Chapter 15

Magnetostatic Problems in Fractal Domains

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We consider a magnetostatic problem in a three-dimensional "cylindrical" domain of Koch type. We prove existence and uniqueness results for both the fractal and pre-fractal problems and we investigate the convergence of the pre-fractal solutions to the limit fractal one. We consider the numerical approximation of the pre-fractal problems via FEM and we give *a priori* error estimates. Some numerical simulations are also shown. Our long-term motivation includes studying problems that appear in quantum physics in fractal domains.

15.1. Introduction

The aim of this chapter is to study a magnetostatic problem in a fractal domain. Trying to understand the magnetic properties of fractal structures is a new challenge from both the practical and theoretical point of views. In general, mathematical physics on fractals is still a young subject, see [2–6] for some results; magnetic operators on fractal spaces have been studied only very recently, [26, 28, 29, 32], as well as heat transfer across fractal layers or boundaries [8, 12, 13, 27, 37, 38, 40, 53]. Our long-term motivation includes a possibility to study non-quantized penetration of magnetic field in the vortex state of superconductors [20] in fractal domains.

A mathematical theory of electrodynamics on domains with fractal boundary still has to be developed. Although many results are well known in the case of Lipschitz domains (see, for instance, [15, Chapter IX]), for such fractal domains even the simplest models and effects have not yet been discussed. Our considerations here should be regarded as a preliminary step in a long-term project, which aims to provide theoretical and numerical studies of related physical phenomena. We believe that, beyond their theoretical interest, such results may also be useful for the construction of concrete prototypes in industrial applications, which aim to maximize (or minimize) physical quantities such as the intensity of the magnetic vector field induced by a given current density.

In the present chapter, we consider a linear magnetostatic problem in a cylindrical three-dimensional domain $Q = \Omega \times I$, where Ω is the two-dimensional snowflake domain with Koch-type boundary F and I is the unit interval. We consider the problem of finding a divergence free magnetic vector potential for given time-independent permeability and time-independent current density, and we assume that the magnetic induction vanishes outside Q.

Using trace and extension techniques from [34], we establish a generalized Stokes formula, see Theorem 15.5. It involves generalized tangential traces that can be expressed as a limit of tangential traces along the boundaries of "polyhedral" approximations. We establish a Friedrichs inequality, Theorem 15.10, and establish existence and uniqueness of weak solutions, Theorem 15.11. For the numerical approximation, we restrict ourselves to the axial-symmetric case, which in turn brings us to solve the problem in the snowflake domain. We consider both the fractal and pre-fractal problems, which we denote with $(\bar{P})$ and $(\bar{P}_n)$, respectively. We prove existence and uniqueness of weak solutions (Propositions 15.17 and 15.19) and regularity results (Proposition 15.18). We show that, in a suitable sense, the pre-fractal solutions converge to the limit fractal one, see Theorem 15.20. We consider the numerical approximation of the pre-fractal problem $(\bar{P}_n)$ by an FEM scheme. To obtain an optimal *a priori* error estimate, we rely on the regularity of the weak solution of problem $(\bar{P}_n)$ in suitable weighted Sobolev spaces, see Theorem 15.21. Since the pre-fractal domain Ω_n is not convex, the solution is not in $H^2(\Omega_n)$, hence the rate of convergence is deteriorated. By using a suitable mesh constructed in [10], which is compliant with the so-called *Grisvard conditions* [24], we can prove optimal *a priori* error estimates. These conditions involve the weight exponent of the weak solution given in Theorem 15.21. We finally present numerical simulations, which describe the behavior of the magnetic field. It turns out that the intensity of the magnetic field increases as the length of the boundary approaches the "length" of $F \times I$. We believe that this effect may be useful for potential applications.

15.2. Fractal Domains

We write $|P - P'|$ to denote the Euclidean distance between two points P and P' in $\mathbb{R}^N$. The *Koch snowflake* $F \subset \mathbb{R}^2$ is the union $F = \bigcup_{i=1}^3 K^{(i)}$ of three co-planar Koch curves $K^{(1)}$, $K^{(2)}$ and $K^{(3)}$, cf. [17, Chapter 8], whose junction points A, B and C are the vertices of a regular triangle. We assume this triangle has unit side length, i.e., $|A - B| = |A - C| = |B - C| = 1$.

The single *Koch curve* $K^{(1)}$ is the uniquely determined self-similar set with respect to a family Ψ^1 of four contractive similarities $\psi_1^{(1)}, \ldots, \psi_4^{(1)}$, all having contraction ratio $\frac{1}{3}$, see [17, 19]. Let $V_0^{(1)} := \{A, B\}$, $\psi_{i_1 \ldots i_n} := \psi_{i_1} \circ \cdots \circ \psi_{i_n}$, $V_{i_1 \ldots i_n}^{(1)} := \psi_{i_1 \ldots i_n}^{(1)}(V_0^{(1)})$ and

$$V_n^{(1)} := \bigcup_{i_1 \ldots i_n = 1}^{4} V_{i_1 \ldots i_n}^{(1)}.$$

We write $i|n = (i_1, i_2, \ldots, i_n)$ and $V_\star^{(1)} := \bigcup_{n \geq 0} V_n^{(1)}$. The closure in $\mathbb{R}^N$ of $V_*^{(1)}$ is just $K^{(1)}$. Now, let $K_0^{(1)}$ denote the unit segment whose endpoints are A and B. We set $K_{i_1 \ldots i_n}^{(1)} = \psi_{i_1 \ldots i_n}(K_0^{(1)})$ and

$$K_n^{(1)} := \bigcup_{i_1 \ldots i_n = 1}^{4} K_{i_1 \ldots i_n}^{(1)}.$$

In a similar way, it is possible to approximate $K^{(2)}$ and $K^{(3)}$ by the sequences $(V_n^{(2)})_{n \geq 0}$ and $(V_n^{(3)})_{n \geq 0}$, we denote their unions by $V_\star^{(2)}$ and $V_\star^{(3)}$, respectively. The polygonal curves associated with $V_n^{(2)}$ and $V_n^{(3)}$ are denoted by $K_n^{(2)}$ and $K_n^{(3)}$, respectively.

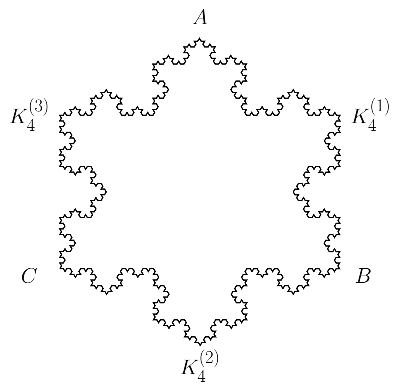

Fig. 15.1. The pre-fractal curve F_4.

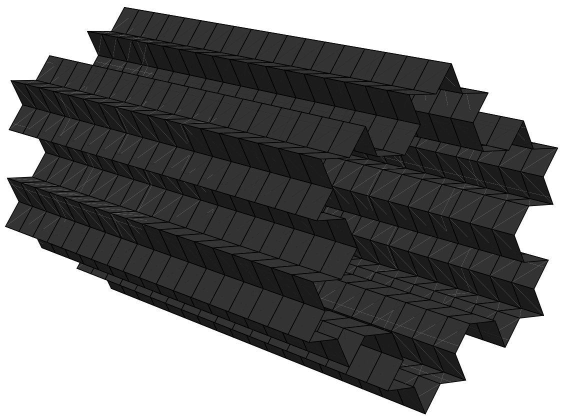

Fig. 15.2. The lateral surface S_2.

The Koch snowflake F itself is approximated by the sequence $(F_n)_{n\geq 1}$ of "pre-fractal" closed polygonal curves F_n, defined by

$$F_n = \bigcup_{i=1}^{3} K_n^{(i)}, \tag{15.1}$$

see Fig. 15.1.

Notation 15.1. By $\Omega_n \subset \mathbb{R}^2$, we denote the bounded open set with boundary F_n and by Q_n, the three-dimensional cylindrical domain having $S_n := F_n \times [0,1]$ as "lateral surface" and the sets $\Omega_n \times \{0\}$ and $\Omega_n \times \{1\}$ as bases. We similarly write Ω for the bounded open domain in $\mathbb{R}^2$ with boundary F ("snowflake domain"), define the cylindrical-type surface $S := F \times I$ and let Q denote the open cylindrical domain having S as lateral surface and the sets $\Omega \times \{0\}$ and $\Omega \times \{1\}$ as bases, see Fig. 15.2.

15.3. A 3D Magnetostatics Problem

We formulate a linear magnetostatic problem on the fractal domain Q. To deduce it and to explain its physical meaning we start by recalling *Maxwell's equations* for classical macroscopic electromagnetic fields. We assume that Q is made up from a *linear* material, i.e., in a material without any magnetization or polarization effects, and we assume it is *dielectric*, i.e., its conductivity can be neglected (see, for instance, [46, Section 1.2.1]). Then *Ampère's law*, $\mathrm{curl}(\mathcal{H}) = \mathcal{J} + \frac{\partial \mathcal{D}}{\partial t}$, tells that the total *magnetic field* $\mathcal{H}$ induced around a closed loop equals the electric current plus the rate of change of the *electric displacement field* $\mathcal{D}$ enclosed by the loop, here $\mathcal{J}$ denotes the *electric current density*, i.e., the vector field describing the directed flow of electric charges. The corresponding *magnetic induction* is $\mathcal{B} = \mu\mathcal{H}$, where μ is a positive and bounded scalar function of space and time, called the *permeability* of the material. By *Faraday's law of induction*, $\mathrm{curl}(\mathcal{E}) = -\frac{\partial \mathcal{B}}{\partial t}$, the voltage induced in a closed loop equals the change of the enclosed magnetic field. Here $\mathcal{E} = \frac{1}{\varepsilon}\mathcal{D}$, where ε is a positive and bounded scalar of space and time referred to as the *permittivity* of the material. These assumptions of μ and ε mean we model an inhomogeneous isotropic material, so practically Q may consist of a mixture of different materials whose electromagnetic properties may depend on the location in space but not on the direction of the fields. *Gauss' law*, $\mathrm{div}(\mathcal{D}) = \rho$, states that the electric flux leaving a volume equals the charge inside, here $\rho \geq 0$ is the *charge density*. According to *Gauss' law for magnetism*, $\mathrm{div}(\mathcal{B}) = 0$, i.e., the magnetic flux through a closed surface is zero.

We now make the following assumptions leading to a much simpler *magnetostatic* setup:

- the permittivities $\varepsilon = \varepsilon(x)$ and $\mu = \mu(x)$ are *time-independent*;
- the charge density is zero, $\rho = 0$;
- the current density $\mathcal{J} \equiv \mathbf{J}(x)$ is *time-independent* and *real-valued*;
- the fields $\mathcal{E} \equiv \mathbf{E}(x)$ and $\mathcal{H} \equiv \mathbf{H}(x)$ are *time-independent* and *real-valued*;
- all the fields vanish outside Q.

Under these assumptions, Maxwell's equations on Q read

$$\mathrm{curl}(\mathbf{H}) = \mathbf{J}, \quad \mathrm{curl}(\mathbf{E}) = \mathbf{0}, \quad \mathrm{div}(\mathbf{D}) = \mathbf{0}, \quad \mathrm{div}(\mathbf{B}) = \mathbf{0}, \tag{15.2}$$

where $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$.

Our assumption that $\mathbf{E}$ vanishes in Q^c means that the surrounding region Q^c is a *perfect conductor*. When passing from one to another medium, the parallel component of the electric field should be continuous; this can be seen by taking a small rectangular loop with long sides parallel to ∂Q, one inside Q, one outside and applying Faraday's law. Since the field vanishes outside Q, this forces to impose what is referred to as the *perfectly conducting boundary condition* $\mathbf{n} \times \mathbf{E} = 0$ on ∂Q.

Since $\mathbf{B}$ is divergence free, there exists a *magnetic vector potential* $\mathbf{u} = (u_1, u_2, u_3)$ such that $\mathbf{B} = \mathrm{curl}(\mathbf{u})$, and we may choose it to be divergence free, $\mathrm{div}\,\mathbf{u} = 0$. Note that Gauss' law for magnetism then becomes trivial.

Also $\mathbf{B}$ is supposed to be zero on Q^c. Therefore, looking at the flux of the magnetic field through small closed loops on ∂Q, which should not differ for the interior and the exterior field, and applying the Kelvin–Stokes theorem, it follows that we should impose $\mathbf{n} \times \mathbf{u} = 0$ on ∂Q. See, for instance, [23, Section 5.4.2] or [58, p. 82].

We now restrict attention to the magnetic field only and pose the following problem: Given μ and $\mathbf{J}$ as above, find a magnetic vector potential $\mathbf{u}$ that satisfies

$$(P) \begin{cases} \mathrm{curl}\left(\dfrac{1}{\mu}\,\mathrm{curl}(\mathbf{u})\right) = \mathbf{J} & \text{in } Q, \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } Q, \\ \mathbf{n} \times \mathbf{u} = 0 & \text{on } \partial Q. \end{cases} \tag{15.3}$$

Note that if μ is constant then the first equation rewrites

$$-\Delta_{\mathrm{vec}}\mathbf{u} = \mu\mathbf{J}, \tag{15.4}$$

where Δ_{vec} denotes the vector Laplacian.

15.4. Trace Theorems, Stokes Formula and Gauss–Green Identity

We discuss measures, function spaces and trace theorems. The latter allow rigorous definitions of boundary conditions and generalizations of classical integral formulas. We write $B(P, r) = \{P' \in \mathbb{R}^N : |P' - P| < r\}$, $P \in \mathbb{R}^N, r > 0$, for the Euclidean ball of radius r centered at P. For the two-dimensional Lebesgue measure, we write $\mathrm{d}x_1\mathrm{d}x_2$ and for the three-dimensional one, we write $\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$.

On the snowflake curve $F = \bigcup_{i=1}^{3} K^{(i)}$, we consider the finite Borel measure μ defined by

$$\mu_F := \mu_1 + \mu_2 + \mu_3,$$

where μ_i denotes the normalized Hausdorff measure of dimension $D_f = \frac{\ln 4}{\ln 3}$, restricted to K_i, $i = 1, 2, 3$. It is well known that $c_1 r^{D_f} \leq \mu_F(B(P, r)) \leq c_2 r^{D_f}$, $P \in F$, $r > 0$, with positive constants c_1 and c_2. If we endow the cylindrical type surface $S = F \times I$ with the measure

$$\mathrm{d}\mu_S := \mathrm{d}\mu_F \times \mathrm{d}x_3,$$

where $\mathrm{d}x_3$ is one-dimensional Lebesgue measure on I, then clearly

$$c_1 r^{D_f+1} \leq \mu_S(B(P, r)) \leq c_2 r^{D_f+1} \tag{15.5}$$

for all $P \in S$ and $r > 0$.

We equip the boundary ∂Q with the measure

$$\mathrm{d}\mu_{\partial Q} = \chi_S \mathrm{d}\mu_S + \chi_{\tilde{\Omega}} \mathrm{d}x_1 \mathrm{d}x_2, \tag{15.6}$$

where $\tilde{\Omega} = (\Omega \times \{0\}) \cup (\Omega \times \{1\})$ is the union of the two bases of the cylinder domain Q in Notation 15.1. In particular, $\operatorname{supp} \mu_{\partial Q} = \partial Q$.

From (15.5) and the quadratic scaling of the two-dimensional Lebesgue measure it follows that

$$\begin{aligned} &\mu_{\partial Q}(B(P, kr)) \leq c_1\, k^{D_f+1} \mu_{\partial Q}(B(P, r)) \quad \text{and} \\ &\mu_{\partial Q}(B(P, kr)) \geq c_2\, k^2 \mu_{\partial Q}(B(P, r)) \end{aligned} \tag{15.7}$$

for all $P \in \partial Q$, $r > 0$, $k \geq 1$ such that $kr \leq 1$.

We write $L^2(Q)$ and $L^2(Q_n)$ for the L^2-spaces with respect to the three-dimensional Lebesgue measure, the spaces $L^2(\Omega)$, $L^2(\Omega_n)$, $L^2(\partial Q_n)$ are taken with respect to the two-dimensional Lebesgue (or Hausdorff) measure (depending on whether considered in $\mathbb{R}^2$ or $\mathbb{R}^3$). For ∂Q, we write $L^2(\partial Q) = L^2(\partial Q, \mu_{\partial Q})$, the L^2-space with respect to $\mu_{\partial Q}$.

The spaces $H^\alpha(\mathbb{R}^N) = H^{\alpha,2}(\mathbb{R}^N)$ denote the usual Bessel potential spaces (see, for instance, [1]), where they are denoted by $L^{\alpha,2}(\mathbb{R}^N)$. Given a domain $O \subset \mathbb{R}^N$, the notation $H^1(O)$ denotes the classical Sobolev space of square integrable functions with finite Dirichlet integral, usually denoted by $W^{1,2}(O)$.

Since the boundary $\partial Q = \tilde{\Omega} \cup S$ is a closed set composed by sets of different Hausdorff dimension, in order to consider the trace space of $H^\alpha(Q)$ on ∂Q, we introduce suitable spaces $\tilde{B}^{2,2}_\alpha(\partial Q)$ as in [34, p. 356]. For any

$$\frac{1}{2} < \alpha < 2 - \frac{D_f}{2}, \tag{15.8}$$

let $\tilde{B}^{2,2}_\alpha(\partial Q)$ denote the class of functions u on ∂Q such that

$$\begin{aligned}\|u\|^2_{\tilde{B}^{2,2}_\alpha(\partial Q)} = \|u\|^2_{L^2(\partial Q)} + \iint_{|x-y|<1} & \frac{|u(x)-u(y)|^2}{|x-y|^{2\alpha-3}(\mu_{\partial Q}(B(x,|x-y|)))^2} \\ & \times \mathrm{d}\mu_{\partial Q}(x)\,\mathrm{d}\mu_{\partial Q}(y)\end{aligned} \tag{15.9}$$

is finite.

We remark that $\mu_{\partial Q}$ defined in (15.6) is not an Ahlfors regular d-measure on ∂Q. That is, the $\mu_{\partial\Omega}$-measure of a ball of radius $r > 0$ cannot be estimated from above and below, respectively, by a constant times r^d. Therefore, the space $\tilde{B}^{2,2}_\alpha(\partial Q)$ does not coincide with the usual Besov space $B^{2,2}_\alpha(\partial Q)$ defined in [35, p. 103] or [57].

We denote by $|A|$ the Lebesgue measure of a subset $A \subset \mathbb{R}^N$. For $f \in H^\alpha(O)$, $O \subset \mathbb{R}^N$ open, we put

$$\gamma_0 f(P) = \lim_{r\to 0} \frac{1}{|B(P,r)\cap O|} \int_{B(P,r)\cap O} f(x)\,\mathrm{d}x \tag{15.10}$$

at every point $P \in \overline{O}$ where the limit exists. This is a typical form of *restriction operator* in the spirit of Lebesgue differentiation.

The following trace theorem is a special case of [34, Theorem 1], see also [34, Proposition 2].

Proposition 15.2. *Let α be as in* (15.8). *$\tilde{B}^{2,2}_\alpha(\partial Q)$ is the trace space of $H^\alpha(\mathbb{R}^3)$, i.e.,*

(i) *$f \mapsto \gamma_0 f$ is a linear and continuous operator from $H^\alpha(\mathbb{R}^3)$ to $\tilde{B}^{2,2}_\alpha(\partial Q)$;*
(ii) *there exists a linear and continuous operator* $\mathrm{Ext}\colon \tilde{B}^{2,2}_\alpha(\partial Q) \to H^\alpha(\mathbb{R}^3)$ *such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator on $\tilde{B}^{2,2}_\alpha(\partial Q)$.*

Combined with trace and extension results between the spaces $H^1(\mathbb{R}^3)$ and $H^1(Q)$, such as, for instance, [35, Chapter VII, Theorem 1, combined with Chapter VIII, Proposition 1], we obtain the following Corollary.

Corollary 15.3. *The space $\tilde{B}^{2,2}_1(\partial Q)$ is the trace space of $H^1(Q)$ on ∂Q, i.e., there exist a continuous linear restriction operator from $H^1(Q)$ to $\tilde{B}^{2,2}_1(\partial Q)$ and a continuous linear extension from $\tilde{B}^{2,2}_1(\partial Q)$ to $H^1(Q)$.*

For the restriction to ∂Q of a function $f \in H^1(Q)$, we write $f|_{\partial Q}$.

More classical trace and extension results cover the case of Lipschitz boundaries, such as the sets $\partial Q_n := S_n \cup \tilde{\Omega}_n$, where $\tilde{\Omega}_n := (\Omega_n \times \{0\}) \cup (\Omega_n \times \{1\})$. For the following result, see [25, 48].

Proposition 15.4. *The space $H^{\frac{1}{2}}(\partial Q_n)$ is the trace space of $H^1(Q_n)$ on ∂Q_n in the following sense*:

(i) γ_0 *is a continuous and linear operator from* $H^1(Q_n)$ *to* $H^{\frac{1}{2}}(\partial Q_n)$;
(ii) *there exists a continuous linear operator* Ext *from* $H^{\frac{1}{2}}(\partial Q_n)$ *to* $H^1(Q_n)$ *such that* $\gamma_0 \circ \mathrm{Ext}$ *is the identity operator in* $H^{\frac{1}{2}}(\partial Q_n)$.

As usual, we write $H^{-\frac{1}{2}}(\partial Q_n)$ to denote the dual space of $H^{\frac{1}{2}}(\partial Q_n)$, see [21, p. 8].

We pass to vector-valued functions. Consider the space

$$H(\mathrm{curl}, Q) := \big\{\mathbf{u} = (u_1, u_2, u_3) \colon Q \to \mathbb{R}^3 : u_1, u_2, u_3 \in L^2(Q) \ \text{ and } \\ \mathrm{curl}\,\mathbf{u} \in L^2(Q)^3\big\}.$$

Endowed with the norm $\|\mathbf{u}\|_{\mathrm{curl},Q} = (\|\mathbf{u}\|^2_{L^2(Q)^3} + \|\mathrm{curl}\,\mathbf{u}\|^2_{L^2(Q)^3})^{1/2}$, it becomes a Hilbert space; see, for instance, [16, 21] or [56].

We now prove a generalized vector Stokes formula. Suppose $\mathbf{u} \in H(\mathrm{curl}, Q)$. For any $\mathbf{v} \in \tilde{B}^{2,2}_1(\partial Q)^3$ let $\mathbf{w} \in H^1(Q)^3$ be such that $\mathbf{w}|_{\partial Q} = \mathbf{v}$, defined component-wise in the sense of Corollary 15.3, and consider the quantity

$$\gamma_\tau \mathbf{u}(\mathbf{v}) := \int_Q \mathbf{u} \cdot \mathrm{curl}\ \mathbf{w}\, \mathrm{d}x - \int_Q \mathbf{w} \cdot \mathrm{curl}\ \mathbf{u}\, \mathrm{d}x.$$

Theorem 15.5. *Let Q be the Koch-type pipe.*

(i) *The map $\mathbf{u} \mapsto \gamma_\tau \mathbf{u}$ is well defined as a bounded linear operator from $H(\mathrm{curl}, Q)$ into $((\tilde{B}^{2,2}_1(\partial Q))')^3$. By setting $\mathbf{u} \times \mathbf{n}|_{\partial Q} := \gamma_\tau \mathbf{u}$, we have*

$$|\left\langle \mathbf{u} \times \mathbf{n}|_{\partial Q}, \mathbf{v}\right\rangle_{((\tilde{B}^{2,2}_1(\partial Q))')^3, \tilde{B}^{2,2}_1(\partial Q)^3}| \le c\, \|\mathbf{u}\|_{\mathrm{curl},Q}\, \|\mathbf{v}\|_{\tilde{B}^{2,2}_1(\partial Q)^3} \tag{15.11}$$

for all $\mathbf{u} \in H(\mathrm{curl}, Q)$ and $\mathbf{v} \in \tilde{B}^{2,2}_1(\partial Q)^3$.

(ii) *Moreover, we have*

$$\begin{aligned}&\langle \mathbf{u}\times\mathbf{n}|_{\partial Q}, \mathbf{w}|_{\partial Q}\rangle_{((\tilde{B}_1^{2,2}(\partial Q))')^3, \tilde{B}_1^{2,2}(\partial Q)^3}\\ &\quad = \lim_{n\to\infty} \langle \mathbf{u}\times\mathbf{n}|_{\partial Q_n}, \mathbf{w}|_{\partial Q_n}\rangle_{H^{-\frac{1}{2}}(\partial Q_n)^3, H^{\frac{1}{2}}(\partial Q_n)^3}\end{aligned} \tag{15.12}$$

and

$$\begin{aligned}&\langle \mathbf{u}\times\mathbf{n}|_{\partial Q}, \mathbf{w}|_{\partial Q}\rangle_{((\tilde{B}_1^{2,2}(\partial Q))')^3, \tilde{B}_1^{2,2}(\partial Q)^3}\\ &\quad = \int_Q \mathbf{u}\cdot \operatorname{curl}\mathbf{w}\,\mathrm{d}x - \int_Q \mathbf{w}\cdot\operatorname{curl}\mathbf{u}\,\mathrm{d}x\end{aligned} \tag{15.13}$$

for all $\mathbf{u}\in H(\operatorname{curl}, Q)$ *and* $\mathbf{w}\in H^1(Q)^3$.

Formula (15.12) provides a suitable approximation of $\mathbf{u}\times\mathbf{n}|_{\partial Q}$ in terms of the tangential traces $\mathbf{u}\times\mathbf{n}|_{\partial Q_n}$ along the Lipschitz boundaries ∂Q_n, see [21, §2, Theorem 2.11] or [56]. In this sense, $\mathbf{u}\times\mathbf{n}|_{\partial Q}$ can be seen as a *generalized tangential trace* and (15.13) is a *generalized Stokes formula*.

Proof. Let $\mathbf{u}\in H(\operatorname{curl}, Q)$. Given $\mathbf{v}\in \tilde{B}_1^{2,2}(\partial Q)^3$, let $\mathbf{w}\in H^1(Q)^3$ be such that $\mathbf{w}|_{\partial Q} = \mathbf{v}$ in $\tilde{B}_1^{2,2}(\partial Q)^3$. Then Cauchy–Schwarz together with the inclusion $H^1(Q)^3\subset H(\operatorname{curl}, Q)$ and Corollary 15.3 lead to the estimate

$$\begin{aligned}|\langle \mathbf{u}\times\mathbf{n}|_{\partial Q}, \mathbf{w}|_{\partial Q}\rangle| &\le \|\mathbf{u}\|_{L^2(Q)^3}\|\operatorname{curl}\mathbf{w}\|_{L^2(Q)^3} + \|\mathbf{w}\|_{L^2(Q)^3}\|\operatorname{curl}\mathbf{u}\|_{L^2(Q)^3}\\ &\le c\,\|\mathbf{w}\|_{H^1(Q)^3}\|\mathbf{u}\|_{\operatorname{curl},Q}\\ &\le c\,\|\mathbf{v}\|_{\tilde{B}_1^{2,2}(\partial Q)^3}\|\mathbf{u}\|_{\operatorname{curl},Q}.\end{aligned}$$

This shows, in particular, that $\gamma_\tau\mathbf{u}(\mathbf{v})$ is independent from the choice of the extension $\mathbf{w}$ of $\mathbf{v}$, and that $\mathbf{u}\times\mathbf{n}$ is an element of $((\tilde{B}_1^{2,2}(\partial Q))')^3$ which satisfies (15.11).

We now consider the sequence of domains $Q_n = \Omega_n\times I$, which are bounded Lipschitz domains and satisfy $Q_n\subset Q_{n+1}$ and $Q=\bigcup_{n=1}^\infty Q_n$. By the vector Stokes formula for Lipschitz domains, cf. [21, §2, Theorem 2.11] or Appendix I in [56], together with the dominated convergence theorem, we have

$$\begin{aligned}&\lim_{n\to\infty} \langle \mathbf{u}\times\mathbf{n}|_{\partial Q_n}, \mathbf{w}|_{\partial Q_n}\rangle_{H^{-\frac{1}{2}}(\partial Q_n)^3, H^{\frac{1}{2}}(\partial Q_n)^3}\\ &\quad = \lim_{n\to\infty}\int_{Q_n} \mathbf{u}\cdot\operatorname{curl}\mathbf{w}\,\mathrm{d}x - \int_{Q_n}\mathbf{w}\cdot\operatorname{curl}\mathbf{u}\,\mathrm{d}x\\ &\quad = \int_Q \mathbf{u}\cdot\operatorname{curl}\mathbf{w}\,\mathrm{d}x - \int_Q\mathbf{w}\cdot\operatorname{curl}\mathbf{u}\,\mathrm{d}x\\ &\quad = \langle \mathbf{u}\times\mathbf{n}, \mathbf{w}|_{\partial Q}\rangle_{((\tilde{B}_1^{2,2}(\partial Q))')^3, \tilde{B}_1^{2,2}(\partial Q)^3}\end{aligned}$$

for all $\mathbf{w} \in H^1(Q)^3$ and n, where $\mathbf{u} \times \mathbf{n}|_{\partial Q_n}$ is defined as an element of $H^{-\frac{1}{2}}(\partial Q_n)^3$. □

Next, consider the space

$$H(\mathrm{div}, Q) := \big\{\mathbf{u} = (u_1, u_2, u_3)\colon Q \to \mathbb{R}^3 : u_1, u_2, u_3 \in L^2(Q) \text{ and } \\ \mathrm{div}\,\mathbf{u} \in L^2(Q)\big\},$$

which is Hilbert when equipped with the norm $\|\mathbf{u}\|_{\mathrm{div},Q} = (\|\mathbf{u}\|^2_{L^2(Q)^3} + \|\mathrm{div}\,\mathbf{u}\|^2_{L^2(Q)})^{1/2}$. Following the same pattern as above, one can establish a generalized Gauss–Green formula. This can be done as in [39].

Suppose $\mathbf{u} \in H(\mathrm{div}, Q)$. For any $v \in \tilde{B}^{2,2}_1(\partial Q)$ let $w \in H^1(Q)$ be such that $w|_{\partial Q} = v$ in the sense of Corollary 15.3 and consider

$$\gamma_\nu \mathbf{u}(v) := \int_Q \mathbf{u} \cdot \nabla w \,\mathrm{d}x + \int_Q (\mathrm{div}\,\mathbf{u}) w \,\mathrm{d}x.$$

By proceeding as in [39, Theorem 3.7], we can prove the following Green formula.

Theorem 15.6. *Let Q be the Koch-type pipe.*

(i) *The map $\mathbf{u} \mapsto \gamma_\nu \mathbf{u}$ is well defined as a bounded linear operator from $H(\mathrm{div}, Q)$ into $((\tilde{B}^{2,2}_1(\partial Q))')$. By setting $\mathbf{u}\cdot\mathbf{n}|_{\partial Q} := \gamma_\nu \mathbf{u}$, we have*

$$|\,\langle \mathbf{u}\cdot\mathbf{n}|_{\partial Q}, v\rangle_{((\tilde{B}^{2,2}_1(\partial Q))'),\tilde{B}^{2,2}_1(\partial Q)}\,| \le c\,\|\mathbf{u}\|_{\mathrm{div},Q}\,\|v\|_{\tilde{B}^{2,2}_1(\partial Q)}$$

for all $\mathbf{u} \in H(\mathrm{div}, Q)$ and $v \in \tilde{B}^{2,2}_1(\partial Q)$.

(ii) *Moreover, we have*

$$\begin{aligned}&\langle \mathbf{u}\cdot\mathbf{n}|_{\partial Q}, w|_{\partial Q}\rangle_{((\tilde{B}^{2,2}_1(\partial Q))'),\tilde{B}^{2,2}_1(\partial Q)} \\ &\quad = \lim_{n\to\infty} \langle \mathbf{u}\cdot\mathbf{n}|_{\partial Q_n}, w|_{\partial Q_n}\rangle_{H^{-\frac{1}{2}}(\partial Q_n), H^{\frac{1}{2}}(\partial Q_n)}\end{aligned} \tag{15.14}$$

and

$$\langle \mathbf{u}\cdot\mathbf{n}|_{\partial Q}, w|_{\partial Q}\rangle_{(\tilde{B}^{2,2}_1(\partial Q))',\tilde{B}^{2,2}_1(\partial Q)} = \int_Q \mathbf{u}\cdot\nabla w\,\mathrm{d}x - \int_Q (\mathrm{div}\,\mathbf{u}) w\,\mathrm{d}x \tag{15.15}$$

for all $\mathbf{u} \in H(\mathrm{div}, Q)$ and $w \in H^1(Q)$.

Similarly as before, formula (15.14) provides a suitable approximation of $\mathbf{u}\cdot\mathbf{n}|_{\partial Q}$ by normal traces $\mathbf{u}\cdot\mathbf{n}|_{\partial Q_n}$ on the Lipschitz boundaries ∂Q_n, which follows again from corresponding results in the Lipschitz case [21, §2, Theorem 2.5].

Remark 15.7. We point out that the results of this section hold not only for the Koch-type pipe. Indeed, these results can be extended to every domain Q having as boundary ∂Q a d-set or an arbitrary closed set of $\mathbb{R}^3$, under the assumption that Q can be approximated by an invading sequence of Lipschitz domains $\{Q_n\}$, as in this case.

15.5. Friedrichs Inequality and Weak Solutions

We discuss (15.3) in terms of weak solutions and the Lax–Milgram theorem, and to do so we introduce the symmetric bilinear form

$$a(\mathbf{u}, \mathbf{w}) = \int_Q \operatorname{curl}(\mathbf{w}) \cdot \left(\frac{1}{\mu} \operatorname{curl}(\mathbf{u})\right) \mathrm{d}x, \quad \mathbf{u}, \mathbf{w} \in H(\operatorname{curl}, Q),$$

where, in agreement with the above assumptions, μ is a real-valued measurable function on Q satisfying $\mu_0 \leq \mu \leq \mu_1$ a.e. in Q with two constants $\mu_0, \mu_1 > 0$. Given $\mathbf{J} \in L^2(Q)^3$, we consider the linear and continuous functional on $H(\operatorname{curl}, Q)$, defined by

$$f(\mathbf{w}) = \int_Q \mathbf{J} \cdot \mathbf{w} \, \mathrm{d}x, \quad \mathbf{w} \in H(\operatorname{curl}, Q).$$

The interpretation as an identity in $((\tilde{B}_1^{2,2}(\partial Q))')^3$ gives a rigorous meaning to the boundary condition $\mathbf{u} \times \mathbf{n} = 0$ in (15.3). To encode it in a suitable function space, we consider the space $H_0(\operatorname{curl}, Q)$, defined as the closure in $H(\operatorname{curl}, Q)$ of all compactly supported smooth vector fields $C_c^\infty(Q)^3$.

Remark 15.8. Taking into account the boundary condition in (15.3), the natural space would be $\operatorname{Ker} \gamma_\tau := \{\mathbf{w} \in H(\operatorname{curl}, Q) \,:\, \mathbf{n} \times \mathbf{w} = 0 \text{ on } \partial Q\}$. The inclusion $H_0(\operatorname{curl}, Q) \subset \operatorname{Ker} \gamma_\tau$ follows from (15.13). The reverse inclusion is not straightforward, and to keep the present note simple we leave its investigation to a later forthcoming paper.

If we agree to say that a weak solution in $H_0(\operatorname{curl}, Q)$ of the equation

$$\operatorname{curl}\left(\frac{1}{\mu} \operatorname{curl}(\mathbf{u})\right) = \mathbf{J} \tag{15.16}$$

is a vector field $\mathbf{u} \in H_0(\operatorname{curl}, Q)$ such that $a(\mathbf{u}, \mathbf{v}) = f(\mathbf{v})$ for all $\mathbf{v} \in H_0(\operatorname{curl}, Q)$, then test vector fields $\mathbf{v}$ can in particular be recruited from

$$\operatorname{Ker}(\operatorname{curl}, Q) := \{\mathbf{w} \in H_0(\operatorname{curl}, Q) : \operatorname{curl} \mathbf{w} = 0\},$$

so that a weak solution of (P) can only exist if $\mathbf{J}$ satisfies the *compatibility condition*

$$f(\mathbf{v}) = \int_Q \mathbf{J} \cdot \mathbf{v} \, \mathrm{d}x = 0 \quad \forall \mathbf{v} \in \mathrm{Ker}(\mathrm{curl}, Q). \tag{15.17}$$

Moreover, since we are also interested in the uniqueness of weak solutions, we restrict ourselves to the quotient space $H_0(\mathrm{curl}, Q)/\,\mathrm{Ker}(\mathrm{curl}, Q)$, which by a simple quadratic variational problem, [21, Corollary 1.2], involving the quotient space norm, see [30, p. 94–95] or [44, Lemma 3.5], is seen to be isometrically isomorphic to the space

$$H_{0,\perp}(\mathrm{curl}, Q) := \Big\{ \mathbf{u} \in H_0(\mathrm{curl}, Q) \,:\, \int_Q \mathbf{u} \cdot \mathbf{w} \, \mathrm{d}x = 0 \\ \text{for all } \mathbf{w} \in \mathrm{Ker}(\mathrm{curl}, Q) \Big\}. \tag{15.18}$$

A second requirement to be incorporated in the function spaces is that a solution $\mathbf{u}$ of (P) should be divergence free. We consider the space $H_0(\mathrm{div}, Q)$, defined as the completion in $H(\mathrm{div}, Q)$ of $C_c^\infty(Q)^3$, and its subspace

$$\mathrm{Ker}(\mathrm{div}, Q) := \{\mathbf{u} \in H_0(\mathrm{div}, Q) : \mathrm{div}\, \mathbf{u} = 0\}.$$

This discussion suggests that one possible way to phrase (P) rigorously could be to look for a weak solution to equation (15.16) in the space $H_{0,\perp}(\mathrm{curl}, Q) \cap \mathrm{Ker}(\mathrm{div}, Q)$. The latter space admits a much simpler description. A proof of the following fact can be found at the end of this section.

Proposition 15.9. *A vector field* $\mathbf{u} \in H_0(\mathrm{curl}, Q) \cap H_0(\mathrm{div}, Q)$ *is an element of* $H_{0,\perp}(\mathrm{curl}, Q)$ *if and only if* $\mathrm{div}\, \mathbf{u} = 0$.

As a next step of simplification, the intersection of the spaces $H_0(\mathrm{curl}, Q)$ and $H_0(\mathrm{div}, Q)$ can be determined in a standard way, see [7, Theorem 2.5] or [21, Lemma 2.5]. As a by-product, we obtain the following *Friedrichs inequality* [55], sometimes also referred to as a *Maxwell inequality* [49], which provides a suitable coercivity bound for our problem. As usual, $H_0^1(Q)$ denotes the closure of $C_c^\infty(Q)$ in $H^1(Q)$.

Theorem 15.10. *We have* $H_0(\mathrm{curl}, Q) \cap H_0(\mathrm{div}, Q) = H_0^1(Q)^3$, *and there exists a constant* $C > 0$ *such that, for any* $\mathbf{u} \in H_0^1(Q)^3$, *we have*

$$\|\mathbf{u}\|_{H^1(Q)} \le C\,(\|\mathrm{curl}\ \mathbf{u}\|_{L^2(Q)^3} + \|\mathrm{div}\ \mathbf{u}\|_{L^2(Q)}). \tag{15.19}$$

In particular, we have $\|\mathbf{u}\|_{\mathrm{curl},Q} \le C\,\|\mathrm{curl}\ \mathbf{u}\|_{L^2(Q)^3}$ *for all* $\mathbf{u} \in H_0^1(Q)^3 \cap \mathrm{Ker}(\mathrm{div}, Q)$.

Proof. We follow the cited references to prove $H_0(\mathrm{curl}, Q) \cap H_0(\mathrm{div}, Q) \subset H_0^1(Q)^3$, the other inclusion is trivial. Given $\mathbf{u} \in H_0(\mathrm{curl}, Q) \cap H_0(\mathrm{div}, Q)$ consider the trivial extension of $\mathbf{u}$ to $\mathbb{R}^3$,

$$\tilde{\mathbf{u}} = \begin{cases} \mathbf{u} & \text{in } Q, \\ 0 & \text{in } \mathbb{R}^3 \setminus \overline{Q}. \end{cases}$$

Since $\mathbf{u} \in H_0(\mathrm{curl}, Q) \cap H_0(\mathrm{div}, Q)$, it evidently follows that $\mathrm{curl}\,\tilde{\mathbf{u}} \in L^2(\mathbb{R}^3)^3$ and $\mathrm{div}\,\tilde{\mathbf{u}} \in L^2(\mathbb{R}^3)$. By definition $\tilde{\mathbf{u}}$ has compact support (in the distributional sense), so that by Schwartz' Paley–Wiener theorem (see [31, Theorem 7.3.1]) the Fourier transform $\hat{\mathbf{u}}$ of $\tilde{\mathbf{u}}$ is analytic. The above properties can be rewritten algebraically as

$$\begin{aligned} &(\xi_2\hat{u}_3 - \xi_3\hat{u}_2, \xi_3\hat{u}_1 - \xi_1\hat{u}_3, \xi_1\hat{u}_2 - \xi_2\hat{u}_1) \in L^2(\mathbb{R}^3)^3 \quad \text{and} \\ &\xi_1\hat{u}_1 + \xi_2\hat{u}_2 + \xi_3\hat{u}_3 \in L^2(\mathbb{R}^3). \end{aligned}$$

It then follows that, for $i, j = 1, 2, 3$,

$$\|\xi_i\hat{u}_j\|_{L^2(\mathbb{R}^3)} \le \|\mathrm{curl}\,\tilde{\mathbf{u}}\|_{L^2(\mathbb{R}^3)^3} + \|\mathrm{div}\,\tilde{\mathbf{u}}\|_{L^2(\mathbb{R}^3)}. \tag{15.20}$$

Note that for instance $(\xi_1\hat{u}_2 - \xi_2\hat{u})^2 \ge (\xi_1\hat{u}_2)^2 - [(\xi_1\hat{u}_1)^2 + (\xi_2\hat{u}_2)^2] + (\xi_2\hat{u}_1)^2$, and by rearranging and summing up we obtain (15.20). It follows that

$$\|\nabla\mathbf{u}\|_{L^2(Q)^3} \le \|\mathrm{curl}\,\mathbf{u}\|_{L^2(Q)^3} + \|\mathrm{div}\,\mathbf{u}\|_{L^2(Q)}.$$

Hence $\mathbf{u} \in H_0^1(Q)^3$, and using Poincaré' inequality for Q we obtain (15.19). □

We say that $\mathbf{u}$ is a *weak solution of* (P) if $\mathbf{u} \in H_0^1(Q)^3 \cap \mathrm{Ker}(\mathrm{div}, Q)$ and $a(\mathbf{u}, \mathbf{v}) = f(\mathbf{v})$ for all $\mathbf{v} \in H_0^1(Q)^3 \cap \mathrm{Ker}(\mathrm{div}, Q)$.

Existence and uniqueness of a solution are now easily seen from the Lax–Milgram theorem (see [52]) together with Theorem 15.10.

Theorem 15.11. *For any* $\mathbf{J} \in L^2(Q)^3$ *satisfying* (15.17) *there exists a unique weak solution* $\mathbf{u}$ *of problem* (P). *Moreover, there exists a positive constant* $C = C(Q, \mu_0, \mu_1)$ *such that*

$$\|\mathbf{u}\|_{\mathrm{curl},Q} \le C\|\mathbf{J}\|_{L^2(Q)^3}.$$

The rest of this section is devoted to the proof of Proposition 15.9. The first observation follows from (15.15) by the same arguments as used to show [21, Theorem 2.6], we recall them for convenience. Let $\operatorname{Ker}\gamma_\nu := \{\mathbf{w} \in H(\operatorname{div}, Q) : \mathbf{n} \cdot \mathbf{w} = 0 \text{ on } \partial Q\}$.

Theorem 15.12. *We have* $H_0(\operatorname{div}, Q) = \operatorname{Ker}\gamma_\nu$.

Proof. It suffices to show that $C_c^\infty(Q)^3$ is dense in $\operatorname{Ker}\gamma_\nu$. Let $l \in (\operatorname{Ker}\gamma_\nu)'$ and let $\mathbf{v} \in \operatorname{Ker}\gamma_\nu$ be such that

$$\langle l, \mathbf{u}\rangle_{(\operatorname{Ker}\gamma_\nu)', \operatorname{Ker}\gamma_\nu} = \int_Q \mathbf{v} \cdot \mathbf{u} \, \mathrm{d}x + \int_Q \widetilde{v} \operatorname{div} \mathbf{u} \, \mathrm{d}x, \quad \mathbf{u} \in \operatorname{Ker}\gamma_\nu,$$

where $\widetilde{v} = \operatorname{div} \mathbf{v}$. Suppose now that $l \equiv 0$ on $C_c^\infty(Q)^3$. Then $\mathbf{v} = \nabla \widetilde{v}$ in distributional sense on Q, and since $\mathbf{v} \in L^2(Q)^3$, it follows that $\widetilde{v} \in H^1(Q)$. By (15.15), therefore, we have

$$\langle l, \mathbf{u}\rangle_{(\operatorname{Ker}\gamma_\nu)', \operatorname{Ker}\gamma_\nu} = \langle \mathbf{u} \cdot \mathbf{n}|_{\partial Q}, \widetilde{v}|_{\partial Q}\rangle_{(\tilde{B}_1^{2,2}(\partial Q))', \tilde{B}_1^{2,2}(\partial Q)} = 0, \quad \mathbf{u} \in \operatorname{Ker}\gamma_\nu.$$

This implies the desired density, see [21, p. 26, property (2.14)]. □

The second item is an adaption of [21, Theorem 2.7] about the complement of $\operatorname{Ker}(\operatorname{div}, Q)$, seen as a closed subspace of $L^2(Q)^3$. Again, we briefly recall the classical proof.

Theorem 15.13. *The space* $L^2(Q)^3$ *admits the orthogonal decomposition*

$$L^2(Q)^3 = \operatorname{Ker}(\operatorname{div}, Q) \oplus \{\nabla q : q \in H^1(Q)\}.$$

Proof. The space $X := \{\nabla q : q \in H^1(Q)\}$ is a closed subspace of $L^2(Q)^3$, so it suffices to show that $X^\perp = H := \operatorname{Ker}(\operatorname{div}, Q)$. If $\mathbf{u} \in H$, then by (15.15) and Theorem 15.12 we have

$$\int_Q \mathbf{u} \cdot \nabla q \, \mathrm{d}x = 0, \quad q \in H^1(Q), \tag{15.21}$$

so that $H \subset X^\perp$. If $\mathbf{u} \in L^2(Q)^3$ satisfies (15.21), then taking $q \in C_c^\infty(Q)^3$ implies $\operatorname{div} \mathbf{u} = 0$ and in particular, $\mathbf{u} \in H(\operatorname{div}, Q)$, so that (15.15) may be applied and yields $\mathbf{u} \cdot \mathbf{n} = 0$, i.e., $\mathbf{u} \in H_0(\operatorname{div}, Q)$ and therefore $\mathbf{u} \in H$. This shows $X^\perp = H$. □

Adaptions of [21, Theorem 2.9 and Corollary 2.9] provide a suitable version of the classical fact that a curl free differentiable vector field in a simply connected domain is a gradient field. We interpret curl as an operator on $L^2(Q)^3$ in the sense of distributions on Q.

Theorem 15.14. *A vector* $\mathbf{u} \in L^2(Q)^3$ *satisfies* $\operatorname{curl} \mathbf{u} = 0$ *if and only if there exists a function* $q \in H^1(Q)/\mathbb{R}$ *such that* $\mathbf{u} = \nabla q$.

Proof. If $\mathbf{u} = \nabla q$ with some $q \in H^1(Q)$ then clearly $\operatorname{curl} \mathbf{u} = 0$.

Suppose $\mathbf{u} \in L^2(Q)^3$ is such that $\operatorname{curl} \mathbf{u} = 0$. Let $\widetilde{\mathbf{u}}$ be the extension of $\mathbf{u}$ to $\mathbb{R}^3$ by zero on Q^c and let $(\varrho_\varepsilon)_{\varepsilon>0} \subset C_c^\infty(\mathbb{R}^3)$ be a standard mollifier. Then we have $\operatorname{curl} \varrho_\varepsilon * \widetilde{\mathbf{u}} = \varrho_\varepsilon * \operatorname{curl} \widetilde{\mathbf{u}}$ and $\varrho_\varepsilon * \widetilde{\mathbf{u}} \in C_c^\infty(\mathbb{R}^3)^3$ for any $\varepsilon > 0$, and $\lim_{\varepsilon \to 0} \varrho_\varepsilon * \widetilde{\mathbf{u}} = \widetilde{\mathbf{u}}$ in $L^2(Q)^3$.

Let $(O_n)_n$ be an increasing sequence of simply connected Lipschitz domains O_n such that $\overline{O}_n \subset Q$ for all n and $Q = \bigcup_{n=1}^\infty O_n$. Because the two-dimensional snowflake domain can be exhausted by increasing simply connected Lipschitz domains whose closures are contained in the snowflake domain, see for instance [27, Section 6], it follows easily that such a sequence $(O_n)_n$ exists.

If now n is fixed and $\varepsilon > 0$ is small enough then $\bigcup_{x \in O_n} B(x, \varepsilon) \subset Q$ and therefore $\operatorname{curl} \varrho_\varepsilon * \widetilde{\mathbf{u}} = 0$ in O_n. Consequently, there is a function $q_\varepsilon \in H^1(O_n)$ such that $\varrho_\varepsilon * \widetilde{\mathbf{u}} = \nabla q_\varepsilon$ in O_n. Since $\lim_{\varepsilon \to 0} \nabla q_\varepsilon = \widetilde{\mathbf{u}} \in L^2(O_n)^3$, the limit $q_n := \lim_{\varepsilon \to 0} q_\varepsilon$ exists in $H^1(O_n)/\mathbb{R}$, and clearly $\mathbf{u} = \nabla q_n$ in O_n.

Varying n, we have $\nabla q_n = \nabla q_{n+1}$ in O_n, i.e., $q_n - q_{n+1}$ is constant on O_n. We can choose these constants so that $q_{n+1} = q_n$ in O_n for all $n \geq 1$, and then consistently define $q := q_n$ on O_n for all $n \geq 1$ to obtain a function q with the desired properties. □

Theorem 15.14 implies a description of $\operatorname{Ker}(\operatorname{curl}, Q)$.

Corollary 15.15. *We have*

$$\operatorname{Ker}(\operatorname{curl}, Q) = \left\{ w \in H_0(\operatorname{curl}, Q) : w = \nabla q \ \textit{ for some } q \in H^1(Q) \right\}.$$

We can now easily prove Proposition 15.9.

Proof. If $\mathbf{u} \in H_0(\operatorname{curl}, Q) \cap H_0(\operatorname{div}, Q)$ is in $H_{0,\perp}(\operatorname{curl}, Q)$, then, by (15.15) and Corollary 15.15, it satisfies

$$\int_Q (\operatorname{div} \mathbf{u}) q \,\mathrm{d}x = \int_Q \mathbf{u} \cdot \nabla q \,\mathrm{d}x = 0$$

for all $q \in H^1(Q)$ such that $\nabla q \in H_0(\operatorname{curl}, Q)$, and in particular, for all $q \in C_c^\infty(Q)$, which implies $\operatorname{div} \mathbf{u} = 0$ in $L^2(Q)$. The opposite inclusion follows similarly from (15.15). □

Remark 15.16. Using [61, Theorem 3], one can show that $H_0^1(Q)$ coincides with the space of all elements of $H^1(Q)$ having zero trace on ∂Q. With Remark 15.8 and Theorem 15.12 in mind, one can therefore view Theorem 15.10 as a rough paraphrase of the statement that if in the formal identity

$\mathbf{u}|_{\partial Q} = \mathbf{n}(\mathbf{u}\cdot\mathbf{n})|_{\partial Q} + \mathbf{n}\times\mathbf{u}|_{\partial Q}$ both summands on the right-hand side are zero, then we have $\mathbf{u}|_{\partial Q} = 0$ in the sense of traces.

15.6. Weak Solutions and Hölder Regularity in 2D

We now reduce the three-dimensional problem (P) to a magnetostatic problem in 2D. If $\mathbf{J}(x) = (0, 0, J(x_1, x_2))$ and $\mu = \mu(x_1, x_2)$, then it is reasonable to assume that also the magnetic induction $\mathbf{B}$ does not depend on the x_3 coordinate. Therefore, it is possible to choose a magnetic vector potential of form $\mathbf{u} = (0, 0, u(x_1, x_2))$. Problem (P) then reduces to finding a function $u = u(x_1, x_2)$ on Ω such that

$$(\bar{P}) \begin{cases} -\operatorname{div}\left(\dfrac{1}{\mu}\nabla u\right) = J & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{15.22}$$

From this two-dimensional problem, we obtain a magnetic induction of form $\mathbf{B} = (u_{x_2}, -u_{x_1}, 0)$. The domain $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, (x_1, x_2, 0) \in Q\}$ is a cross-section of Q, i.e., $\Omega \times \{0\} = Q \cap \{x \in \mathbb{R}^3 \,:\, x_3 = 0\}$, and the differential operator ∇u (applied to the scalar function u) operates only on the variables x_1 and x_2, i.e., $\nabla u = (u_{x_1}, u_{x_2})$.

The energy form associated with $(\bar{P})$ is

$$a(u, v) = \int_\Omega \frac{1}{\mu(x)} \nabla u \nabla v \,\mathrm{d}x, \quad u, v \in H_0^1(\Omega), \tag{15.23}$$

where, as usual, $H_0^1(\Omega)$ denotes the closure in $H^1(\Omega)$ of the smooth functions with compact support in Ω.

Proposition 15.17. *For every given $J \in L^2(\Omega)$, there exists a unique weak solution in $H_0^1(\Omega)$ of problem $(\bar{P})$, i.e., a function $u \in H_0^1(\Omega)$ such that*

$$a(u, v) = \int_\Omega J\, v \,\mathrm{d}x, \quad v \in H_0^1(\Omega).$$

We recall some regularity results for the weak solution of problem $(\bar{P})$.

Proposition 15.18. *Suppose that μ is constant. Then the weak solution u of problem $(\bar{P})$ belongs to $W_0^{1,3}(\Omega) \cap C^{0,1/3}(\overline{\Omega})$. Moreover $\nabla^2 u \in L^2(\Omega, d)$, where d is the distance from the boundary. In particular, it follows that $\mathbf{B} \in (L^3(Q))^3$.*

Here $W_0^{1,3}(\Omega)$ and $C^{0,1/3}(\overline{\Omega})$ denote, respectively, the usual Sobolev space and the space of Hölder continuous functions of exponent $\frac{1}{3}$, while $\nabla^2 u$ denotes the Hessian of u. The statement $\nabla^2 u \in L^2(\Omega, d)$ means that

$$\int_\Omega |\nabla^2 u|^2 d(x, \partial\Omega)^2 \, \mathrm{d}x < \infty.$$

For the proof of Proposition 15.18, we refer to Theorem 1.3 (part B) and Proposition 7.1 in [50] (which is also related to [51]). These references also explain the appearance of the exponents 1/3 and 3 in this proposition in relation to the geometry of the Koch snowflake. The proof of Nystrom's result is very technical and it is strictly related to the Koch snowflake and to certain sophisticated estimates. We mentioned this result only for the sake of completeness, since we do not use it for our results in the paper and do not need this type of a deeper analysis.

We now consider the approximating problems on the pre-fractal domains Ω_n introduced in Section 15.2.

Let us assume that μ is a positive constant and $J \in L^2(\Omega)$. For every fixed $n \in \mathbb{N}$, we consider the following problems $(\bar{P}_n)$:

$$(\bar{P}_n) \begin{cases} -\mathrm{div}\left(\dfrac{1}{\mu}\nabla u_n\right) = J & \text{in } \Omega_n, \\ u_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{15.24}$$

We set $H_0^1(\Omega_n) := \overline{\{w \in C_0^1(\Omega) : \mathrm{supp}\, w \subset \Omega_n\}}^{H^1(\Omega)}$. For every $u_n, v \in H_0^1(\Omega_n)$, let

$$a_n(u_n, v) = \int_{\Omega_n} \frac{1}{\mu} \nabla u_n \nabla v \, \mathrm{d}x$$

be the energy form associated with problem $(\bar{P}_n)$.

Proposition 15.19. *For every given $J \in L^2(\Omega)$, there exists a unique weak solution $u_n \in H_0^1(\Omega_n)$ of problem $(\bar{P}_n)$.*

The following result states the convergence of the pre-fractal solutions u_n to the solution u of problem $(\bar{P})$ in a suitable sense. We recall that, for any compact subset $E \subset \Omega$, its relative capacity with respect to Ω is defined by

$$\mathrm{cap}_{2,\Omega}(E) = \inf\{\|\varphi\|^2_{H^1(\Omega)} : \varphi \in C_c^\infty(\Omega) \text{ and } \varphi \geq 1 \text{ on } E\},$$

see [47, p. 531].

Theorem 15.20. *Let u and u_n be the solutions of problems $(\bar{P})$ and $(\bar{P}_n)$, respectively. Then u_n strongly converges to u in $H^1_0(\Omega)$ as $n \to \infty$.*

Proof. The result follows from [47] since Ω_n is an increasing sequence of sets invading Ω and $\mathrm{cap}_{2,\Omega}(\Omega' \backslash \Omega_n) \to 0$ when $n \to \infty$ for any compact subset Ω' of Ω. □

15.7. Numerical Approximation in 2D

In this section, we perform a numerical approximation of problem (P) by a finite element method. For the sake of simplicity, we put $\mu = 1$. Hence, problem $(\bar{P}_n)$ reduces to the following form:

$$(\tilde{P}_n) \begin{cases} -\Delta u_n = J & \text{in } \Omega_n, \\ u_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{15.25}$$

In order to obtain the optimal rate of convergence of the numerical scheme, we use the theory of regularity in weighted Sobolev spaces developed by Grisvard. Let us introduce the weighted Sobolev space

$$H^2_\eta(\Omega_n) = \{v \in H^1(\Omega_n) \,:\, r^\eta \, D^\beta v \in L^2(\Omega_n),\, |\beta| = 2\},$$

where $r = r(x)$ is the distance from the vertices of $\partial\Omega_n$ whose angles are "reentrant".

This space is endowed with the norm

$$\|u\|_{H^2_\eta(\Omega)} = \left(\|u\|^2_{H^1(\Omega)} + \sum_{|\beta|=2} \int_\Omega r^{2\eta} |D^\beta u(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

From Kondrat'ev results [36; 33, Proposition 4.15] and Sobolev embedding theorem we deduce the following.

Theorem 15.21. *Let u_n be the weak solution of problem $(\tilde{P}_n)$. Then $u_n \in H^2_\eta(\Omega_n)$ for $\eta > \frac{1}{4}$. Moreover, $u_n \in H^s(\Omega_n)$ for $s < \frac{7}{4}$ and $u_n \in C^{0,\delta}(\overline{\Omega}_n)$ for $\delta = \frac{3}{4} - \varepsilon$ for every $\varepsilon > 0$.*

We point out that $u_n \notin H^2(\Omega_n)$ since it has a singular behavior in small neighborhoods of the reentrant corners of $\partial\Omega_n$. Hence, we have to construct a suitable mesh compliant with the so-called *Grisvard conditions* [24] in order to obtain the optimal rate of convergence. We refer to [9, 10], where such mesh algorithm was developed (see [11] for the case of fractal mixtures). We point out that this mesh algorithm produces a sequence of nested refinements.

The mesh refinement process generates a *conformal* and *regular* family of triangulations $\{T_{n,h}\}$, where $h = \max\{\text{diam}(S),\, S \in T_{n,h}\}$ is the size of the triangulation, which is also compliant with the Grisvard conditions (see [10, Section 5] for the case of interest). We define the finite-dimensional space of piecewise linear functions

$$X_{n,h} := \{v \in C^0(\overline{\Omega_n}) \,:\, v|_{\mathcal{T}} \in \mathbb{P}_1 \;\forall \mathcal{T} \in T_{n,h}\}.$$

We set $V_{n,h} := X_{n,h} \cap H^1_0(\Omega_n)$. Hence $V_{n,h}$ is a finite-dimensional space of dimension $N_h = \{\text{number of inner nodes of } T_{n,h}\}$. The discrete approximation problem is the following: given $J \in L^2(\Omega_n)$, find $u_{n,h} \in V_{n,h}$ such that

$$(\nabla u_{n,h}, \nabla v_h)_{L^2(\Omega_n)} = (J, v_h)_{L^2(\Omega_n)} \quad \forall v_h \in V_{n,h}. \tag{15.26}$$

The existence and uniqueness of the semi-discrete solution $u_{n,h} \in V_{n,h}$ of the variational problem (15.26) follows from the Lax–Milgram theorem (see, e.g., [52]).

Theorem 15.22. *Let u_n be the solution of problem $(\tilde{P}_n)$ and $u_{n,h}$ be the solution of the discrete problem* (15.26). *Then*

$$\|u_n - u_{n,h}\|^2_{H^1_0(\Omega_n)} \leq C\, h^2 \|J\|^2_{L^2(\Omega_n)}, \tag{15.27}$$

where C is a suitable constant independent of h.

For the proof, see [24, Theorem 8.4.1.6].

We now show some numerical simulations for problem $(\tilde{P}_n)$. We choose the source J as follows:

$$J(x_1, x_2) = 10^5 \; e^{-5((x_1-\bar{x}_1)^2+(x_2-\bar{x}_2)^2)},$$

where $(\bar{x}_1, \bar{x}_2)$ are the center coordinates of the domain (Fig. 15.3).

In our simulations, Ω_0 is the circle of radius $\frac{1}{2}$, while Ω_n, $n = 1, \ldots, 5$, are the domains having as boundary the nth approximation of the Koch snowflake. We suppose that all the domains are centered at the same point.

Denoting by u_n the solution of problem $(\tilde{P}_n)$, we define the vector $\mathbf{u}_n = (0, 0, u_n)$ and we compute the magnetic field $\mathbf{B}$ generated by the current $\mathbf{J} := (0, 0, J(x_1, x_2))$. In other words, $\mathbf{B} = \operatorname{curl} \mathbf{u}_n = \nabla \times \mathbf{u}_n$.

In Table 15.1, we write in the second column the value of the L^∞-norm of $\mathbf{B}$ in Ω_n, while in the third column we write the length $\ell(n)$ of the

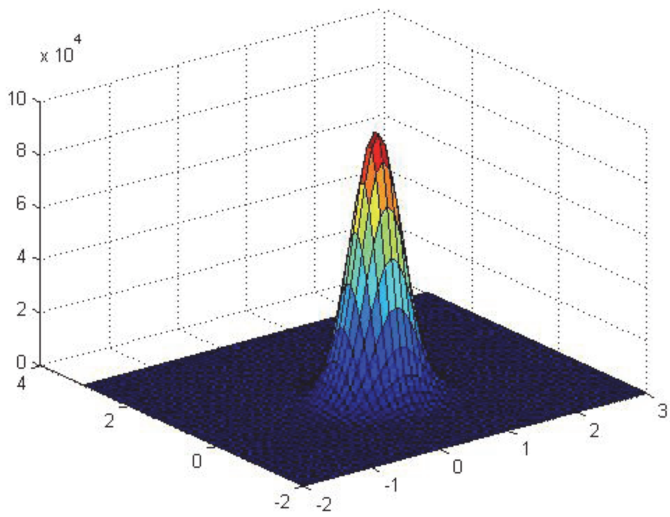

Fig. 15.3. The source J.

Table 15.1. The values obtained in our simulations.

Ω_n	$\|\mathbf{B}\|_\infty$	$\ell(n)$
Ω_0	17.946	π
Ω_1	26.688	4
Ω_2	35.575	$\frac{16}{3}$
Ω_3	47.124	$\frac{64}{9}$
Ω_4	63.504	$\frac{256}{27}$
Ω_5	85.43	$\frac{1024}{81}$

boundary $\partial\Omega_n$. In the first column, we write the domain we consider in the simulation.

As one can notice from Table 15.1, the magnetic field increases as the length of the boundary of the domain increases.

Remark 15.23. We note that our numerical results (see Fig. 15.4) compare well with numerical results of Lapidus *et al.* [14, 22, 42] on eigenfunctions of the scalar Dirichlet Laplacian in the Koch snowflake domain, and with some earlier physics results, such as [54]. In particular, one can expect that the localization and other properties of the electromagnetic fields can be analyzed using similar methods as for the scalar Laplacian (see, for instance,

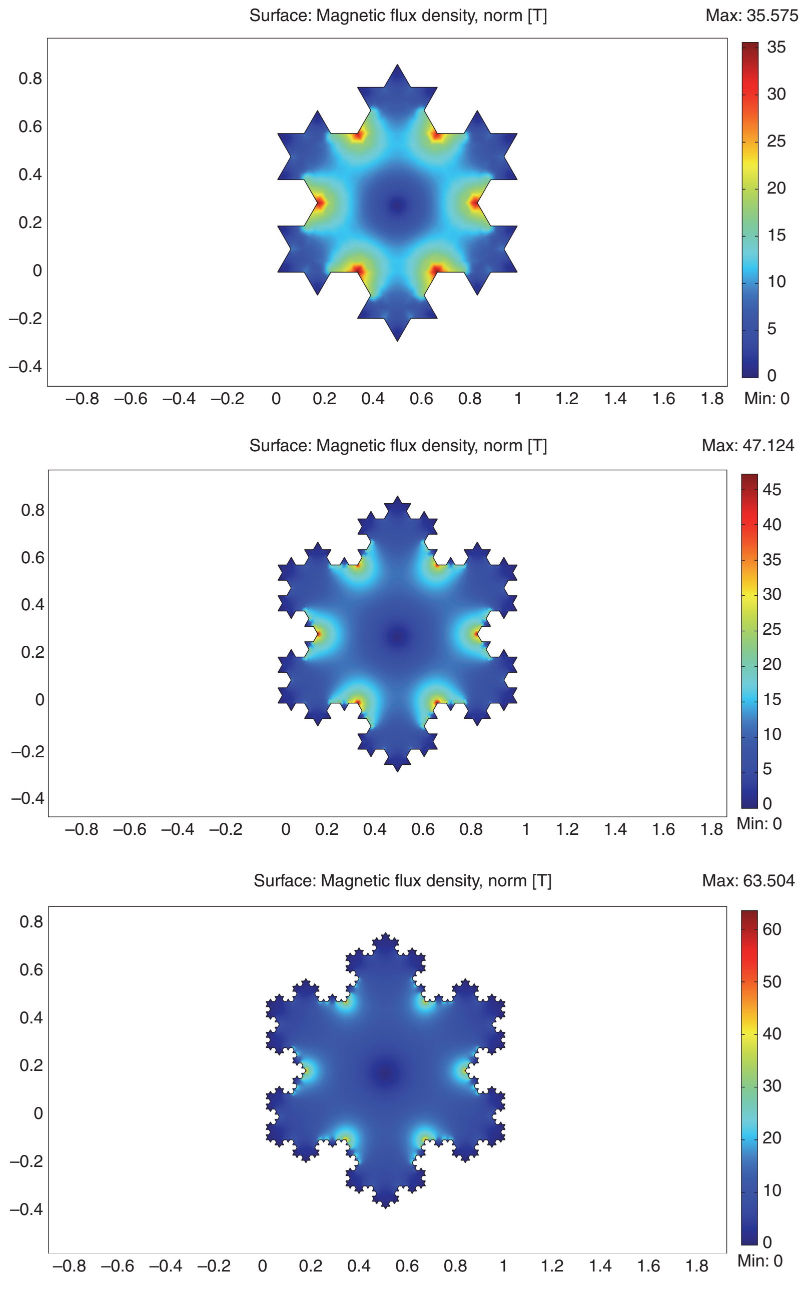

Fig. 15.4. The magnetic field generated by J in Ω_2, Ω_3, Ω_4.

[18, 41, 43, 45, 59, 60]). This connection lies outside of the scope of our chapter and will be the subject of future research.

Acknowledgments

S. C., M. R. L. and P. V. have been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). M. H. has been supported by the DFG IRTG 2235. A. T. has been supported by the NSF DMS 1613025. The authors thank the anonymous referee of this paper for the insightful and helpful comments and suggestions on its earlier version.

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