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Asymptotics for time-fractional Venttsel′ problems in fractal domains

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Abstract: We consider fractional-in-time Venttsel′ problems in fractal domains of Koch type. Well-posedness and regularity results are given. In view of numerical approximation, we consider the associated approximating pre-fractal problems. **Our main result is the convergence of the solutions of such problems towards the solution of the fractional-in-time Venttsel′ problem in the corresponding fractal domain.** This is achieved via the convergence (in the Mosco-Kuwae-Shioya sense) of the approximating energy forms in varying Hilbert spaces.

Keywords: Fractional Caputo time derivative; Venttsel′ problems; fractal domains; asymptotic behavior; varying Hilbert spaces: resolvent families.

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1. Introduction

Aim of this paper is to study the asymptotic behavior of the solution of time-fractional Venttsel′ problems $(\overline{P}_h)$ in Koch-type pre-fractal domains Ω_h, and to prove that the limit is the solution of the corresponding problem $(\overline{P})$ in the Koch domain Ω. Beyond the interest in itself, this result is a preliminary step towards the numerical approximation of problem $(\overline{P})$, following the approach of [11].
Fractal geometries are good models for irregular media, and many diffusion phenomena take place across irregular layers. This motivates the study of fractional heat diffusion across irregular boundaries.
From the mathematical point of view, the problem can be viewed as the coupling of an evolution equation in the bulk and an evolution equation on the boundary. These problems are also known as boundary value problems (BVPs) with dynamical boundary conditions. In the present setting, the resulting boundary condition is of second order, which is in some sense unusual for BVPs involving second order operators.

We formally state the model problem $(\overline{P})$ as:

$$(\overline{P})\begin{cases}\partial_t^\alpha u(t,P)-\Delta u(t,P)=f(t,P) & \text{in } (0,T)\times\Omega,\\ \partial_t^\alpha u(t,P)-\Delta_K u(t,P)+b(P)u(t,P)+\frac{\partial u(t,P)}{\partial n}=f(t,P) & \text{in } (0,T)\times K,\\ u(0,P)=\varphi(P) & \text{in } \overline{\Omega},\end{cases}$$

where $\Omega\subset\mathbb{R}^2$ is the two-dimensional open bounded domain with boundary $K=\partial\Omega$ the Koch snowflake (see Section 2.1), $0<\alpha\leq 1$, ∂_t^α is the fractional Caputo time derivative (see Section 2.5 for the definition), Δ_K **is the Laplace operator defined on the fractal** K **(see (8) in Section 3.1),** b **is a continuous strictly positive function on** $\overline{\Omega}$, $\frac{\partial u}{\partial n}$ **denotes the normal derivative across** K, f **and** φ **are given data in suitable functional spaces (see Section 4).**

For $h \in \mathbb{N}$, we denote by $\Omega_h \subset \mathbb{R}^2$ the pre-fractal domain with boundary $\partial\Omega_h = K_h$, where K_h is the polygonal curve approximating K at the h-th step (see Section 2.1). We consider the problems $(\overline{P}_h)$ defined on Ω_h. For every $h \in \mathbb{N}$, we formally present problem $(\overline{P}_h)$ as:

$$(\overline{P}_h)\begin{cases} \partial_t^\alpha u_h(t,P) - \Delta u_h(t,P) = f_h(t,P) & \text{in } (0,T)\times\Omega_h, \\ \delta_h \partial_t^\alpha u_h(t,P) - \Delta_{K_h} u_h(t,P) + \delta_h b(P) u_h(t,P) + \frac{\partial u_h(t,P)}{\partial n_h} = \delta_h f_h(t,P) & \text{in } (0,T)\times K_h, \\ u_h(0,P) = \varphi_h(P) & \text{in } \overline{\Omega}_h, \end{cases}$$

where Δ_{K_h} is the piecewise tangential Laplacian defined on K_h (see Section 3.2), $\frac{\partial u_h}{\partial n_h}$ the normal derivative across K_h and $f_h(t,P)$ and $\varphi_h(P)$ are given data in suitable functional spaces. The positive constant δ_h will have a key role in the asymptotic behavior as $h \to +\infty$ (see Section 5). The choice of this constant allows us to overcome the difficulties arising from the jump of dimension in the asymptotic analysis from the pre-fractal case to the fractal one.

We remark that Venttsel' problems in fractal domains and their approximation have been firstly studied in [34], see also [9,13,33]. These problems have been later generalized to the case of quasi-linear and/or fractional-in-space operators, see e.g. [12,14].

The literature on Venttsel' problems in smooth domains is huge, starting from the pioneering work of Venttsel' of 1959 [39], where he introduced a new class of boundary conditions for elliptic operators given by second order integro-differential equations (see also [2,3,17,24,37]). We refer the reader to the introduction of [34] for the physical motivations, see also [20].

As to the literature on time-fractional problems, the existing literature is wide. Among the others, we refer to [4,5,15,22,29,31] and the references therein and to [19] for time-fractional Venttsel' problems in Lipschitz domains; for time-fractional equations in fractal domains, we refer e.g. to [7,8].

Our goal is to prove well-posedness results for problems $(\overline{P})$ and $(\overline{P}_h)$ and to prove that the "fractal" solution of problem $(\overline{P})$ can be approximated by the sequence $\{u_h\}$ of the "smoother" solutions of problems $(\overline{P}_h)$.

More precisely, in Section 4.1 we introduce abstract Cauchy problems (P) and (P_h) and we prove that problem $(\overline{P})$ is the "strong formulation" of problem (P) (see Theorem 3) and that, for every $h \in \mathbb{N}$, problem $(\overline{P}_h)$ is the "strong formulation" of problem (P_h) (see Theorem 4). Existence and uniqueness results of the "strong solution" are obtained by the well-posedness results for fractional-in-time Cauchy problems [19].

We emphasize that the natural functional **framework** for studying problems (P_h) is that of the varying spaces $L^2(\overline{\Omega}_h, m_h)$ (see Section 5.1).

The asymptotic analysis of the solutions of problems $(\overline{P}_h)$ is performed by using the Mosco-Kuwae-Shioya (M-K-S) convergence. In [34] it has been proved that the energy forms $E^{(h)}$, associated to problems $(\overline{P}_h)$, converge in the M-K-S sense to the fractal energy form E, associated to problem $(\overline{P})$. This implies the convergence of associated semigroups and resolvents and it turns out to be crucial for the proof of Theorem 6.

The plan of the paper is the following.
In Section 2 we recall the geometry, the functional setting, the definition of convergence of varying Hilbert spaces as well as the definition of fractional Caputo time derivative.
In Section 3 we introduce the energy forms E and $E^{(h)}$, see (11) and (17) respectively, and the associated resolvents and semigroups.
**In Section 4 we study existence and uniqueness of the solutions of the evolution problems (P) and (P_h). Moreover, we give the strong formulations of problems (P) and (P_h).
In Section 5 we state the convergence of the energy forms and of the Hilbert spaces and in Theorem 6 we prove the convergence of the pre-fractal solutions to the fractal solution in a suitable weak sense**.

2. Preliminaries

2.1. Geometry

In this paper we denote points in $\mathbb{R}^2$ by $P = (x_1, x_2)$, the Euclidean distance by $|P - P_0|$ and the Euclidean ball by $B(P_0, r) = \{P \in \mathbb{R}^2 : |P - P_0| < r\}$ for $P_0 \in \mathbb{R}^2$ and $r > 0$. The Koch snowflake K [16] is the union of three com-planar Koch curves K_1, K_2 and K_3, see Figure 1.

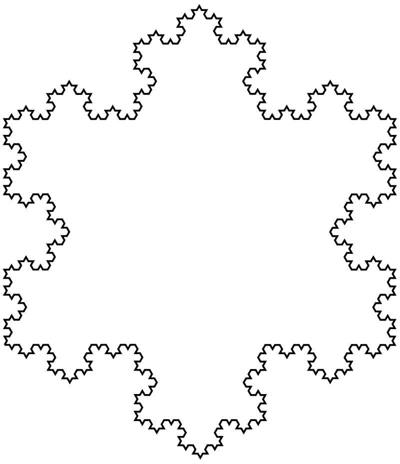

Figure 1. The Koch snowflake K.

The Hausdorff dimension of the Koch snowflake is $d_f = \frac{\ln 4}{\ln 3}$.
The natural finite Borel measure μ supported on K is defined as

$$\mu := \mu_1 + \mu_2 + \mu_3, \tag{1}$$

where μ_i denotes the normalized d_f-dimensional Hausdorff measure, restricted to K_i, $i = 1, 2, 3$.

We denote by

$$K_{h+1} = \bigcup_{i=1}^{3} K_i^{(h+1)} \tag{2}$$

the closed polygonal curve approximating K at the $(h+1)$-th step. We denote by $K_i^{(h+1)}$ the pre-fractal (polygonal) curve approximating K_i.

The measure μ enjoys the following property: there exist two positive constants c_1, c_2 such that

$$c_1 r^{d_f} \le \mu(B(P,r) \cap K) \le c_2 r^{d_f} \quad \forall P \in K. \tag{3}$$

Since μ is supported on K, in (3) we replace $\mu(B(P,r) \cap K)$ with $\mu(B(P,r))$.

Let Ω denote the two-dimensional open bounded domain with boundary K and, for every $h \in \mathbb{N}$, let Ω_h be the pre-fractal polygonal domains approximating Ω at the n-th step, and let $K_h = \partial\Omega_h$ be the pre-fractal curves. **We denote by M and by $\overset{\circ}{M}$ any segment of K_h and the related open segment respectively. We note that the sequence $\{\Omega_h\}_{h \in \mathbb{N}}$ is an invading sequence of sets exhausting Ω.**

2.2. Sobolev spaces

Throughout the paper, C will denote possibly different positive constants. The dependence of such constants on some parameters will be given in parentheses.
Let $\mathcal{G}$ (resp. $\mathcal{S}$) be an open (resp. a closed) set of $\mathbb{R}^N$. **For $p \ge 1$, we denote the Lebesgue space with respect to the Lebesgue measure $\mathrm{d}\mathcal{L}_N$ by $L^p(\mathcal{G})$ and the Lebesgue space on $\partial\mathcal{G}$ with respect to an invariant Hausdorff measure μ supported on $\partial\mathcal{G}$ by $L^p(\partial\mathcal{G}, \mu)$. For $s \in \mathbb{R}^+$, we denote the usual (possibly fractional) Sobolev spaces by $H^s(\mathcal{G})$ [36]. We**

denote the space of infinitely differentiable functions with compact support on $\mathcal{G}$ by $\mathcal{D}(\mathcal{G})$ and the space of continuous functions on $\mathcal{S}$ by $C(\mathcal{S})$.

In the following, we will make use of trace spaces on boundaries of polygonal domains of $\mathbb{R}^2$; for more details, we refer the reader to [6].
By $H^1(K_h)$ we denote the set

$$\{v \in C(K_h) : u|_{\mathring{M}} \in H^1(\mathring{M})\},$$

with the norm

$$\|u\|^2_{H^1(K_h)} = \|u\|^2_{L^2(K_h)} + \|\nabla u\|^2_{L^2(K_h)}.$$

By $H^s(K_h)$, for $0 < s \leq 1$, we denote the Sobolev space on K_h, defined by local Lipschitz charts as in [36]. We point out that for $s = 1$ the two definitions coincide with equivalent norms.

By $|A|$ we denote the Lebesgue measure of a measurable subset $A \subset \mathbb{R}^N$. For f in $H^s(\mathcal{G})$, the trace operator γ_0 is defined as

$$\gamma_0 f(P) := \lim_{r \to 0} \frac{1}{|B(P,r) \cap \mathcal{G}|} \int_{B(P,r)\cap\mathcal{G}} f(Q)\, \mathrm{d}\mathcal{L}_N(Q) \tag{4}$$

at every point $P \in \overline{\mathcal{G}}$ where the limit exists. The limit (4) exists at quasi every $P \in \overline{\mathcal{G}}$ with respect to the $(s,2)$-capacity (see [1], Definition 2.2.4 and Theorem 6.2.1 page 159). In the following, sometimes we omit the trace symbol leaving the interpretation to the reader.

We now recall the results of Theorem 2.24 in [6], referring to [23] for a more general discussion.

Proposition 1. *Let Ω_h and K_h be as above and let $\frac{1}{2} < s < \frac{3}{2}$. Then $H^{s-\frac{1}{2}}(K_h)$ is the trace space to K_h of $H^s(\Omega_h)$ in the following sense:*

i) *γ_0 is a linear and continuous operator from $H^s(\Omega_h)$ to $H^{s-\frac{1}{2}}(K_h)$;*

ii) *there exists a linear and continuous operator* Ext *from $H^{s-\frac{1}{2}}(K_h)$ to $H^s(\Omega_h)$ such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator in $H^{s-\frac{1}{2}}(K_h)$.*

In the sequel we denote by the symbol $f|_{K_h}$ the trace $\gamma_0 f$ to K_h.

2.3. Besov spaces

We start by giving the definition of d-set.

Definition 1. *Let $\mathcal{S} \subset \mathbb{R}^N$ be closed and non-empty. $\mathcal{S}$ is a* d-set*, for $0 < d \leq N$, if there exist a Borel measure $\tilde{\mu}$ with* supp $\tilde{\mu} = \mathcal{S}$ *and two constants $c_1 = c_1(\mathcal{S}) > 0$ and $c_2 = c_2(\mathcal{S}) > 0$ such that*

$$c_1 r^d \leq \tilde{\mu}(B(P,r)) \leq c_2 r^d \quad \forall\, P \in \mathcal{S},\, 0 < r \leq 1. \tag{5}$$

Such measure $\tilde{\mu}$ is called a d-measure *on $\mathcal{S}$.*

The following result follows from [16].

Proposition 2. *Let $d = d_f$. Then the measure μ defined in (3) is a d-measure, hence the Koch snowflake K is a d-set.*

We recall the definition of Besov spaces specialized to our case. For generalities on Besov spaces, we refer the reader to [38] and [26].

Definition 2. *Let $\mathcal{S}$ be a d-set in $\mathbb{R}^N$ and $0 < \gamma < 1$. We say that $f \in B^{2,2}_{\gamma}(\mathcal{S})$ if*

$$\|f\|^2_{B^{2,2}_{\gamma}(\mathcal{S})} := \|f\|^2_{L^2(\mathcal{S},\tilde{\mu})} + \iint_{|P-P'|<3^{-n}} \frac{|f(P)-f(P')|^2}{|P-P'|^{d+2\gamma}} \mathrm{d}\tilde{\mu}(P)\,\mathrm{d}\tilde{\mu}(P') < \infty.$$

We now state the trace theorem specialized to our case.

Proposition 3. *$B^{2,2}_{\frac{d_f}{2}}(K)$ is the trace space to K of $H^1(\Omega)$ in the following sense:*

i) *γ_0 is a linear and continuous operator from $H^1(\Omega)$ to $B^{2,2}_{\frac{d_f}{2}}(K)$;*

ii) *there exists a linear and continuous operator* Ext *from $B^{2,2}_{\frac{d_f}{2}}(K)$ to $H^1(\Omega)$ such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator in $B^{2,2}_{\frac{d_f}{2}}(K)$.*

For the proof we refer to Theorem 1 of Chapter VII in [26], see also [38]. The symbol$f|_K$ will denote the trace $\gamma_0 f$ to K.

As to the dual of Besov spaces on K, we refer to [27], where it is shown that they coincide with a subspace of Schwartz distributions $\mathcal{D}'(\mathbb{R}^2)$, supported on K. For a complete discussion and description of duals of Besov spaces on d-sets see [27].

2.4. Convergence of Hilbert spaces

In this subsection, we recall the definition of convergence of varying real and separable Hilbert spaces (for definitions and proofs, see [32] and [30]).

Definition 3. *A sequence of Hilbert spaces $\{H_h\}_{h\in\mathbb{N}}$ converges to a Hilbert space H if there exists a dense subspace $C \subset H$ and a sequence $\{Z_h\}_{h\in\mathbb{N}}$ of linear operators $Z_h\colon C \subset H \to H_h$ such that*

$$\lim_{h\to\infty} \|Z_h u\|_{H_h} = \|u\|_H \quad \textit{for any } u \in C.$$

In the following, we assume that $\{H_h\}_{h\in\mathbb{N}}$, H and $\{Z_h\}_{h\in\mathbb{N}}$ are as in Definition 3. Let be $\mathcal{H} = \{\cup_h H_h\} \cup H$. We recall the definition of strong convergence in $\mathcal{H}$.

Definition 4 (Strong convergence in $\mathcal{H}$)**.** *A sequence of vectors $\{u_h\}_{h\in\mathbb{N}}$ strongly converges to u in $\mathcal{H}$ if $u_h \in H_h$, $u \in H$ and there exists a sequence $\{\tilde{u}_m\}_{m\in\mathbb{N}} \in C$ tending to u in H such that*

$$\lim_{m\to\infty} \overline{\lim_{h\to\infty}} \|Z_h \tilde{u}_m - u_h\|_{H_h} = 0.$$

We recall the definition of strong convergence in $\mathcal{H}$.

Definition 5 (Weak convergence in $\mathcal{H}$)**.** *A sequence of vectors $\{u_h\}_{h\in\mathbb{N}}$ weakly converges to u in $\mathcal{H}$ if $u_h \in H_h$, $u \in H$ and*

$$(u_h, v_h)_{H_h} \to (u, v)_H$$

for every sequence $\{v_h\}_{h\in\mathbb{N}}$ strongly tending to v in $\mathcal{H}$.

We point out that the strong convergence implies the weak convergence [32].

Lemma 1. *Let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence weakly converging to u in $\mathcal{H}$. Then*

$$\sup_h \|u_h\|_{H_h} < \infty, \qquad \|u\|_H \le \underline{\lim}_{h\to\infty} \|u_h\|_{H_h}.$$

Moreover, $u_h \to u$ strongly if and only if $\|u\|_H = \lim_{h\to\infty} \|u_h\|_{H_h}$.

We recall some useful properties of the strong convergence of a sequence of vectors $\{u_h\}_{h\in\mathbb{N}}$ in $\mathcal{H}$.

Lemma 2. *Let $u \in H$ and let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence of vectors $u_h \in H_h$. Then $\{u_h\}_{h\in\mathbb{N}}$ strongly converges to u in $\mathcal{H}$ if and only if*

$$(u_h, v_h)_{H_h} \to (u, v)_H$$

for every sequence $\{v_h\}_{h\in\mathbb{N}}$ with $v_h \in H_h$ weakly converging to a vector v in $\mathcal{H}$.

Lemma 3. *A sequence of vectors $\{u_h\}_{h\in\mathbb{N}}$ with $u_h \in H_h$ strongly converges to a vector u in $\mathcal{H}$ if and only if*

$$\begin{aligned} \|u_h\|_{H_h} &\to \|u\|_H \quad \text{and} \\ (u_h, Z_h(\varphi))_{H_h} &\to (u, \varphi)_H \quad \text{for every } \varphi \in C. \end{aligned}$$

Lemma 4. *Let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence with $u_h \in H_h$. If $\|u_h\|_{H_h}$ is uniformly bounded, then there exists a subsequence of $\{u_h\}_{h\in\mathbb{N}}$ which weakly converges in $\mathcal{H}$.*

Lemma 5. *For every $u \in H$ there exists a sequence $\{u_h\}_{h\in\mathbb{N}}$, with $u_h \in H_h$, strongly converging to u in $\mathcal{H}$.*

We denote by $\mathfrak{L}(X)$ the space of linear and continuous operators on a Hilbert space X. **We now recall the notion of strong convergence of operators.**

Definition 6. *A sequence of bounded operators $\{B_h\}_{h\in\mathbb{N}}$, with $B_h \in \mathfrak{L}(H_h)$, strongly converges to an operator $B \in \mathfrak{L}(H)$ if for every sequence of vectors $\{u_h\}_{h\in\mathbb{N}}$ with $u_h \in H_h$ strongly converging to a vector u in $\mathcal{H}$, the sequence $\{B_h u_h\}$ strongly converges to Bu in $\mathcal{H}$.*

2.5. Fractional-in-time derivatives

We recall the notion of fractional-in-time derivatives in the sense of Riemann-Liouville and Caputo by using the notations of the monograph [19].
Let $\alpha \in (0,1)$. We define

$$g_\alpha(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

where Γ is the usual Gamma function.

Definition 7. *Let Y be a Banach space, $T > 0$ and let $f \in C([0,T];Y)$ be such that $g_{1-\alpha} * f \in W^{1,1}((0,T);Y)$.*

i) *The* Riemann-Liouville *fractional derivative of order $\alpha \in (0,1)$ is defined as follows:*

$$D_t^\alpha f(t) := \frac{\mathrm{d}}{\mathrm{d}t}(g_{1-\alpha} * f)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t g_{1-\alpha}(t-\tau) f(\tau)\,\mathrm{d}\tau,$$

for a.e. $t \in (0,T]$.

ii) *The* Caputo-type *fractional derivative of order $\alpha \in (0,1)$ is defined as follows:*

$$\partial_t^\alpha f(t) := D_t^\alpha (f(t) - f(0)),$$

for a.e. $t \in (0,T]$.

We stress the fact that Definition 7-*ii*) gives a weaker definition of (Caputo) fractional derivative with respect to the original one (see [10]), since f is not assumed to be differen-

tiable. Moreover, it holds that $\partial_t^\alpha(c) = 0$ for every constant $c \in \mathbb{R}$.
We refer to the book [15] for further details on fractional derivatives.

In the next sections we will consider problems of the following type:

$$(\tilde{P})\begin{cases} \partial_t^\alpha u - Au = f & \text{a.e. in } \Omega, \text{ for all } t \in (0,T), \\ u(0) = \varphi & \text{in } \Omega. \end{cases}$$

Here, A is a closed linear operator with domain $D(A)$ in a Banach space Y, $f\colon [0,\infty) \to Y$ and $\varphi \in Y$ are given.
According to [19, Definition 2.1.4], we give the following notion of strong solution for problem $(\tilde{P})$.

Definition 8. *Let $0 < T_1 \le T_2 < T$. We say that u is a* strong solution *of $(\tilde{P})$ on the interval $I = [0,T]$ if the following conditions are satisfied.*

i) *(The case $\alpha = 1$) The function $u \in C([0,T);Y)$ is such that $u(0) = \varphi$, $u(t) \in D(A)$ for all $t \in [T_1,T_2] \subset I$, and $\partial_t u \in C([T_1,T_2];Y)$. Moreover, the equation $\partial_t u(t) = Au(t) + f(t)$ is satisfied on $[T_1,T_2] \subset I$.*

ii) *(The case $\alpha \in (0,1)$) The function $u \in C([0,T);Y)$ is such that $u(0) = \varphi$, $u(t) \in D(A)$ for $t \in [T_1,T_2]$, and $\partial_t^\alpha u \in C([T_1,T_2];Y)$. Moreover, the equation $\partial_t^\alpha u(t) = Au(t) + f(t)$ is satisfied on $[T_1,T_2] \subset I$.*

3. The energy forms

We now introduce energy forms associated to the formal problems $(\overline{P})$ and $(\overline{P}_h)$ respectively. From now on, let Ω, K, Ω_h and K_h be as defined in Section 2.1 and let b denote a strictly positive continuous function in $\overline{\Omega}$.

3.1. The fractal energy form

As in [34, Section 3.1], we introduce a Lagrangian measure $\mathcal{L}_K$ on K and the corresponding energy form E_K as

$$E_K(u,v) = \int_K \mathrm{d}\mathcal{L}_K(u,v) \tag{6}$$

with domain $D(K)$; this space is a Hilbert space with norm

$$\|u\|_{D(K)} = \left(\|u\|^2_{L^2(K)} + E_K(u,u)\right)^{\frac{1}{2}} \tag{7}$$

and it has been characterized in terms of the domains of the energy forms on K_i.
In the following we will omit the subscript K, the Lagrangian measure will be simply denoted by $\mathcal{L}(u,v)$ and we will set $\mathcal{L}[u] = \mathcal{L}(u,u)$.
As in Proposition 3.1 of [34], the following result holds.

Proposition 4. *In the previous notations and assumptions, the form E_K with domain $D(K)$ is a regular Dirichlet form in $L^2(K)$ and the space $D(K)$ is a Hilbert space under the intrinsic norm (7).*

For the definition and properties of Dirichlet forms, see [18].
We now introduce the Laplace operator on K. Since $(E_K, D(K))$ is a densely defined regular Dirichlet form on $L^2(K)$, from [28, Chap. 6, Theorem 2.1] there exists a unique self-adjoint, non-positive operator Δ_K on $L^2(K)$, with domain $D(\Delta_K) \subseteq D(K)$ dense in $L^2(K)$, such that

$$E_K(u,v) = -\int_K (\Delta_K u)\, v \,\mathrm{d}\mu, \quad u \in D(\Delta_K),\, v \in D(K). \tag{8}$$

We denote by $(D(K))'$ the dual space of $D(K)$. We now introduce the Laplace operator on K as a variational operator from $D(K)$ to $(D(K))'$ by

$$E_K(u,w) = -\langle \Delta_K z, w\rangle_{(D(K))',D(K)}, \quad z \in D(K),\, w \in D(K), \tag{9}$$

where $\langle\cdot,\cdot\rangle_{(D(K))',D(K)}$ denotes the duality pairing between $(D(K))'$ and $D(K)$. In the following Δ_K will denote the Laplace operator both as the self-adjoint operator (see (8)) and as the variational operator (see (9)), leaving the interpretation to the context.

We now define the space of functions

$$V(\Omega,K) = \Big\{u \in H^1(\Omega) \,:\, u|_K \in D(K)\Big\}. \tag{10}$$

We remark that the space $V(\Omega,K)$ is non trivial.
We introduce the energy form

$$E[u] = \int_\Omega |\nabla u|^2 \mathrm{d}\mathcal{L}_2 + E_K[u|_K] + \int_K b|u|_K|^2\,\mathrm{d}\mu \tag{11}$$

defined on the domain $V(\Omega,K)$. In the following we denote by $L^2(\overline{\Omega},m)$ the Lebesgue space with respect to the measure m with

$$\mathrm{d}m = \mathrm{d}\mathcal{L}_2 + \mathrm{d}\mu. \tag{12}$$

By $E(u,v)$, for $u,v \in V(\Omega,K)$, we denote the corresponding bilinear form

$$E(u,v) = \int_\Omega \nabla u \nabla v\,\mathrm{d}\mathcal{L}_2 + E_K(u|_K, v|_K) + \int_K bu|_K v|_K\,\mathrm{d}\mu. \tag{13}$$

Proposition 5. *The form E defined in (11) is a Dirichlet form in $L^2(\overline{\Omega},m)$ and the space $V(\Omega,K)$ is a Hilbert space equipped with the scalar product*

$$(u,v)_{V(\Omega,K)} = (u,v)_{H^1(\Omega)} + E_K(u,v) + (u,v)_{L^2(K)}. \tag{14}$$

We denote by $\|u\|_{V(\Omega,K)}$ the norm in $V(\Omega,K)$ associated with (14), i.e.

$$\|u\|_{V(\Omega,K)} = \left(\|u\|^2_{H^1(\Omega)} + \|u\|^2_{D(K)}\right)^{\frac{1}{2}}. \tag{15}$$

3.2. The pre-fractal energy forms

For each $h \in \mathbb{N}$, we construct the energy forms E_{K_h} on the pre-fractal boundaries K_h. By ℓ we denote the natural arc-length coordinate on each segment of the polygonal curve K_h and we introduce the coordinates $x_1 = x_1(\ell)$, $x_2 = x_2(\ell)$, on every segment $M_h^{(j)}$ of K_h, $j = 1,\ldots,4^h$. By $\mathrm{d}\ell$ we denote the one-dimensional measure given by the arc-length ℓ. Let $u \in H^1(K_h)$, where we recall that $H^1(K_h)$ is the Sobolev space on the piecewise affine set K_h (see Section 2.2). We define $E_{K_h}[u]$ by setting

$$E_{K_h}[u] = \sum_{j=1}^{4^h} \int_{M_h^{(j)}} \sigma_h |\nabla_\ell u|_{K_h}|^2\,\mathrm{d}\ell, \tag{16}$$

where σ_h is a positive constant and ∇_ℓ denotes the tangential derivative along the pre-fractal K_h. We denote the corresponding bilinear form by $E_{K_h}(u,v)$.

Let $V(\Omega_h,K_h)$ be the space of restrictions to Ω_h of functions u defined on Ω for which the following norm is finite:

$$\|u\|^2_{V(\Omega_h,K_h)} = \|u\|^2_{H^1(\Omega_h)} + \|u\|^2_{H^1(K_n)}.$$

We point out that this space is not trivial as it contains $C^\infty(\Omega) \cap H^1(\Omega)$ (see [25]).

We consider now the following energy form defined on $V(\Omega_h, K_h)$:

$$E^{(h)}[u] = \int_{\Omega_h} |\nabla u|^2 \, \mathrm{d}\mathcal{L}_2 + E_{K_h}[u|_{K_h}] + \delta_h \int_{K_h} b|u|_{K_h}|^2 \, \mathrm{d}\ell, \tag{17}$$

where δ_h is a positive constant.
By $E^{(h)}(u,v)$ we denote the corresponding bilinear form defined on $V(\Omega_h, K_h) \times V(\Omega_h, K_h)$:

$$E^{(h)}(u,v) = \int_{\Omega_h} \nabla u \nabla v \, \mathrm{d}\mathcal{L}_2 + E_{K_h}(u|_{K_h}, v|_{K_h}) + \delta_h \int_{K_h} bu|_{K_h} v|_{K_h} \, \mathrm{d}\ell. \tag{18}$$

In the following we consider also the space $L^2(\overline{\Omega}_h, m_h)$, where m_h is the measure given by

$$\mathrm{d}m_h = \mathrm{d}\mathcal{L}_2 + \chi_{K_h} \delta_h \mathrm{d}\ell. \tag{19}$$

Proposition 6. *The form $E^{(h)}$ with domain $V(\Omega_h, K_h)$, defined in (17), is a Dirichlet form in $L^2(\overline{\Omega}_h, m_h)$ and the space $V(\Omega_h, K_h)$ is a Hilbert space equipped with the norm*

$$\|u\|_{V(\Omega_h, K_h)} = \left(\int_{\Omega_h} |\nabla u|^2 \, \mathrm{d}\mathcal{L}_2 + E_{K_h}[u|_{K_h}] + \|u\|^2_{L^2(\overline{\Omega}_h, m_h)} \right)^{\frac{1}{2}}. \tag{20}$$

3.3. Resolvents and associated semigroups

Since $(E, V(\Omega, K))$ is a densely defined closed bilinear form on $L^2(\overline{\Omega}, m)$, from [28, Chapter 6, Theorem 2.1] there exists a unique self-adjoint non-positive operator A on $L^2(\overline{\Omega}, m)$, with domain $D(A) \subseteq V(\Omega, K)$ dense in $L^2(\overline{\Omega}, m)$, such that

$$E(u,v) = (-Au, v)_{L^2(\overline{\Omega}, m)}, \quad u \in D(A), \, v \in V(\Omega, K). \tag{21}$$

Moreover, in Theorem 13.1 of [18] it is proved that to each closed symmetric form E can be associated a family of linear operators $\{G_\lambda, \, \lambda > 0\}$ with the property

$$E(G_\lambda u, v) + \lambda (G_\lambda u, v)_{L^2(\overline{\Omega}, m)} = (u, v)_{L^2(\overline{\Omega}, m)}, \quad u \in L^2(\overline{\Omega}, m), \, v \in V(\Omega, K).$$

This family $\{G_\lambda, \, \lambda > 0\}$ is a *strongly continuous resolvent* with generator A, which also generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$.

Proceeding as above, we denote by $\{G^h_\lambda, \, \lambda > 0\}$, A_h and $\{T_h(t)\}_{t \geq 0}$ the resolvents, the generators and the semigroups associated to $E^{(h)}$, for every $h \in \mathbb{N}$, respectively.

We recall the main properties of the semigroups $\{T(t)\}_{t \geq 0}$ and $\{T_h(t)\}_{t \geq 0}$ in the following Proposition.

Proposition 7. *Let $\{T(t)\}_{t \geq 0}$ and $\{T_h(t)\}_{t \geq 0}$ be the semigroups generated by the operators A and A_h associated to the energy forms in (11) and in (17) respectively. Then $\{T(t)\}_{t \geq 0}$ and $\{T_h(t)\}_{t \geq 0}$ are analytic contraction semigroups in $L^2(\overline{\Omega}, m)$ and $L^2(\overline{\Omega}_h, m_h)$ respectively.*

The proof follows as in Proposition 3.4 in [34].

4. Existence and uniqueness results

4.1. The abstract Cauchy problems

Let T be a fixed positive real number. We consider the Cauchy problem

$$(P) \begin{cases} \partial^\alpha_t u(t) = Au(t) + f(t), & 0 < t < T, \\ u(0) = \varphi, \end{cases}$$

where $A\colon D(A)\subset H\to H$ is the generator associated to the energy form E introduced in (11), and f and φ are given functions in suitable Banach spaces.
We consider also, for every $h\in\mathbb{N}$, the Cauchy problems

$$(P_h)\begin{cases}\partial_t^\alpha u_h(t)=A_h u_h(t)+f_h(t), & 0<t<T,\\ u_h(0)=\varphi_h,\end{cases}$$

where $A_h\colon D(A_h)\subset H_h\to H_h$ is the generator associated to the energy form $E^{(h)}$ introduced in (17), and f_h and φ_h are given functions in suitable Banach spaces.

We want to prove existence and uniqueness results for the strong solutions of problems (P) and (P_h), for every $h\in\mathbb{N}$, in the sense of Definition 8. Firstly, recall the definition of the Wright type function (see [21, Formula (28)]):

$$\Phi_\alpha(z):=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!(-\alpha n+1-\alpha)},\quad 0<\alpha<1,\, z\in\mathbb{C}.$$

From [5, page 14], it holds that $\Phi_\alpha(t)$ is a probability density function, i.e.

$$\Phi_\alpha(t)\ge 0\quad\text{if } t>0,\qquad \int_0^{+\infty}\Phi_\alpha(t)\,\mathrm{d}t=1.$$

For more properties about the Wright function, among the others we refer to [5], [21], [40].

We recall that the operators A and A_h generate strongly continuous, analytic, contraction semigroups $\{T(t)\}$ and $\{T_h(t)\}$ on H and H_h respectively. For $t>0$, we define the operators $S_\alpha(t)\colon H\to H$ and $P_\alpha(t)\colon H\to H$ as follows:

$$S_\alpha(t)v:=\int_0^{+\infty}\Phi_\alpha(\tau)T(\tau t^\alpha)v\,\mathrm{d}\tau,$$

$$P_\alpha(t)v:=\alpha t^{\alpha-1}\int_0^{+\infty}\tau\Phi_\alpha(\tau)T(\tau t^\alpha)v\,\mathrm{d}\tau.$$

The operators S_α and P_α are known in the literature as *resolvent families*. We note that the semigroup property does not hold for the operators S_α and P_α unless $\alpha=1$.

We can define in an analogous way, for every $h\in\mathbb{N}$, resolvent families $S_\alpha^h(t)$ and $P_\alpha^h(t)$ on H_h associated to the semigroup $\{T_h(t)\}$.
We now give the existence and uniqueness results for the strong solutions of problems (P) and (P_h) respectively. For both cases, we refer to [19, Theorem 2.1.7].

Theorem 1. *Let $\varphi\in\overline{D(A)}$. Let $f\in C^{0,\beta}((0,T);H)$ for $0<\beta<1$ satisfy one of the following two properties:*

i) *(The case $\alpha=1$)*

$$\int_0^{T_0}\|f(t)\|_H\,\mathrm{d}t<\infty$$

for some $T_0>0$;

ii) *(The case $\alpha\in(0,1)$) there exists $q\in(\frac{1}{\alpha},\infty)$ such that*

$$\int_0^{T_0}\|f(t)\|_H^q\,\mathrm{d}t<\infty$$

for some $T_0>0$.

Then there exists a unique strong solution u of problem (P) in the sense of Definition 8 given by

$$u(t)=T(t)\varphi+\int_0^t T(t-\tau)f(\tau)\,\mathrm{d}\tau \tag{22}$$

if $\alpha = 1$*, and by*

$$u(t) = S_\alpha(t)\varphi + \int_0^t P_\alpha(t-\tau) f(\tau)\,\mathrm{d}\tau \tag{23}$$

if $0 < \alpha < 1$*, respectively.*

Theorem 2. *For every* $h \in \mathbb{N}$*, let* $\varphi_h \in \overline{D(A_h)}$*. Let* $f_h \in C^{0,\beta}((0,T); H_h)$ *for* $0 < \beta < 1$ *satisfy one of the following two properties:*

i) *(The case* $\alpha = 1$*)*

$$\int_0^{T_0} \|f_h(t)\|_{H_h}\,\mathrm{d}t < \infty$$

for some $T_0 > 0$*;*

ii) *(The case* $\alpha \in (0,1)$*) there exists* $q \in (\frac{1}{\alpha}, \infty)$ *such that*

$$\int_0^{T_0} \|f_h(t)\|^q_{H_h}\,\mathrm{d}t < \infty$$

for some $T_0 > 0$*.*

Then, for every $h \in \mathbb{N}$ *there exists a unique strong solution* u_h *of problem* (P_h) *in the sense of Definition 8 given by*

$$u_h(t) = T_h(t)\varphi_h + \int_0^t T_h(t-\tau) f_h(\tau)\,\mathrm{d}\tau \tag{24}$$

in $\alpha = 1$*, and by*

$$u_h(t) = S^h_\alpha(t)\varphi + \int_0^t P^h_\alpha(t-\tau) f_h(\tau)\,\mathrm{d}\tau \tag{25}$$

in $0 < \alpha < 1$*, respectively.*

4.2. The Venttsel' boundary value problems

In this section we prove that the strong solutions of problems (P) and (P_h) solve respectively problems $(\bar{P})$ and $(\bar{P}_h)$ formally stated in the Introduction. We start with the fractal case.

Theorem 3. *Let* u *be the solution of problem* (P)*. Then we have, for every fixed* $t \in (0,T)$*,*

$$\begin{cases} \partial^\alpha_t u(t,P) - \Delta u(t,P) = f(t,P) & \text{for a.e. } P \in \Omega, \\ \langle \partial^\alpha_t u, z\rangle_{L^2(K),L^2(K)} + E_K(u,z) + \left\langle \frac{\partial u}{\partial n}, z\right\rangle_{(D(K))',D(K)} & \\ +\langle bu, z\rangle_{L^2(K),L^2(K)} = \langle f, z\rangle_{L^2(K),L^2(K)} & \text{for every } z \in D(K), \\ u(0,P) = \varphi(P) & \text{for } P \in \overline{\Omega}. \end{cases}$$

Moreover, $\frac{\partial u}{\partial n} \in C((0,T); (B^{2,2}_{\frac{d_f}{2}}(K))')$*.*

Proof. Following the approach of the proof of Theorem 6.1 in [34] and taking into account Theorem 1, we obtain the thesis. ☐

As to the pre-fractal case, the following result holds.

Theorem 4. *For every $h \in \mathbb{N}$, let u_h be the solution of problem (P_h). Then we have, for every fixed $t \in (0,T)$,*

$$\begin{cases} \partial_t^\alpha u_h(t,P) - \Delta u_h(t,P) = f_h(t,P) & \text{for a.e. } P \in \Omega_h, \\ \delta_h \langle \partial_t^\alpha u_h, z \rangle_{L^2(K_h),L^2(K_h)} + E_{K_h}(u_h,z) + \left\langle \frac{\partial u_h}{\partial n_h}, z \right\rangle_{H^{-\frac{1}{2}}(K_h),H^{\frac{1}{2}}(K_h)} & \\ +\delta_h \langle b u_h, z \rangle_{L^2(K_h),L^2(K_h)} = \delta_h \langle f_h, z \rangle_{L^2(K_h),L^2(K_h)} & \text{for every } z \in H^{\frac{1}{2}}(K_h), \\ u_h(0,P) = \varphi_h(P) & \text{for } P \in \overline{\Omega}_h. \end{cases}$$

Moreover, $\frac{\partial u_h}{\partial n_h} \in C((0,T); L^2(K_h))$.

Proof. Following the approach of the proof of Theorem 6.2 in [34] and taking into account Theorem 2, we obtain the thesis. □

5. Convergence results

In this section we study the asymptotic behavior of the solution u_h of the following homogeneous problem associated to (P_h), i.e.

$$(P_h^0) \begin{cases} \partial_t^\alpha u_h(t) = A_h u_h(t), & 0 < t < T, \\ u_h(0) = \varphi_h, \end{cases}$$

for every $h \in \mathbb{N}$. Namely, we will prove that $\{u_h\}$ converges to the unique strong solution of the homogeneous problem associated to (P):

$$(P^0) \begin{cases} \partial_t^\alpha u(t) = Au(t), & 0 < t < T, \\ u(0) = \varphi. \end{cases}$$

The convergence will be achieved by the Mosco-Kuwae-Shioya convergence of the energy forms. To this aim, we recall some preliminary definitions and results.

5.1. Convergence of spaces and M-convergence of the energy forms

We define the space $H := L^2(\overline{\Omega}, m)$ where m is the measure in (12). We also introduce the sequence $\{H_h\}_{h\in\mathbb{N}}$ with $H_h := \{L^2(\Omega) \cap L^2(\overline{\Omega}_h, m_h)\}$ where m_h is the measure in (19). We endow these spaces with the norms

$$\|u\|_H^2 = \|u\|^2_{L^2(\Omega)} + \|u|_K\|^2_{L^2(K,\mu)}, \quad \|u\|^2_{H_h} = \|u\|^2_{L^2(\Omega_h)} + \|u|_{K_h}\|^2_{L^2(K_h,\delta_h \ell)}$$

Proposition 8. *Let $\delta_h = \left(\frac{3}{4}\right)^h$. The sequence of Hilbert spaces $\{H_h\}_{h\in\mathbb{N}}$ converges in the sense of Definition 3 to the Hilbert space H.*

For the proof, see Proposition 4.1 in [34].

We now introduce the notion of M-K-S convergence of forms, firstly given by Mosco in [35] for a fixed Hilbert space and then extended by Kuwae and Shioya (see [32, Definition 2.11]) to the case of varying Hilbert spaces .
We extend the forms E defined in (11) and $E^{(h)}$ defined in (17) to the whole spaces H and H_h respectively by setting

$$E[u] = +\infty \quad \text{if } u \in H \setminus V(\Omega, K)$$

and

$$E^{(h)}[u] = +\infty \quad \text{if } u \in H_h \setminus V(\Omega_h, K_h).$$

Definition 9. *Let H_h be a sequence of Hilbert spaces converging to a Hilbert space H. A sequence of forms $\left\{E^{(h)}\right\}$ defined in H_h M-K-S-converges to a form E defined in H if the following conditions hold:*

i) for every $\{v_h\} \in H_h$ weakly converging to $u \in H$ in $\mathcal{H}$

$$\varliminf_{h\to\infty} E^{(h)}[v_h] \geq E[u];$$

ii) for every $u \in H$ there exists a sequence $\{w_h\}$, with $w_h \in H_h$ strongly converging to u in $\mathcal{H}$, such that

$$\varlimsup_{h\to\infty} E^{(h)}[w_h] \leq E[u].$$

We now state the convergence of the approximating energy forms $E^{(h)}$ in the context of varying Hilbert spaces.

Theorem 5. *Let $\delta_h = \left(\frac{3}{4}\right)^h$ and $\sigma_h = \delta_h^{-1}$. Then the sequence $\{E^{(h)}\}$ defined in (17) converges in the sense of Definition 9 to the form E defined in (11).*

For the proof, we refer to Theorem 4.3 in [34].

5.2. Convergence of the solutions of the abstract Cauchy problems

We are now ready to prove the main theorem of this section, i.e. the convergence of the sequence $\{u_h\}$ of strong solutions of problems (P^0_h) to the unique strong solution u of problem (P^0). Crucial tools will be the Mosco-Kuwae-Shioya convergence of the energy forms and the use of the representation formulas for the strong solutions given by (23) and (25). We remark that here we extend to the setting of varying Hilbert spaces the results in [7].

We consider the one-dimensional Lebesgue measure $\mathrm{d}t$ on $[T_1, T_2]$. Let m_h be the measure introduced in (19) and m be the measure introduced in (12). The space $L^2([T_1, T_2] \times \Omega, \mathrm{d}t \times \mathrm{d}m_h)$ is isomorphic to $L^2([T_1, T_2]; H_h)$ and $L^2([T_1, T_2] \times \overline{\Omega}, \mathrm{d}t \times \mathrm{d}m)$ is isomorphic to $L^2([T_1, T_2]; H)$. If we denote by $F_h = L^2([T_1, T_2]; H_h)$ and by $F = L^2([T_1, T_2]; H)$, it holds that F_h converges to F in the sense of Definition 3, where the set C is now $C([T_1, T_2] \times \overline{\Omega})$ and Z_h is the identity operator on C.
We denote by $\mathcal{F} = \{\cup_h F_h\} \cup F$. **In the following Proposition, we recall the characterization of strong convergence in $\mathcal{F}$ (by using Lemma 2 and 3).**

Proposition 9. *A sequence of vectors $\{u_h\}_{h\in\mathbb{N}}$ strongly converges to u in $\mathcal{F}$ if one of the following holds:*

$$i) \quad \begin{cases} \displaystyle\int_{T_1}^{T_2} \|u_h(t)\|^2_{H_h}\,\mathrm{d}t \xrightarrow[h\to+\infty]{} \int_{T_1}^{T_2} \|u(t)\|^2_H\,\mathrm{d}t \\ \displaystyle\int_{T_1}^{T_2} (u_h(t), \psi(t))_{H_h}\,\mathrm{d}t \xrightarrow[h\to+\infty]{} \int_{T_1}^{T_2} (u(t), \psi(t))_H\,\mathrm{d}t \end{cases} \tag{26}$$

for every $\psi \in C([T_1, T_2] \times \overline{\Omega})$;

$$ii) \quad \int_{T_1}^{T_2} (u_h(t), v_h(t))_{H_h}\,\mathrm{d}t \xrightarrow[h\to+\infty]{} \int_{T_1}^{T_2} (u(t), v(t))_H\,\mathrm{d}t \tag{27}$$

for every sequence $\{v_h\}_{h\in\mathbb{N}}$ strongly converging to v in $\mathcal{F}$.

Theorem 6. *Let $u(t,x) = S_\alpha(t)\varphi(x)$ and $u_h(t,x) = S^h_\alpha(t)\varphi_h(x)$ be the unique strong solutions of problems (P^0) and (P^0_h), for every $h \in \mathbb{N}$, according to Theorems 1 and 2 respectively. Let δ_h be as in Theorem 5. If $\{\varphi_h\}$ strongly converges to φ in $\mathcal{H}$ and there exists a constant $C > 0$ such that*

$$\|\varphi_h\|_{\overline{D(A_h)}} < C \quad \textit{for every } h \in \mathbb{N}, \tag{28}$$

then:

i) $\{u_h(t)\}$ *converges to* $u(t)$ *in* $\mathcal{H}$ *for every fixed* $t \in [T_1, T_2] \subset [0, T]$*;*
ii) $\{u_h\}$ *converges to* u *in* $\mathcal{F}$*.*

Proof. If $\alpha = 1$, the proof follows as in Theorem 5.3 in [34] with small changes. Let now $0 < \alpha < 1$.

First, we prove $i)$**. By using the characterization of the strong convergence given in Lemma 2, we have to prove that for every** $t \in [T_1, T_2] \subset [0, T]$

$$(u_h, v_h)_{H_h} \xrightarrow[n \to +\infty]{} (u, v)_H$$

for every sequence $\{v_h\}_{h\in\mathbb{N}}$ with $v_h \in H_h$ weakly converging in $\mathcal{H}$ to a vector $v \in H$.

We first point out that, from Theorem 5, Theorem 2.8 in [30] and Theorem 2.4 in [32], it follows that for every $t \in [T_1, T_2]$

$$T_h(t)\varphi_h \xrightarrow[n \to +\infty]{} T(t)\varphi \quad \text{in } \mathcal{H} \tag{29}$$

since $\varphi_h \to \varphi$ in $\mathcal{H}$ (see Definition 6).

From the representation formula (25) of Theorem 2 we have

$$(u_h, v_h)_{H_h} = \int_{\Omega_h} S^h_\alpha \varphi_h\, v_h \,\mathrm{d}\mathcal{L}_2 + \delta_h \int_{K_h} S^h_\alpha \varphi_h\, v_h \,\mathrm{d}\ell$$

and

$$(u, v)_H = \int_\Omega S_\alpha \varphi\, v \,\mathrm{d}\mathcal{L}_2 + \int_K S_\alpha \varphi\, v \,\mathrm{d}\mu.$$

Recalling the definitions of S^h_α and S_α, we obtain that

$$\begin{aligned}(u_h, v_h)_{H_h} - (u, v)_H &= \int_0^\infty \Phi_\alpha(\tau)\left(\int_{\Omega_h} T_h(\tau t^\alpha)\varphi_h\, v_h \,\mathrm{d}\mathcal{L}_2 - \int_\Omega T(\tau t^\alpha)\varphi\, v \,\mathrm{d}\mathcal{L}_2\right)\mathrm{d}\tau \\ &+ \int_0^\infty \Phi_\alpha(\tau)\left(\delta_h \int_{K_h} T_h(\tau t^\alpha)\varphi_h\, v_h \,\mathrm{d}\ell - \int_K T(\tau t^\alpha)\varphi\, v \,\mathrm{d}\mu\right)\mathrm{d}\tau = \\ &= \int_0^\infty \Phi_\alpha(\tau)\left[(T_h(\tau t^\alpha)\varphi_h, v_h)_{H_h} - (T(\tau t^\alpha)\varphi, v)_H\right]\mathrm{d}\tau.\end{aligned}$$

From (29) and the weak convergence of v_h to v, we have that for every $t \in [T_1, T_2]$

$$(T_h(\tau t^\alpha)\varphi_h, v_h)_{H_h} \to (T(\tau t^\alpha)\varphi, v)_H.$$

By using Lemma 1, (28) and the contraction property of T_h **we have that there exists a constant** $C > 0$ **(independent from** h**) such that**

$$\left|(T_h(\tau t^\alpha)\varphi_h, v_h)_{H_h}\right| \leq C.$$

From the dominated convergence theorem, the claim follows directly.

Now we prove $ii)$. From Proposition 9 we have to prove that

$$\|u_h\|_{F_h} \to \|u\|_F, \tag{30}$$

$$(u_h, \psi)_{F_h} \to (u, \psi)_F \quad \forall\, \psi \in C([T_1, T_2] \times \overline{\Omega}). \tag{31}$$

We note that

$$\|u_h(t)\|_{H_h} \leq \int_0^{+\infty} \Phi_\alpha(\tau)\|T_h(\tau t^\alpha)\varphi_h\|_{H_h} \,\mathrm{d}\tau \leq C \quad \forall\, t \in [T_1, T_2],$$

where the last inequality follows from the properties of the Wright function Φ_α, Proposition 7 and (28).

Thus, the sequence $\left\{\|u_h(t)\|_{H_h}\right\}$ is equibounded in $[T_1, T_2]$. Moreover, from $i)$ we have that for every $t \in [T_1, T_2]$

$$\|u_h(t)\|_{H_h} \to \|u(t)\|_H.$$

Hence, from the dominated convergence theorem, (30) is achieved.

We now go to (31). From $i)$ we have that for every $t \in [T_1, T_2]$

$$(u_h(t), \psi(t))_{H_h} \xrightarrow[n\to+\infty]{} (u(t), \psi(t))_H \quad \forall\, \psi \in C([T_1, T_2] \times \overline{\Omega}).$$

Since

$$\left|(u_h(t), \psi(t))_{H_h}\right| \le C\|\psi\|_{C([T_1,T_2]\times\overline{\Omega})},$$

the dominated convergence theorem yields

$$(u_h, \psi)_{F_h} \xrightarrow[n\to+\infty]{} (u, \psi)_F.$$

□

Remark 1. *We note that the convergence of φ_h to φ in $\mathcal{H}$ and the equi-boundeness hypothesis (28) imply the convergence in $\mathcal{F}$.*

Remark 2. *We stress the fact that the geometry considered in this paper is a prototype. Actually, our results can be extended to the case of domains whose boundaries are quasi-filling variable Koch curves. Indeed, Theorem 5 can be extended to these geometries by adapting Theorem 3.2 in [9] to the framework of varying Hilbert spaces, thus allowing us to state a result analogous to Theorem 6.*

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