcal{F}}$ satisfying (1.4). The Besov space $\tilde{B}^{p,p}_{\gamma}(\tilde{\mathcal{F}})$ with respect to $\mu_{\tilde{\mathcal{F}}}$ is the space of functions such that the following norm is finite:*

$$\|u\|^p_{\tilde{B}^{p,p}_{\gamma}(\tilde{\mathcal{F}})} = \|u\|^p_{L^p(\tilde{\mathcal{F}})} + \iint\limits_{|x-y|<1} \frac{|u(x) - u(y)|^p}{|x-y|^{\gamma p - N} (\mu_{\tilde{\mathcal{F}}}(B(x, |x-y|)))^2} \,\mathrm{d}\mu_{\tilde{\mathcal{F}}}(x)\, \mathrm{d}\mu_{\tilde{\mathcal{F}}}(y). \tag{1.5}$$

We define the measure $\tilde{\mu}$ supported on ∂Q as

$$\mathrm{d}\tilde{\mu} = \chi_S \mathrm{d}g + \chi_{\tilde{\Omega}} \mathrm{d}\mathcal{L}_2.$$

The measure $\tilde{\mu}$ satisfies condition (1.4) with $d_1 = 2$ and $d_2 = d_f + 1$.
We now state a trace theorem for functions in $W^{s,p}(Q)$. For the proof, we refer to Theorem 1 in [28].

Proposition 1.3. *Let $\frac{1}{p} < s < 1$. $\tilde{B}^{p,p}_s(\partial Q)$ is the trace space of $W^{s,p}(Q)$ in the following sense:*

(i) γ_0 is a continuous linear operator from $W^{s,p}(Q)$ to $\tilde{B}^{p,p}_s(\partial Q)$;

(ii) there exists a continuous linear operator Ext from $\tilde{B}^{p,p}_s(\partial Q)$ to $W^{s,p}(Q)$ such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator in $\tilde{B}^{p,p}_s(\partial Q)$.

By $(\tilde{B}^{p,p}_s(\partial Q))'$ we denote the dual space of $\tilde{B}^{p,p}_s(\partial Q)$, see [30].
From now on we suppose that
$$\frac{1}{p} < s < 1.$$

1.3 Varying Hilbert spaces

In this subsection, we introduce the notion of convergence of varying Hilbert spaces. We refer to [32] and [31] for definitions and proofs. The Hilbert spaces we consider are real and separable.

Definition 1.4. *A sequence of Hilbert spaces $\{H_n\}_{n\in\mathbb{N}}$ converges to a Hilbert space H if there exists a dense subspace $C \subset H$ and a sequence $\{Z_n\}_{n\in\mathbb{N}}$ of linear operators $Z_n \colon C \subset H \to H_n$ such that*

$$\lim_{n\to\infty} \|Z_n u\|_{H_n} = \|u\|_H \text{ for any } u \in C.$$

We set $\mathcal{H} = \{\cup_n H_n\} \cup H$ and define strong and weak convergence in $\mathcal{H}$. From now on we assume that $\{H_n\}_{n\in\mathbb{N}}$, H and $\{Z_n\}_{n\in\mathbb{N}}$ are as in Definition 1.4.

Definition 1.5 (Strong convergence in $\mathcal{H}$)**.** *A sequence of vectors $\{u_n\}_{n\in\mathbb{N}}$ strongly converges to u in $\mathcal{H}$ if $u_n \in H_n$, $u \in H$ and there exists a sequence $\{\widetilde{u}_m\}_{m\in\mathbb{N}} \in C$ tending to u in H such that*

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \|Z_n \widetilde{u}_m - u_n\|_{H_n} = 0.$$

Definition 1.6 (Weak convergence in $\mathcal{H}$)**.** *A sequence of vectors $\{u_n\}_{n\in\mathbb{N}}$ weakly converges to u in $\mathcal{H}$ if $u_n \in H_n$, $u \in H$ and*

$$(u_n, v_n)_{H_n} \to (u, v)_H$$

for every sequence $\{v_n\}_{n\in\mathbb{N}}$ strongly tending to v in $\mathcal{H}$.

We remark that the strong convergence implies the weak convergence (see [32]).

Lemma 1.7. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence weakly converging to u in $\mathcal{H}$. Then*

$$\sup_n \|u_n\|_{H_n} < \infty, \quad \|u\|_H \le \underline{\lim}_{n\to\infty} \|u_n\|_{H_n}.$$

Moreover, $u_n \to u$ strongly if and only if $\|u\|_H = \lim_{n\to\infty} \|u_n\|_{H_n}$.

Let us recall some characterizations of the strong convergence of a sequence of vectors $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$.

Lemma 1.8. *Let $u \in H$ and let $\{u_n\}_n \in \mathbb{N}$ be a sequence of vectors $u_n \in H_n$. Then $\{u_n\}_{n\in\mathbb{N}}$ strongly converges to u in $\mathcal{H}$ if and only if*

$$(u_n, v_n)_{H_n} \to (u, v)_H$$

for every sequence $\{v_n\}_{n\in\mathbb{N}}$ with $v_n \in H_n$ weakly converging to a vector v in $\mathcal{H}$.

Lemma 1.9. *A sequence of vectors $\{u_n\}_{n\in\mathbb{N}}$ with $u_n \in H_n$ strongly converges to a vector u in $\mathcal{H}$ if and only if*

$$\begin{aligned} \|u_n\|_{H_n} &\to \|u\|_H && \text{and} \\ (u_n, Z_n(\varphi))_{H_n} &\to (u, \varphi)_H && \text{for every } \varphi \in C. \end{aligned}$$

Lemma 1.10. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence with $u_n \in H_n$. If $\|u_n\|_{H_n}$ is uniformly bounded, then there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ which weakly converges in $\mathcal{H}$.*

Lemma 1.11. *For every $u \in H$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$, with $u_n \in H_n$, strongly converging to u in $\mathcal{H}$.*

We now define the G-convergence of operators (see Definition 7.20 in [45]).

Definition 1.12. *Let $n \in \mathbb{N}$, $A_n \colon H_n \to 2^{H_n}$, $A \colon H \to 2^H$ be multivalued operators. We say that A_n G-converges to A, $A_n \xrightarrow{G} A$, if for every $[x, y] \in A$ (i.e. $x \in D(A)$ and $y \in A(x)$) there exists $[x_n, y_n] \in A_n$, $n \in \mathbb{N}$ such that $x_n \to x$ and $y_n \to y$ strongly in $\mathcal{H}$.*

In the following we denote by $L^2(\overline{Q}, m)$ the Lesbegue space with respect to the measure m defined as

$$\mathrm{d}m = \mathrm{d}\mathcal{L}_3 + \mathrm{d}g, \tag{1.6}$$

and by the space $L^2(Q, m_n)$ the Lebesgue space with respect to the measure m_n given, for every $n \in \mathbb{N}$, by

$$\mathrm{d}m_n = \chi_{Q_n}\mathrm{d}\mathcal{L}_3 + \chi_{S_n}\delta_n\mathrm{d}\sigma, \tag{1.7}$$

where δ_n is a given positive number and χ_{Q_n} and χ_{S_n} denote the characteristic function of Q_n and S_n respectively.

Throughout the paper, we define $H = L^2(\overline{Q}, m)$, where m is the measure defined in (1.6), and the sequence $\{H_n\}_{n\in\mathbb{N}}$ by $H_n = \{L^2(Q) \cap L^2(Q, m_n)\}$, where m_n is the measure defined in (1.7). We endow these spaces with the following norms:

$$\|u\|_H^2 = \|u\|^2_{L^2(Q)} + \|u\|^2_{L^2(S)}, \;\; \|u\|^2_{H_n} = \|u\|^2_{L^2(Q_n)} + \delta_n\|u\|^2_{L^2(S_n)}.$$

Proposition 1.13. *Let $\delta_n = \left(\frac{3}{4}\right)^n$. Then the sequence $\{H_n\}_{n\in\mathbb{N}}$ converges in the sense of Definition 1.4 to H.*

For the proof, we refer to Proposition 5.13 in [34], where C and Z_n in Definition 1.4 are respectively $C(\overline{Q})$ and the identity operator on $C(\overline{Q})$.

2 Energy functionals and M-convergence

From now on, let $p \geq 2$. Let $b \in C(\overline{Q})$ be a strictly positive continuous function on $\overline{Q}$. We define the following energy functional for every $u \in H$:

$$E_{p,s}[u] := \begin{cases} \dfrac{C_{3,p,s}}{2p} \displaystyle\iint\limits_{Q \times Q} \frac{|u(x)-u(y)|^p}{|x-y|^{sp+3}} \, \mathrm{d}\mathcal{L}_3(x) \mathrm{d}\mathcal{L}_3(y) + \frac{1}{p} \int\limits_S b|u|^p \, \mathrm{d}g & \text{if } u \in \mathcal{D}(E_{p,s}), \\ +\infty & \text{if } u \in H \setminus \mathcal{D}(E_{p,s}), \end{cases} \tag{2.1}$$

where the effective domain is

$$\mathcal{D}(E_{p,s}) := \{u \in W^{s,p}(Q) \, : \, u = 0 \text{ on } \tilde{\Omega}\},$$

and $C_{3,p,s}$ is a suitable positive constant depending on p and s, see Section 3.

Proposition 2.1. *$E_{p,s}$ is a weakly lower semicontinuous, proper and convex functional in H. Moreover, its subdifferential $\partial E_{p,s}$ is single-valued.*

Proof. The functional $E_{p,s}$ is clearly convex and proper. As to the weakly lower semicontinuity, the thesis follows from the weak lower semicontinuity of the $L^p(S)$-norm and by proceeding as in [47, Theorem 3.1], see also [35, Proposition 2.3]. Moreover, from Proposition 2.40 in [5], $\partial E_{p,s}$ is single-valued. □

Similarly, for every $n \in \mathbb{N}$ and for every $u \in H_n$, we define

$$E^{(n)}_{p,s}[u] := \begin{cases} \dfrac{C_{3,p,s}}{2p} \displaystyle\iint\limits_{Q \times Q} \chi_{Q_n}(x)\chi_{Q_n}(y) \frac{|u(x)-u(y)|^p}{|x-y|^{sp+3}} \, \mathrm{d}\mathcal{L}_3(x) \mathrm{d}\mathcal{L}_3(y) + \frac{\delta_n}{p} \int\limits_{S_n} b|u|^p \, \mathrm{d}\sigma & \\ & \text{if } u \in \mathcal{D}(E^{(n)}_{p,s}), \\ +\infty & \text{if } u \in H_n \setminus \mathcal{D}(E^{(n)}_s), \end{cases} \tag{2.2}$$

with effective domain

$$\mathcal{D}(E^{(n)}_{p,s}) := \{u \in W^{s,p}(Q) \, : \, u = 0 \text{ on } \tilde{\Omega}_n\}.$$

By proceeding as in the proof of Proposition 2.1, the following result holds.

Proposition 2.2. *$E^{(n)}_{p,s}$ is a weakly lower semicontinuous, proper and convex functional in H_n. Moreover, its subdifferential $\partial E^{(n)}_{p,s}$ is single-valued.*

We point out that Propositions 2.1 and 2.2 can be proved also for $1 < p < 2$.

We now introduce the notion of *M-convergence*. The definition of M-convergence of quadratic energy forms is due to Mosco [42] for a fixed Hilbert space. This notion has been adapted to the case of varying Hilbert spaces by Kuwae and Shioya (see Definition 2.11 in [32]) and then extended to the case of proper convex functionals in Banach spaces by Tölle (see Section 7.5, Definition 7.26 in [45]).

Definition 2.3. *Let H_n be a sequence of Hilbert spaces converging to a Hilbert space H. A sequence of proper and convex functionals $\left\{E^{(n)}_{p,s}\right\}$ defined in H_n M-converges to a functional $E_{p,s}$ defined in H if the following conditions hold:*

a) *for every $\{v_n\} \in H_n$ weakly converging to $u \in H$ in $\mathcal{H}$*

$$\varliminf_{n\to\infty} E^{(n)}_{p,s}[v_n] \geq E_{p,s}[u];$$

b) *for every $u \in H$ there exists a sequence $\{w_n\}$, with $w_n \in H_n$ strongly converging to u in $\mathcal{H}$, such that*

$$\varlimsup_{n\to\infty} E^{(n)}_{p,s}[w_n] \leq E_{p,s}[u].$$

We now prove two preliminary results.

Proposition 2.4. *If $\{v_n\}_{n\in\mathbb{N}}$ weakly converges to a vector u in $\mathcal{H}$, then $\{v_n\}_{n\in\mathbb{N}}$ weakly converges to u in $L^2(Q)$ and $\lim\limits_{n\to\infty} \delta_n \int\limits_{S_n} \varphi v_n \,\mathrm{d}\sigma = \int\limits_{S} \varphi u \,\mathrm{d}g$ for every $\varphi \in C(\overline{Q})$.*

For the proof see Proposition 6.6 in [34].

Proposition 2.5. *Let $v_n \rightharpoonup u$ in $W^{s,p}(Q)$ and $b \in C(\overline{Q})$. Then*

$$\delta_n \int\limits_{S_n} b|v_n|^p \,\mathrm{d}\sigma \to \int\limits_{S} b|u|^p \,\mathrm{d}g.$$

Proof. We adapt the proof of Proposition 7.4 in [15] to our case. It holds that

$$\left|\delta_n \int\limits_{S_n} b\,|v_n|^p \,\mathrm{d}\sigma - \int\limits_{S} b\,|u|^p \,\mathrm{d}g\right| \leq \left|\delta_n \int\limits_{S_n} b\,|v_n|^p \,\mathrm{d}\sigma - \delta_n \int\limits_{S_n} b\,|u|^p \,\mathrm{d}\sigma\right|$$

$$+ \left|\delta_n \int\limits_{S_n} b\,|u|^p \,\mathrm{d}\sigma - \int\limits_{S} b\,|u|^p \,\mathrm{d}g\right| =: A_n + B_n.$$

For the term A_n, we have the following estimate:

$$A_n \leq C\,\delta_n \|b\|_{C(\overline{Q})} \,\|v_n - u\|_{L^p(S_n)} \left(\|v_n\|_{L^p(S_n)} + \|u\|_{L^p(S_n)}\right)^{p-1}.$$

Since v_n weakly converges to u in $W^{s,p}(Q)$ by hypothesis, v_n is equibounded in $W^{s,p}(Q)$; hence v_n strongly converges to u in $W^{l,p}(Q)$ for every $0 < l < s$.
We now consider the extension of $v_n - u$ in $W^{l,p}(\mathbb{R}^3)$. From Theorem 3.6 in [15] it follows that, if $w \in W^{\tilde{\beta},s}(\mathbb{R}^3)$, for $\frac{1}{p} < \tilde{\beta} \le \frac{3}{p}$,

$$\|w\|^p_{L^p(S_n)} \le \frac{C_{\tilde{\beta}}}{\delta_n} \|w\|^p_{W^{\tilde{\beta},p}(\mathbb{R}^3)}, \tag{2.3}$$

where $C_{\tilde{\beta}}$ is independent of n. Moreover, from Theorem 1 on page 103 in [29], it follows that, for $0 < \tilde{\beta} < 1$, there exists a linear extension operator $\mathcal{E}\text{xt} : W^{\tilde{\beta},p}(Q) \to W^{\tilde{\beta},p}(\mathbb{R}^3)$ such that

$$\|\mathcal{E}\text{xt}\, w\|^p_{W^{\tilde{\beta},p}(\mathbb{R}^3)} \le \bar{C}_{\tilde{\beta}} \|w\|^p_{W^{\tilde{\beta},p}(Q)}, \tag{2.4}$$

with $\bar{C}_{\tilde{\beta}}$ depending on $\tilde{\beta}$. Therefore we get

$$\delta_n \|v_n - u\|_{L^p(S_n)} \le C \,\|\mathcal{E}\text{xt}(v_n - u)\|_{W^{l,p}(\mathbb{R}^3)} \le C \,\|v_n - u\|_{W^{l,p}(Q)},$$

hence $A_n \to 0$ when $n \to +\infty$.
We now focus on B_n. Since $u \in W^{s,p}(Q)$, from [29, page 213] there exists a sequence $\{w_m\} \in C(\overline{Q}) \cap W^{s,p}(Q)$ such that $\|w_m - u\|_{W^{s,p}(Q)} \to 0$ as $m \to +\infty$. We then get

$$B_n \le \left| \delta_n \int\limits_{S_n} b\,|u|^p \,\mathrm{d}\sigma - \delta_n \int\limits_{S_n} b\,|w_m|^p \,\mathrm{d}\sigma \right| + \left| \delta_n \int\limits_{S_n} b\,|w_m|^p \,\mathrm{d}\sigma - \int\limits_{S} b\,|w_m|^p \,\mathrm{d}g \right|$$
$$+ \left| \int\limits_{S} b\,|w_m|^p \,\mathrm{d}g - \int\limits_{S} b\,|u|^p \,\mathrm{d}g \right|.$$

We proceed as above and we estimate the first term in the right-hand side with $\|u - w_m\|_{W^{s,p}(Q)}$. We estimate the third term with the same quantity by Theorem 1, Chapter V in [29]. Hence, for every $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that these two terms are smaller than $c\,\varepsilon$. Since $b\,w_m$ is a continuous function, if we take $m > m_\varepsilon$, the second term in the right-hand side goes to zero for $n \to +\infty$ from Proposition 1.13. $\square$

We now prove the main Theorem.

Theorem 2.6. *Let $\delta_n = (3^{1-d_f})^n = \left(\frac{3}{4}\right)^n$. Let $E_{p,s}$ and $E^{(n)}_{p,s}$ be defined as in (2.1) and (2.2) respectively. Then $E^{(n)}_{p,s}$ M-converges to the functional $E_{p,s}$.*

Proof. We have to prove conditions a) and b) in Definition 2.3.
Proof of condition a). Let $v_n \in H_n$ be a weakly converging sequence in $\mathcal{H}$ to $u \in H$. We can suppose $v_n \in \mathcal{D}(E^{(n)}_{p,s})$ and

$$\varliminf_{n\to\infty} E^{(n)}_{p,s}[v_n] < \infty,$$

otherwise the thesis follows trivially. Thus there exists a constant independent from n such that

$$\frac{C_{3,p,s}}{2p}\iint\limits_{Q\times Q}\chi_{Q_n}(x)\chi_{Q_n}(y)\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{sp+3}}\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y)+\frac{\delta_n}{p}\int\limits_{S_n}b|v_n|^p\,\mathrm{d}\sigma\leq C. \tag{2.5}$$

In particular we have that $\|v_n\|_{W^{s,p}(Q_n)}<C$. From Theorem 1 page 103 in [29], for every $n\in\mathbb{N}$ there exists a bounded linear operator $\mathrm{Ext}\colon W^{s,p}(Q_n)\to W^{s,p}(\mathbb{R}^3)$ such that

$$\|\mathrm{Ext}\,v_n\|_{W^{s,p}(\mathbb{R}^3)}\leq C_{\mathrm{Ext}}\,\|v_n\|_{W^{s,p}(Q_n)}\leq C_{\mathrm{Ext}}\,C,$$

with C_{Ext} independent of n.
We define $\hat{v}_n=\mathrm{Ext}\,v_n|_Q$. Then $\hat{v}_n\in W^{s,p}(Q)$ and $\|\hat{v}_n\|_{W^{s,p}(Q)}\leq C_{\mathrm{Ext}}\,C$. Therefore, there exists a subsequence (which we still denote by $\hat{v}_n$) weakly converging to some $\hat{v}$ in $W^{s,p}(Q)$; moreover, $\hat{v}_n$ strongly converges to $\hat{v}$ in $L^p(Q)$ (and hence also in $L^2(Q)$ since $p\geq 2$). From Proposition 2.4, v_n weakly converges to u in $L^2(Q)$. We now prove that $\hat{v}=u$ in $L^2(Q)$, that is

$$\int\limits_Q(\hat{v}-u)\varphi\,\mathrm{d}\mathcal{L}_3=0$$

for every $\varphi\in L^2(Q)$.
We first note that

$$\begin{aligned}\int\limits_Q(\hat{v}-u)\varphi\,\mathrm{d}\mathcal{L}_3&=\int\limits_Q(\hat{v}-\hat{v}_n+\hat{v}_n-u)\varphi\,\mathrm{d}\mathcal{L}_3\\&=\int\limits_Q(\hat{v}-\hat{v}_n)\varphi\,\mathrm{d}\mathcal{L}_3+\int\limits_{Q_n}(v_n-u)\varphi\,\mathrm{d}\mathcal{L}_3+\int\limits_{Q\setminus Q_n}(\hat{v}_n-u)\varphi\,\mathrm{d}\mathcal{L}_3.\end{aligned} \tag{2.6}$$

We claim that each term on the right-hand side of (2.6) tends to zero as $n\to+\infty$. From the strong convergence of $\hat{v}_n$ to $\hat{v}$ in $L^2(Q)$ and the weak convergence of v_n to u in $L^2(Q)$, we deduce our claim for the first two terms. As to the third, from Hölder inequality we deduce that

$$\int\limits_{Q\setminus Q_n}|(\hat{v}_n-u)\varphi|\,\mathrm{d}\mathcal{L}_3\leq\|\varphi\|_{L^2(Q\setminus Q_n)}(\|\hat{v}_n\|_{L^2(Q)}+\|u\|_{L^2(Q)})\xrightarrow[n\to+\infty]{}0$$

since $|Q\setminus Q_n|\to 0$ as $n\to+\infty$ and $\hat{v}_n$ is equibounded in $W^{s,p}(Q)$ and in $L^2(Q)$. Hence $\hat{v}_n\rightharpoonup u$ in $W^{s,p}(Q)$ and $\hat{v}_n\to u$ in $L^p(Q)$.
We now prove that

$$\varliminf_{n\to\infty}\iint\limits_{Q\times Q}\chi_{Q_n}(x)\chi_{Q_n}(y)\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{sp+3}}\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y)\geq\iint\limits_{Q\times Q}\frac{|u(x)-u(y)|^p}{|x-y|^{sp+3}}\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y). \tag{2.7}$$

We prove a preliminary fact. We recall that $\hat{v}_n$ converges to u weakly in $W^{s,p}(Q)$ and strongly in $L^2(Q)$.
We set

$$\tilde{v}_n(x,y) := \chi_{Q_n}(x)\chi_{Q_n}(y)\frac{\hat{v}_n(x)-\hat{v}_n(y)}{|x-y|^{\frac{sp+3}{p}}}.$$

Since $\hat{v}_n$ belongs to $W^{s,p}(Q)$ and is equibounded, $\tilde{v}_n$ belongs to $L^p(Q\times Q)$ and is equibounded. Hence there exists a subsequence (still denoted by $\tilde{v}_n$) which weakly converges to $\tilde{v}$ in $L^p(Q\times Q)$. We claim that

$$\tilde{v}(x,y) = \tilde{u}(x,y) := \frac{u(x)-u(y)}{|x-y|^{\frac{sp+3}{p}}} \quad \text{a.e.}, \tag{2.8}$$

where u is the weak limit of $\hat{v}_n$ in $W^{s,p}(Q)$. We have to prove that

$$\iint_{Q\times Q} (\tilde{v}(x,y)-\tilde{u}(x,y))\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) = 0 \quad \forall\,\varphi\in L^{p'}(Q\times Q), \tag{2.9}$$

where p' denotes the conjugate exponent of p. We remark that, since $p\geq 2$, we have $p'\leq 2$. Moreover, we point out that we can suppose that $\varphi\in C(Q\times Q)$; the thesis will then follow by density.
We add and subtract the following two terms on the left-hand side of (2.9):

$$\iint_{Q\times Q} \tilde{v}_n(x,y)\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) \quad \text{and} \quad \iint_{Q\times Q} \frac{\hat{v}_n(x)-\hat{v}_n(y)}{|x-y|^{\frac{sp+3}{p}}}\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y).$$

Hence the following holds:

$$\begin{aligned}
&\iint_{Q\times Q} (\tilde{v}(x,y)-\tilde{u}(x,y))\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) = \iint_{Q\times Q} (\tilde{v}-\tilde{v}_n)\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y)\\
&+\iint_{Q\times Q} \frac{\hat{v}_n(x)-\hat{v}_n(y)}{|x-y|^{\frac{sp+3}{p}}}(\chi_{Q_n}(x)\chi_{Q_n}(y)-\chi_Q(x)\chi_Q(y))\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y)\\
&+\iint_{Q\times Q}\left(\frac{\hat{v}_n(x)-\hat{v}_n(y)}{|x-y|^{\frac{sp+3}{p}}}-\frac{u(x)-u(y)}{|x-y|^{\frac{sp+3}{p}}}\right)\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) =: I_1^{(n)}+I_2^{(n)}+I_3^{(n)}.
\end{aligned}$$

We study these three terms separately. As to $I_1^{(n)}$, since $\tilde{v}_n$ weakly converges to $\tilde{v}$ in $L^p(Q\times Q)$,

$$I_1^{(n)} \xrightarrow[n\to+\infty]{} 0.$$

As to $I_2^{(n)}$, we point out that $\chi_Q(x)\chi_Q(y)-\chi_{Q_n}(x)\chi_{Q_n}(y) = \chi_{(Q\times Q)\setminus(Q_n\times Q_n)}(x,y)$, since $Q_n\subset Q$. Hence, from Hölder inequality it follows that

$$I_2^{(n)} = \iint_{(Q\times Q)\setminus(Q_n\times Q_n)} \frac{\hat{v}_n(x)-\hat{v}_n(y)}{|x-y|^{\frac{sp+3}{p}}}\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) \leq \|\hat{v}_n\|_{W^{s,p}(Q)}\|\varphi\|_{L^{p'}((Q\times Q)\setminus(Q_n\times Q_n))},$$

and the right-hand side tends to zero as $n \to +\infty$ since $\hat{v}_n$ is equibounded in $W^{s,p}(Q)$.
As to $I_3^{(n)}$, we can rewrite it in the following way:

$$
\begin{aligned}
I_3^{(n)} &= \iint\limits_{Q\times Q} \left(\frac{\hat{v}_n(x)-\hat{v}_n(y)}{|x-y|^{\frac{sp+3}{p}}}\varphi(x,y) - \frac{u(x)-u(y)}{|x-y|^{\frac{sp+3}{p}}}\varphi(x,y) \right) \mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) \\
&= \iint\limits_{Q\times Q} \frac{\hat{v}_n(x)-u(x)}{|x-y|^{\frac{sp+3}{p}}}\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) - \iint\limits_{Q\times Q} \frac{\hat{v}_n(y)-u(y)}{|x-y|^{\frac{sp+3}{p}}}\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) \\
&= \int\limits_Q (\hat{v}_n(x)-u(x))\phi_1(x)\,\mathrm{d}\mathcal{L}_3(x) - \int\limits_Q (\hat{v}_n(y)-u(y))\phi_2(y)\,\mathrm{d}\mathcal{L}_3(y),
\end{aligned}
$$

where

$$
\phi_1(x) := \int\limits_Q \frac{\varphi(x,y)}{|x-y|^{\frac{sp+3}{p}}}\,\mathrm{d}\mathcal{L}_3(y), \quad \phi_2(y) := \int\limits_Q \frac{\varphi(x,y)}{|x-y|^{\frac{sp+3}{p}}}\,\mathrm{d}\mathcal{L}_3(x).
$$

We point out that both ϕ_1 and ϕ_2 belong to $L^{p'}(Q)$. Hence, since $\hat{v}_n$ converges strongly to u in $L^p(Q)$, from Hölder inequality also $I_3^{(n)}$ tends to zero as $n \to +\infty$, thus proving (2.9).
We remark that $\hat{v}_n = v_n$ on Q_n. Hence we have that

$$
\chi_{Q_n}(x)\chi_{Q_n}(y)\frac{v_n(x)-v_n(y)}{|x-y|^{\frac{sp+3}{p}}} \rightharpoonup \frac{u(x)-u(y)}{|x-y|^{\frac{sp+3}{p}}}
$$

in $L^p(Q\times Q)$. From the lower semicontinuity of the norm, we get (2.7).
The thesis then follows from (2.7), Proposition 2.5 and liminf properties of the sum.
Proof of condition b). We prove that for every $u \in H$ we can construct a sequence $\{w_n\}_{n\in\mathbb{N}}$ strongly converging to u in $\mathcal{H}$ such that

$$
E_{p,s}[u] \geq \varlimsup_{n\to\infty} E_{p,s}^{(n)}[w_n].
$$

We suppose that $u \in \mathcal{D}(E_{p,s})$, otherwise $E_{p,s}[u] = +\infty$ and the thesis follows trivially from Lemma 1.11.
We set $w_n := u|_{Q_n}$. By proceeding as in the proof of condition (b) in Theorem 7.1 in [14], we have that w_n strongly converges to u in $\mathcal{H}$.

We now prove condition b) of Definition 2.3 for w_n. We have that

$$\overline{\lim_{n\to\infty}} E_{p,s}^{(n)}[w_n] = \overline{\lim_{n\to\infty}} \left(\frac{C_{3,p,s}}{2p} \iint\limits_{Q_n\times Q_n} \frac{|w_n(x)-w_n(y)|^p}{|x-y|^{sp+3}} \,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) + \frac{\delta_n}{p} \int\limits_{S_n} b|w_n|^p \,\mathrm{d}\sigma \right)$$

$$= \overline{\lim_{n\to\infty}} \left(\frac{C_{3,p,s}}{2p} \iint\limits_{Q_n\times Q_n} \frac{|u(x)-u(y)|^p}{|x-y|^{sp+3}} \,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) + \frac{\delta_n}{p} \int\limits_{S_n} b|u|^p \,\mathrm{d}\sigma \right)$$

$$= \frac{C_{3,p,s}}{2p} \iint\limits_{Q\times Q} \frac{|u(x)-u(y)|^p}{|x-y|^{sp+3}} \,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) + \frac{1}{p} \int\limits_{S} b|u|^p \,\mathrm{d}g = E_{p,s}[u],$$

where the second-last equality follows from Proposition 2.5 and from the properties of the pre-fractal domains Q_n, see Section 1.1. This concludes the proof. $\square$

The M-convergence of the energy functionals is equivalent to the G-convergence of the associated subdifferentials, as stated in the following result.

Theorem 2.7. *$E_{p,s}^{(n)}$ M-converges to $E_{p,s}$ in $\mathcal{H}$ if and only if $\partial E_{p,s}^{(n)}$ G-converges to $\partial E_{p,s}$.*

For the proof see Theorem 7.46 in [45].

3 The regional fractional p-Laplacian and the Green formula

The energy functionals introduced in the previous section naturally arise when considering Robin BVPs for the regional fractional p-Laplacian. In order to consider a suitable weak formulation, a key step is to suitably generalize the notion of *p-fractional normal derivative* to irregular sets (Lipschitz and fractals) via Green formulas. We remark that the fractional normal derivative for smooth domains has been introduced in [24, 25] for the case $p = 2$ and it has then been extended to the case $p \geq 2$ in [49].
We first recall the definition of the regional fractional p-Laplacian. We refer to [49] and the references listed in.
Let $s \in (0, 1)$ and $p > 1$. For $\mathcal{G} \subseteq \mathbb{R}^N$, we define the space

$$\mathcal{L}_s^{p-1}(\mathcal{G}) := \left\{ u\colon \mathcal{G} \to \mathbb{R} \text{ measurable } : \int\limits_{\mathcal{G}} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+sp}} \,\mathrm{d}\mathcal{L}_N(x) < \infty \right\}.$$

The regional fractional Laplacian $(-\Delta_p)^s_{\mathcal{G}}$ is defined as follows, for $x \in \mathcal{G}$:

$$\begin{aligned}
(-\Delta_p)^s_{\mathcal{G}} u(x) &= C_{N,p,s} \mathrm{P.V.} \int\limits_{\mathcal{G}} |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+sp}} \,\mathrm{d}\mathcal{L}_N(y) \\
&= C_{N,p,s} \lim_{\varepsilon \to 0^+} \int\limits_{\{y \in \mathcal{G}\,:\,|x-y|>\varepsilon\}} |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+sp}} \,\mathrm{d}\mathcal{L}_N(y),
\end{aligned} \tag{3.1}$$

provided that the limit exists, for every function $u \in \mathcal{L}^{p-1}_s(\mathcal{G})$. The positive constant $C_{N,p,s}$ is defined as follows:

$$C_{N,p,s} = \frac{s 2^{2s} \Gamma(\frac{ps+p+N-2}{2})}{\pi^{\frac{N}{2}} \Gamma(1-s)},$$

where Γ is the Euler function.

Following Section 3 in [14], we prove a p-fractional Green formula for domains with fractal boundary, which, in turn, allows us to define the p-fractional normal derivative on non-smooth domains. A key tool is the use of a limit argument since the fractal domain Q can be approximated by the increasing sequence of pre-fractal domains Q_n.

We define the space

$$V((-\Delta_p)^s_Q, Q) := \{u \in W^{s,p}(Q) \,:\, (-\Delta_p)^s_Q u \in L^{p'}(Q) \text{ in the sense of distributions and } u = 0 \text{ on } \tilde{\Omega}\},$$

which is a Banach space equipped with the norm

$$\|u\|_{V((-\Delta_p)^s_Q, Q)} := \|u\|_{W^{s,p}(Q)} + \|(-\Delta_p)^s_Q u\|_{L^{p'}(Q)}.$$

Analogously, for every $n \in \mathbb{N}$, we define the space $V((-\Delta_p)^s_{Q_n}, Q_n)$ on Q_n as follows:

$$V((-\Delta_p)^s_{Q_n}, Q_n) := \{u \in W^{s,p}(Q) \,:\, (-\Delta_p)^s_{Q_n} u \in L^{p'}(Q_n) \text{ in the sense of distributions and } u = 0 \text{ on } \tilde{\Omega}_n\}.$$

We introduce also the space

$$W^{s-\frac{1}{p},p}_{0,0}(S_n) := \{u \in L^p(S_n) \,:\, \exists\, v \in W^{s,p}(Q) \text{ s.t. } v = 0 \text{ on } \tilde{\Omega} \text{ and } \gamma_0 v = u \text{ on } S_n\}.$$

We now define the p-fractional normal derivative on Lipschitz domains.

Definition 3.1. *Let $n \in \mathbb{N}$ and $u \in V((-\Delta_p)^s_{Q_n}, Q_n)$. We say that u has a weak p-fractional normal derivative in $(W^{s-\frac{1}{p},p}_{0,0}(S_n))'$ if there exists $g \in (W^{s-\frac{1}{p},p}_{0,0}(S_n))'$ such that*

$$\begin{aligned}
\langle g, v \rangle_{(W^{s-\frac{1}{p},p}_{0,0}(S_n))', W^{s-\frac{1}{p},p}_{0,0}(S_n)} &= -\int\limits_{Q_n} (-\Delta_p)^s_{Q_n} u \, v \,\mathrm{d}\mathcal{L}_3 \\
&+ \frac{C_{3,p,s}}{2} \iint\limits_{Q_n \times Q_n} |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp+3}} \,\mathrm{d}\mathcal{L}_3(x) \mathrm{d}\mathcal{L}_3(y)
\end{aligned} \tag{3.2}$$

for every $v \in W^{s,p}(Q_n)$. *In this case,* g *is uniquely determined and we call* $C_{p,s}\mathcal{N}_p^{p'(1-s)}u := g$ *the weak p-fractional normal derivative of* u*, where*

$$C_{p,s} := \frac{(p-1)C_{1,p,s}}{(sp-(p-2))(sp-(p-2)-1)} \int\limits_0^\infty \frac{|t-1|^{(p-2)+1-sp} - (t \vee 1)^{p-sp-1}}{t^{p-sp}}\, \mathrm{d}t.$$

We point out that, when $s \to 1^-$ in (3.2), we recover the quasi-linear Green formula for Lipschitz domains [10].

We introduce the space

$$B^{p,p}_{\eta,0}(S) := \{w \in L^p(S) \,:\, \exists\, v \in W^{s,p}(Q) \text{ s.t. } v = 0 \text{ on } \tilde{\Omega} \text{ and } \gamma_0 v = w \text{ on } S\},$$

where $\eta := s - \frac{2-d_f}{p} > 0$.

Theorem 3.2 (Fractional Green formula)**.** *There exists a bounded linear operator* $\mathcal{N}_p^{p'(1-s)}$ *from* $V((-\Delta_p)^s_Q, Q)$ *to* $(B^{p,p}_{\eta,0}(S))'$.
The following generalized Green formula holds for all $u \in V((-\Delta_p)^s_Q, Q)$ *and* $v \in W^{s,p}(Q)$*:*

$$\begin{aligned} C_{p,s} \left\langle \mathcal{N}_p^{p'(1-s)}u, v \right\rangle_{(B^{p,p}_{\eta,0}(S))', B^{p,p}_{\eta,0}(S)} &= -\int\limits_Q (-\Delta_p)^s_Q u\, v \,\mathrm{d}\mathcal{L}_3 \\ &+ \frac{C_{3,p,s}}{2} \iint\limits_{Q\times Q} |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp+3}}\, \mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y). \end{aligned} \tag{3.3}$$

Proof. For $u \in V((-\Delta_p)^s_Q, Q)$ and $v \in W^{s,p}(Q)$, we define

$$\langle l(u), v\rangle := -\int\limits_Q (-\Delta_p)^s_Q u\, v\,\mathrm{d}\mathcal{L}_3 + \frac{C_{3,p,s}}{2}(u,v)_{p,s}$$

From Hölder inequality and trace theorems, we get

$$\begin{aligned} |\langle l(u), v\rangle| &\le \|(-\Delta_p)^s_Q u\|_{L^{p'}(Q)} \|v\|_{L^p(Q)} + \frac{C_{3,p,s}}{2}\|u\|_{W^{s,p}(Q)}\|v\|_{W^{s,p}(Q)} \\ &\le C\,\|u\|_{V((-\Delta_p)^s_Q,Q)}\|v\|_{W^{s,p}(Q)} \le C\,\|u\|_{V((-\Delta_p)^s_Q,Q)}\|v\|_{B^{p,p}_{\eta,0}(S)}. \end{aligned}$$

This shows in particular that the operator is independent from the choice of v and it is an element of the dual space of $B^{p,p}_{\eta,0}(S)$.
From (3.2) we have that

$$\begin{aligned} C_{p,s} \left\langle \mathcal{N}_p^{p'(1-s)}u, v \right\rangle_{(W^{s-\frac{1}{p},p}_{0,0}(S_n))', W^{s-\frac{1}{p},p}_{0,0}(S_n)} &= -\int\limits_Q \chi_{Q_n}(-\Delta_p)^s_Q u\, v\,\mathrm{d}\mathcal{L}_3 \\ &+ \frac{C_{3,p,s}}{2}\iint\limits_{Q\times Q} \chi_{Q_n}(x)\chi_{Q_n}(y)|u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp+3}}\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y). \end{aligned}$$

From the dominated convergence theorem, we have

$$\lim_{n\to\infty} C_{p,s}\left\langle \mathcal{N}_p^{p'(1-s)}u, v\right\rangle_{(W_{0,0}^{s-\frac{1}{p},p}(S_n))', W_{0,0}^{s-\frac{1}{p},p}(S_n)}$$

$$= \lim_{n\to\infty}\left(-\int_{Q_n}(-\Delta_p)_Q^s u\, v\, \mathrm{d}\mathcal{L}_3 + \frac{C_{3,p,s}}{2}\iint_{Q_n\times Q_n}|u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp+3}}\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y)\right)$$

$$= -\int_Q(-\Delta_p)_Q^s u\, v\, \mathrm{d}\mathcal{L}_3 + \frac{C_{3,p,s}}{2}(u,v)_{p,s} = \langle l(u), v\rangle$$

for every $u\in V((-\Delta_p)_Q^s, Q)$ and $v\in W^{s,p}(Q)$. Hence, we define the fractional normal derivative on Q as

$$\langle C_{p,s}\mathcal{N}_p^{p'(1-s)}u, v\rangle_{(B^{p,p}_{\eta,0}(S))', B^{p,p}_{\eta,0}(S)} := -\int_Q(-\Delta_p)_Q^s u\, v\, \mathrm{d}\mathcal{L}_3 + \frac{C_{3,p,s}}{2}(u,v)_{p,s}.$$

□

Remark 3.3. *We note that* $p'(1-s)$ *can be recast as* $2-\beta$*, where* $\beta = \frac{ps-1}{p-1}+1$*, thus recovering the usual notation for the* p*-fractional normal derivative.*
Moreover, we recover the Green formula proved in [38] for fractal domains when $s\to 1^-$ *in* (3.3)*.*
Let $u\in V(-\Delta_p, Q) := \{u\in W^{1,p}(Q) : -\Delta_p u\in L^{p'}(Q)$ *in the sense of distributions*$\}$ *and* $v\in W^{1,p}(Q)$*. It holds that*

$$\lim_{s\to 1^-}\int_Q(-\Delta_p)_Q^s u\, v\, \mathrm{d}\mathcal{L}_3 = \int_Q |\nabla u|^{p-2}\nabla u\,\nabla v\,\mathrm{d}\mathcal{L}_3.$$

As first step, we take $v=u$ *and* $u\in C^\infty(\overline{Q})$*. In particular then* $u\in C^\infty(\overline{Q}_n)$ *for every* n *and* $\mathcal{N}_p^{p'(1-s)}u = 0$ *on* ∂Q_n *pointwise (see [49]). From Definition 3.1 we have*

$$\lim_{s\to 1^-}\int_Q \chi_{Q_n}u\,(-\Delta_p)_Q^s u\,\mathrm{d}\mathcal{L}_3 = \lim_{s\to 1^-}\frac{(1-s)C_{3,p,s}}{2(1-s)}\iint_{Q_n\times Q_n}\frac{|u(x)-u(y)|^p}{|x-y|^{sp+3}}\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y). \tag{3.4}$$

From [6, 7] and the properties of the Euler function, the limit in the right-hand side of (3.4) *is equal to* $\displaystyle\int_{Q_n}|\nabla u|^p\,\mathrm{d}\mathcal{L}_3$*. Then passing to the limit as* $n\to+\infty$ *we get*

$$\lim_{n\to+\infty}\lim_{s\to 1^-}\int_Q\chi_{Q_n}u\,(-\Delta_p)_Q^s u\,\mathrm{d}\mathcal{L}_3 = \lim_{n\to+\infty}\int_{Q_n}|\nabla u|^p\,\mathrm{d}\mathcal{L}_3 = \int_Q|\nabla u|^p\,\mathrm{d}\mathcal{L}_3.$$

The claim then follows by density arguments.

4 The evolution problems

4.1 Abstract Cauchy problems

Let T be a fixed positive number. We now consider the abstract homogeneous Cauchy problems

$$(P) \begin{cases} \frac{\partial u}{\partial t} + \mathcal{A}_s u \ni 0, \quad t \in [0,T] \\ u(0) = u_0, \end{cases}$$

and, for every $n \in \mathbb{N}$ fixed,

$$(P_n) \begin{cases} \frac{\partial u_n}{\partial t} + \mathcal{A}_s^{(n)} u_n \ni 0, \quad t \in [0,T] \\ u_n(0) = u_0^{(n)}, \end{cases}$$

where $\mathcal{A}_s$ and $\mathcal{A}_s^{(n)}$ are the subdifferentials of $E_{p,s}$ and $E_{p,s}^{(n)}$ respectively, and u_0 and $u_0^{(n)}$ are given functions.
According to [4, Section 2.1, chapter II], we give the following definition.

Definition 4.1. *A function $u : [0,T] \to H$ is a strong solution of problem (P) if $u \in C([0,T]; H)$, $u(t)$ is differentiable a.e. in $(0,T)$, $u(t) \in D(-\mathcal{A}_s)$ a.e and $\frac{\partial u}{\partial t} + \mathcal{A}_s u \ni 0$ for a.e. $t \in [0,T]$.*

Similarly, an analogous definition of strong solution u_n of problem (P_n) holds with suitable changes.
From Theorem 1 and Remark 2 in [8] (see also [4]) we have the following result.

Theorem 4.2. *Let $\varphi : \ H \to (-\infty, +\infty]$ be a proper, convex, lower semicontinuous functional on a real Hilbert space H, with effective domain $D(\varphi)$. Then the subdifferential $\partial\varphi$ is a maximal monotone m-accretive operator. Moreover, $\overline{D(\varphi)} = \overline{D(\partial\varphi)}$ and $-\partial\varphi$ generates a nonlinear C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\overline{D(\varphi)}$ in the following sense: for each $u_0 \in \overline{D(\varphi)}$, the function $u := T(\cdot)u_0$ is the unique strong solution of the problem*

$$\begin{cases} u \in C(\mathbb{R}_+; H) \cap W^{1,\infty}_{loc}((0,\infty); H) \ \ \textit{and} \ \ u(t) \in D(\varphi) \ \ \textit{a.e.}, \\ \dfrac{\partial u}{\partial t} + \partial\varphi(u) \ \ni \ 0 \quad \textit{a.e. on} \ \mathbb{R}_+, \\ u(0,x) \ = u_0(x). \end{cases}$$

In addition, $-\partial\varphi$ generates a nonlinear semigroup $\{\tilde{T}(t)\}_{t\geq 0}$ on H where, for every $t \geq 0$, $\tilde{T}(t)$ is the composition of the semigroup $T(t)$ on $\overline{D(\varphi)}$ with the projection on the convex set $\overline{D(\varphi)}$.

From Proposition 2.1, Proposition 2.2 and Theorem 4.2, we have that the subdifferentials $\partial E_{p,s}$ and $\partial E_{p,s}^{(n)}$ are maximal, monotone and m-accretive operators on H and H_n respectively.

We denote by $T_{p,s}(t)$ and $T^{(n)}_{p,s}(t)$ the nonlinear semigroups generated by $-\partial E_{p,s}$ and $-\partial E^{(n)}_{p,s}$ respectively.

Proposition 4.3. *$T_{p,s}(t)$ and $T^{(n)}_{p,s}(t)$ are strongly continuous and contractive semigroups on H and H_n respectively.*

For the proof see Proposition 3.2, page 176 in [44].
Actually, we can prove that the semigroups $T_{p,s}(t)$ and $T^{(n)}_{p,s}(t)$ enjoy "stronger" properties. We recall some definitions (see [11] for details).

Definition 4.4. *Let X be a locally compact metric space and $\tilde{\mu}$ be a Radon measure on X. Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on $L^2(X,\tilde{\mu})$. The semigroup is* order-preserving *if, for every $u, v \in L^2(X,\tilde{\mu})$ such that $u \leq v$,*

$$T(t)u \leq T(t)v \quad \forall\, t \geq 0.$$

The semigroup is non-expansive *on $L^q(X,\tilde{\mu})$ if for every $t \geq 0$*

$$\|T(t)u - T(t)v\|_{L^\infty(X,\tilde{\mu})} \leq \|u - v\|_{L^\infty(X,\tilde{\mu})} \quad \forall\, u \in L^2(X,\tilde{\mu}) \cap L^q(X,\tilde{\mu}).$$

The semigroup is Markovian *if it is order-preserving and non-expansive on $L^\infty(X,\tilde{\mu})$.*

Theorem 4.5. *The semigroup $\{T_{p,s}(t)\}_{t\geq 0}$ is Markovian on H, i.e. it is order-preserving and non-expansive on $L^\infty(\Omega, m)$. The semigroup $\{T^{(n)}_{p,s}(t)\}_{t\geq 0}$ is Markovian on H_n, i.e. it is order-preserving and non-expansive on $L^\infty(\Omega, m_n)$.*

Proof. By proceeding as in the proof of Theorem 3.1 in [35], see also [47, Theorem 3.4], the thesis follows. □

From Theorem 2.1, chapter IV in [4] the following existence and uniqueness results for the strong solutions of problems (P) and (P_n) follow.

Theorem 4.6. *If $u_0 \in \overline{D(-\mathcal{A}_s)}$, then problem (P) has a unique strong solution $u \in C([0,T]; H)$ defined as $u = T_{p,s}(\cdot)u_0$ such that $u \in W^{1,2}((\delta, T); H)$ for every $\delta \in (0,T)$. Moreover $u \in D(-\mathcal{A}_s)$ a.e. for $t \in (0,T)$, $\sqrt{t}\frac{\partial u}{\partial t} \in L^2(0,T;H)$ and $E_{p,s}[u] \in L^1(0,T)$.*

Theorem 4.7. *If $u^{(n)}_0 \in \overline{D(-\mathcal{A}^{(n)}_s)}$, then for every $n \in \mathbb{N}$ problem (P_n) has a unique strong solution $u_n \in C([0,T]; H_n)$ defined as $u_n = T^{(n)}_{p,s}(\cdot)u^{(n)}_0$ such that $u_n \in W^{1,2}((\delta,T); H_n)$ for every $\delta \in (0,T)$. Moreover $u_n \in D(-\mathcal{A}^{(n)}_s)$ a.e. for $t \in (0,T)$, $\sqrt{t}\frac{\partial u_n}{\partial t} \in L^2(0,T;H_n)$ and $E^{(n)}_{p,s}[u_n] \in L^1(0,T)$.*

4.2 Strong formulations

We now prove that the strong solutions of problems (P) and (P_n) actually solve problems $(\tilde{P})$ and $(\tilde{P}_n)$. To this aim, a characterization of the subdifferentials of $E_{p,s}$ and $E^{(n)}_{p,s}$ is a starting point.

We first consider the fractal case and provide a characterization of $\mathcal{A}_s$. We recall that $\tilde{\Omega} = (\Omega \times \{0\}) \cup (\Omega \times \{1\})$.

Theorem 4.8. *Let u belong to $\mathcal{D}(E_{p,s})$ for a.e. $t \in (0,T]$, and let f be in H. Then $f \in \partial E_{p,s}[u]$ if and only if u solves the following problem:*

$$(\bar{P}) \begin{cases} (-\Delta_p)^s_Q u = f & \text{in } L^{p'}(Q), \\ C_{p,s} \left\langle \mathcal{N}_p^{p'(1-s)} u, v \right\rangle_{(B^{p,p}_{\eta,0}(S))', B^{p,p}_{\eta,0}(S)} + \langle b|u|^{p-2}u, v\rangle_{L^{p'}(S), L^p(S)} = \langle f, v\rangle_{L^2(S), L^2(S)} & \forall\, v \in B^{p,p}_{\eta,0}(S), \\ u = 0 & \text{in } W^{s-\frac{1}{p},p}(\tilde{\Omega}). \end{cases}$$

Proof. Let $f \in \partial E_{p,s}$, i.e. $E_{p,s}[\psi] - E_{p,s}[u] \geq (f, \psi - u)_H$ for every $\psi \in \mathcal{D}(E_{p,s})$. This means that

$$\int_Q f(\psi - u)\, \mathrm{d}\mathcal{L}_3 + \int_S f(\psi - u)\, \mathrm{d}g \leq$$
$$\frac{C_{3,p,s}}{2p} \iint_{Q\times Q} \frac{|\psi(x) - \psi(y)|^p - |u(x) - u(y)|^p}{|x-y|^{sp+3}}\, \mathrm{d}\mathcal{L}_3(x) \mathrm{d}\mathcal{L}_3(y) + \frac{1}{p} \int_S b(|\psi|^p - |u|^p)\, \mathrm{d}g. \tag{4.1}$$

By choosing $\psi = u + tv$, with $v \in \mathcal{D}(E_{p,s})$ and $0 < t \leq 1$ in (4.1), we obtain

$$t\int_Q f\, v\, \mathrm{d}\mathcal{L}_3 + t\int_S f\, v\, \mathrm{d}g \leq$$
$$\frac{C_{3,p,s}}{2p} \iint_{Q\times Q} \frac{|(u+tv)(x) - (u+tv)(y)|^p - |u(x) - u(y)|^p}{|x-y|^{sp+3}}\, \mathrm{d}\mathcal{L}_3(x) \mathrm{d}\mathcal{L}_3(y) + \frac{1}{p} \int_S b(|u+tv|^p - |u|^p)\, \mathrm{d}g. \tag{4.2}$$

If we take $v \in D(Q)$, from (4.2) we get

$$\int_Q f\, v\, \mathrm{d}\mathcal{L}_3 \leq \frac{C_{3,p,s}}{2p} \iint_{Q\times Q} \frac{1}{t} \frac{|(u+tv)(x) - (u+tv)(y)|^p - |u(x) - u(y)|^p}{|x-y|^{sp+3}}\, \mathrm{d}\mathcal{L}_3(x) \mathrm{d}\mathcal{L}_3(y).$$

By passing to the limit for $t \to 0^+$, we get

$$\int\limits_Q f\, v\, \mathrm{d}\mathcal{L}_3 \leq \frac{C_{3,p,s}}{2} \iint\limits_{Q\times Q} |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp+3}}\, \mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y).$$

By taking $-v$ in (4.2) we obtain the opposite inequality, and hence we get

$$\int\limits_Q f\, v\, \mathrm{d}\mathcal{L}_3 = \frac{C_{3,p,s}}{2} \iint\limits_{Q\times Q} |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp+3}}\, \mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y).$$

Since $v \in D(Q)$ and $p' \leq 2$ (since $p \geq 2$), it turns out that in particular $f \in L^{p'}(Q)$. Hence, the p-fractional Green formula for fractal domains given by Theorem 3.2 yields that

$$(-\Delta_p)^s_Q u = f \quad \text{in } L^{p'}(Q) \tag{4.3}$$

(and in particular in $L^2(Q)$).

We go back to (4.2). Dividing by $t > 0$ and passing to the limit for $t \to 0^+$, we get

$$\int\limits_Q f\, v\, \mathrm{d}\mathcal{L}_3 + \int\limits_S f\, v\, \mathrm{d}g \leq$$

$$\frac{C_{3,p,s}}{2} \iint\limits_{Q\times Q} |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp+3}}\, \mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) + \int\limits_S b|u|^{p-2} u\, v\, \mathrm{d}g.$$

As above, by taking $-v$ we obtain the opposite inequality, hence we get the equality. Then, by Theorem 3.2 and (4.3) we have

$$\int\limits_S f v\, \mathrm{d}g = C_{p,s} \left\langle \mathcal{N}_p^{p'(1-s)} u, v \right\rangle_{(B^{p,p}_{\eta,0}(S))', B^{p,p}_{\eta,0}(S)} + \int\limits_S b|u|^{p-2} u\, v\, \mathrm{d}g. \tag{4.4}$$

Hence (4.4) holds in $(B^{p,p}_{\eta,0}(S))'$ and since $u \in \mathcal{D}(E_{p,s})$ we have $u = 0$ in $W^{s-\frac{1}{p},p}(\tilde{\Omega})$.

We prove the converse. Let then $u \in \mathcal{D}(E_{p,s})$ be the weak solution of problem $(\bar{P})$. We have then to prove that $E_{p,s}[v] - E_{p,s}[u] \geq (f, v-u)_H$ for every $v \in \mathcal{D}(E_{p,s})$. By using the inequality

$$\frac{1}{p}(|a|^p - |b|^p) \geq |b|^{p-2} b(a-b)$$

one gets

$$E_{p,s}[v] - E_{p,s}[u] \geq \frac{C_{3,p,s}}{2} \iint\limits_{Q\times Q} |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp+3}}\, \mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y)$$

$$+ \int\limits_S b|u|^{p-2} u\, v\, \mathrm{d}g - \frac{C_{3,p,s}}{2} \iint\limits_{Q\times Q} \frac{|u(x)-(u)(y)|^p}{|x-y|^{sp+3}}\, \mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) - \int\limits_S b|u|^p\, \mathrm{d}g. \tag{4.5}$$

Since u is the weak solution of $(\bar{P})$, by using as test functions v and u respectively we get

$$E_{p,s}[v] - E_{p,s}[u] \geq \int_Q f\, v \,\mathrm{d}\mathcal{L}_3 + \int_S f\, v \,\mathrm{d}g - \int_Q f\, u \,\mathrm{d}\mathcal{L}_3 - \int_S f\, u \,\mathrm{d}g = (f, v)_H - (f, u)_H,$$

i.e. the thesis. □

Theorem 4.8 implies that the unique strong solution u of the abstract Cauchy problem (P) solves the following Robin problem $(\tilde{P})$ on Q for a.e. $t \in (0, T]$ in the following weak sense:

$$(\tilde{P}) \begin{cases} \frac{\partial u}{\partial t}(t,x) + (-\Delta_p)^s_Q u(t,x) = 0 & \text{for a.e. } x \in Q, \\ \left\langle \frac{\partial u}{\partial t}, v \right\rangle_{L^2(S), L^2(S)} + C_{p,s} \left\langle \mathcal{N}_p^{p'(1-s)} u, v \right\rangle_{(B^{p,p}_{\eta,0}(S))', B^{p,p}_{\eta,0}(S)} & \\ + \langle b|u|^{p-2}u, v\rangle_{L^{p'}(S), L^p(S)} = 0 & \forall\, v \in B^{p,p}_{\eta,0}(S), \\ u(t,x) = 0 & \text{in } W^{s-\frac{1}{p},p}(\tilde{\Omega}), \\ u(0,x) = u_0(x) & \text{in } H, \end{cases}$$

where $\eta := s - \frac{2-d_f}{p} > 0$.

We proceed similarly in the pre-fractal case. We first provide a characterization of $\mathcal{A}_s^{(n)}$. We recall that $\tilde{\Omega}_n = (\Omega_n \times \{0\}) \cup (\Omega_n \times \{1\})$.

Theorem 4.9. *For every $n \in \mathbb{N}$ fixed, let $\delta_n = \left(\frac{3}{4}\right)^n$, let u belong to $\mathcal{D}(E^{(n)}_{p,s})$ for a.e. $t \in (0, T]$, and let f be in H_n. Then $f \in \partial E^{(n)}_{p,s}[u]$ if and only if u solves the following problem:*

$$(\bar{P}_n) \begin{cases} (-\Delta_p)^s_{Q_n} u = f & \textit{in } L^{p'}(Q_n), \\ C_{p,s} \left\langle \mathcal{N}_p^{p'(1-s)} u, v \right\rangle_{(W^{s-\frac{1}{p},p}_{0,0}(S_n))', W^{s-\frac{1}{p},p}_{0,0}(S_n)} + \delta_n \langle b|u|^{p-2}u, v\rangle_{L^{p'}(S_n), L^p(S_n)} & \\ = \delta_n \langle f, v\rangle_{L^2(S_n), L^2(S_n)} & \forall\, v \in W^{s-\frac{1}{p},p}_{0,0}(S_n), \\ u = 0 & \textit{in } W^{s-\frac{1}{p},p}(\tilde{\Omega}_n). \end{cases}$$

From Theorem 4.9, it follows that the unique strong solution u_n of the abstract Cauchy problem (P_n) solves (for each $n \in \mathbb{N}$) the following Robin problem $(\tilde{P}_n)$ on Q_n for a.e.

$t \in (0, T]$ in the following weak sense:

$$(\tilde{P}_n) \begin{cases} \frac{\partial u_n}{\partial t}(t,x) + (-\Delta_p)^s_{Q_n} u_n(t,x) = 0 & \text{for a.e. } x \in Q_n, \\ \delta_n \left\langle \frac{\partial u_n}{\partial t}, v \right\rangle_{L^2(S_n), L^2(S_n)} + C_{p,s} \left\langle \mathcal{N}_p^{p'(1-s)} u_n, v \right\rangle_{(W^{s-\frac{1}{p},p}_{0,0}(S_n))', W^{s-\frac{1}{p},p}_{0,0}(S_n)} & \\ +\delta_n \langle b|u_n|^{p-2} u_n, v\rangle_{L^{p'}(S_n), L^p(S_n)} = 0 & \forall\, v \in W^{s-\frac{1}{p},p}_{0,0}(S_n), \\ u_n(t,x) = 0 & \text{in } W^{s-\frac{1}{p},p}(\tilde{\Omega}_n), \\ u_n(0,x) = u_0^{(n)}(x) & \text{in } H_n. \end{cases}$$

4.3 Convergence results

We now aim to prove the convergence of the pre-fractal solutions to the fractal one. Let m and m_n be the measures defined in (1.6) and (1.7) respectively. We denote by $\mathrm{d}t$ the one-dimensional Lebesgue measure on $[0, T]$. We observe that $L^2([0,T] \times Q, \mathrm{d}t \times \mathrm{d}m_n)$ is isomorphic to $L^2([0,T]; H_n)$ and $L^2([0,T] \times \overline{Q}, \mathrm{d}t \times \mathrm{d}m)$ is isomorphic to $L^2([0,T]; H)$. If we define

$$K_n = L^2([0,T]; H_n) \text{ for every } n \in \mathbb{N} \text{ and } K = L^2([0,T]; H),$$

K_n converges to K in the sense of Definition 1.4, where the set C is now $C([0,T] \times \overline{Q})$ and Z_n is the identity operator on C.
We denote by $\mathcal{K} = \{\cup_n K_n\} \cup K$. We define strong and weak convergence in $\mathcal{K}$ according to Definition 1.5 and 1.6 respectively. In the following we use either the characterization of strong convergence in $\mathcal{K}$ given in Lemma 1.8 or the one given in Lemma 1.9. For the sake of clarity, we recall them.

Proposition 4.10. *A sequence of vectors $\{u_n\}_{n\in\mathbb{N}}$ strongly converges to u in $\mathcal{K}$ if one of the following holds:*

$$\text{a)} \quad \begin{cases} \int_0^T \|u_n(t)\|^2_{H_n}\, \mathrm{d}t \xrightarrow[n\to+\infty]{} \int_0^T \|u(t)\|^2_H\, \mathrm{d}t \\ \int_0^T (u_n(t), \varphi(t))_{H_n}\, \mathrm{d}t \xrightarrow[n\to+\infty]{} \int_0^T (u(t), \varphi(t))_H\, \mathrm{d}t \end{cases} \tag{4.6}$$

for every $\varphi \in C([0,T] \times \overline{Q})$;

$$\text{b)} \quad \int_0^T (u_n(t), v_n(t))_{H_n}\, \mathrm{d}t \xrightarrow[n\to+\infty]{} \int_0^T (u(t), v(t))_H\, \mathrm{d}t \tag{4.7}$$

for every sequence $\{v_n\}_{n\in\mathbb{N}}$ strongly converging to v in $\mathcal{K}$.

Remark 4.11. *By proceeding as in Proposition 2.4, it follows that the weak convergence in $\mathcal{K}$ implies the weak convergence in $L^2([0,T] \times Q)$.*

Theorem 4.12. *Let H_n, H, $E_{p,s}^{(n)}$, $E_{p,s}$ and δ_n be as in Theorem 2.6. Let $T_{p,s}^{(n)}(t)$, $T_{p,s}(t)$, $u_0^{(n)}$ and u_0 be as in Theorems 4.6 and 4.7. If $u_0^{(n)} \to u_0$ strongly in $\mathcal{H}$ and there exists a constant C such that*

$$\|u_0^{(n)}\|_{\overline{D(A_s^{(n)})}} < C \quad \text{for every } n \in \mathbb{N}, \tag{4.8}$$

then

i) $\{u_n(t)\}$ converges to $u(t)$ strongly in $\mathcal{H}$ for every fixed $t \in [0,T]$;

ii) $\{u_n\}$ converges to u in $\mathcal{K}$.

Proof. The convergence in $\mathcal{H}$ follows from Theorem 2.6, Theorem 2.7 and Theorem 7.24 in [45].
We now prove ii). From Proposition 4.10 a), this amounts to prove

$$\|u_n\|_{K_n} \to \|u\|_K \tag{4.9}$$

and

$$(u_n, \phi)_{K_n} \to (u, \phi)_K \quad \forall\, \phi \in C([0,T] \times \overline{Q}). \tag{4.10}$$

Since $T_{p,s}^{(n)}$ is a contraction semigroup, it follows that for a.e. t

$$\|u_n\|_{H_n} = \|T_{p,s}^{(n)}(t) u_0^{(n)}\|_{H_n} \le \|u_0^{(n)}\|_{\overline{D(-\mathcal{A}_s^{(n)})}} < C, \tag{4.11}$$

where C is independent from n. Thus the sequence $\{u_n\}$ is equibounded for $t \in [0,T]$, and from i) we get

$$\|u_n(t)\|_{H_n} \to \|u(t)\|_H \,.$$

Hence, from the dominated convergence Theorem (4.9) holds.
We come to (4.10). From i), it follows in particular that for every $t \in [0,T]$

$$(u_n(t), \phi(t))_{H_n} \to (u(t), \phi(t))_H \quad \forall\, \phi \in C([0,T] \times \overline{Q}).$$

Since

$$|(u_n(t), \phi(t))_{H_n}| \le C\, \|\phi\|_{C([0,T]\times\overline{Q})} \,,$$

again from the dominated convergence Theorem we deduce

$$(u_n, \phi)_{K_n} \xrightarrow[n \to +\infty]{} (u, \phi)_K,$$

thus proving (4.10). ∎

5 Conclusions and remarks

There are some differences between the linear case $p = 2$ [14] and the nonlinear case $p > 2$ investigated in this paper.
As in [14, Sections 5 and 6], existence and uniqueness results for the strong solutions of problems (P) and (P_n) (and consequently of the strong problems $(\tilde{P})$ and $(\tilde{P}_n)$) can be proved also in the non-homogeneous case. When $p > 2$, we have neither an explicit representation formula for the solutions of the abstract Cauchy problems nor a priori estimates for the solutions. Thus, we cannot use the techniques of Theorem 8.1 in [14] to study the asymptotic behavior of the solutions for the non-homogeneous problems. Moreover, in Theorem 4.12 we studied the convergence of the pre-fractal solution u_n to the fractal solution u in the homogeneous case. Further convergence results, such as the convergence of time derivatives and/or of the p-fractional normal derivatives as in [14, Section 8], are still an open problem since the proves deeply rely on a priori estimates for the strong solutions u and u_n.
We now stress the fact that, as in the linear case [14, Section 3], the p-fractional Green formula, proved in Theorem 3.2, actually holds for more general extension domains having as boundary either a d-set or an arbitrary closed set (see Definition 1.2). Moreover, it also holds for more general fractal geometries such as fractal mixture cylinders in $\mathbb{R}^3$ [33].
We also point out that our problem is a prototype. The results of the present paper can be extended to more general nonlinear fractional operators; this is object of an ongoing research.

Acknowledgements. The authors have been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).
M. R. L. would like to acknowledge networking support by the COST Action CA18232.

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