Es 6:

$$A(K) = \begin{pmatrix} 0 & K & K^2 \\ -K-1 & K-1 & 1-K \\ 1 & (K-1)^2 & K-1 \end{pmatrix}$$

Calcolare $\operatorname{rg} A(K)$ con il teorema degli orlati.

Sol.: c'è un minore 1×1 diverso da zero $\forall K$?

Si:

$$A(K) = \begin{pmatrix} 0 & K & K^2 \\ -K-1 & K-1 & 1-K \\ 1 & (K-1)^2 & K-1 \end{pmatrix}$$

righe colonne

$A(K)([3],[1])$

2 2

1 3

Quindi $\operatorname{rg} A(K) \geq 1 \quad \forall K$.

Orliamolo in tutti i modi possibili:

1) $$A(K) = \begin{pmatrix} 0 & K & K^2 \\ -K-1 & K-1 & 1-K \\ 1 & (K-1)^2 & K-1 \end{pmatrix} \qquad A(K)([3]\cup[2],[1]\cup[2])$$

$$\det\begin{pmatrix} -K-1 & K-1 \\ 1 & (K-1)^2 \end{pmatrix} = \det\begin{pmatrix} 0 & K-1+(K+1)(K-1)^2 \\ 1 & (K-1)^2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 0 & K-1+(K^2-1)(K-1) \\ 1 & (K-1)^2 \end{pmatrix} = \det\begin{pmatrix} 0 & (K-1)K^2 \\ 1 & (K-1)^2 \end{pmatrix}$$

1) $$A(K) = \begin{pmatrix} 0 & K & K^2 \\ -K-1 & K-1 & 1-K \\ 1 & (K-1)^2 & K-1 \end{pmatrix}$$

$$\det\begin{pmatrix} -K-1 & K-1 \\ 1 & (K-1)^2 \end{pmatrix} = \det\begin{pmatrix} 0 & K-1+(K+1)(K-1)^2 \\ 1 & (K-1)^2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 0 & K-1+(K^2-1)(K-1) \\ 1 & (K-1)^2 \end{pmatrix} = \det\begin{pmatrix} 0 & (K-1)K^2 \\ 1 & (K-1)^2 \end{pmatrix}$$

$$= (K-1)\det\begin{pmatrix} 0 & K^2 \\ 1 & K-1 \end{pmatrix} = -K^2(K-1)$$

2) $$A(K) = \begin{pmatrix} 0 & K & K^2 \\ -K-1 & K-1 & 1-K \\ 1 & (K-1)^2 & K-1 \end{pmatrix}$$

$A(K)([3]\cup[1],[1]\cup[2])$

$A(K)([1,3],[1,2])$

$$\det\begin{pmatrix} 0 & K \\ 1 & (K-1)^2 \end{pmatrix} = -K$$

3) $$A(K) = \begin{pmatrix} 0 & K & K^2 \\ -K-1 & K-1 & 1-K \\ 1 & (K-1)^2 & K-1 \end{pmatrix}$$

$$\det\begin{pmatrix} 0 & K^2 \\ 1 & K-1 \end{pmatrix} = -K^2$$

4) $$A(K) = \begin{pmatrix} 0 & K & K^2 \\ -K-1 & K-1 & 1-K \\ 1 & (K-1)^2 & K-1 \end{pmatrix}$$

$$\det \begin{pmatrix} -K-1 & 1-K \\ 1 & K-1 \end{pmatrix} = (K-1) \det \begin{pmatrix} -K-1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= (K-1) \det \begin{pmatrix} -K & -1 \\ 0 & 1 \end{pmatrix} = (K-1)(-K)$$

I minori che orlano $\det A(K)([3],[1])$ sono
$-K^2(K-1)$, $-K$, $-K^2$, $-K(K-1)$.

Per il teorema dei minori orlati,
$rg(A(K)) = 1 \iff$ questi 4 minori sono $= 0$.
$\iff K = 0$.

Se $K \neq 0$, il minore $-K = \det A(K)([1,3],[1,2])$ è diverso da zero. Orliamolo:

$$A(K) = \begin{pmatrix} 0 & K & K^2 \\ -K-1 & K-1 & 1-K \\ 1 & (K-1)^2 & K-1 \end{pmatrix}$$

C'è un solo orlato che è $\det A(K)$;
Calcoliamolo:

$$A(k) = \begin{pmatrix} 0 & k & k^2 \\ -k-1 & k-1 & 1-k \\ 1 & (k-1)^2 & k-1 \end{pmatrix}$$

$(k \neq 0)$

$$\det A(k) = \det \begin{pmatrix} 0 & k & k^2 \\ 0 & k^2(k-1) & k(k-1) \\ 1 & (k-1)^2 & k-1 \end{pmatrix}$$

$$= \det \begin{pmatrix} k & k^2 \\ k^2(k-1) & k(k-1) \end{pmatrix}$$

$$= k^2(k-1) \det \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix}$$

$$= k^2(k-1)(1-k^2)$$

$$= k^2(k-1)(1+k)(1-k)$$

$$= -k^2(k-1)^2(k+1)$$

$\det A(k) = 0 \iff (k=0)$ oppure $k=1$
oppure $k=-1$

$rg\, A(k) = 2 \iff k=1$ oppure $k=-1$.

$rg\, A(k) = 3 \iff k \in \mathbb{R} \setminus \{0,1,-1\} \iff k \neq 0$ e $k \neq 1$ e $k \neq -1$.

Es1.2:

$\det B =$

a) $-(1+i)\det\begin{pmatrix} 2i & 2+2i \\ -2+i & 3i \end{pmatrix} +$

$+ (-i)\det\begin{pmatrix} 2-i & 1-i \\ -2+i & 3i \end{pmatrix} +$

$-(1+i)\det\begin{pmatrix} 2-i & 1-i \\ 2i & 2+2i \end{pmatrix} =$

$= -(1+i)\det\begin{pmatrix} 2i & 2 \\ -2+i & 2i+2 \end{pmatrix}$

$+(-i)\quad\det\begin{pmatrix} 2-i & 1-i \\ 0 & 2i+1 \end{pmatrix}$

$-(1+i)\det\begin{pmatrix} 1 & 1-i \\ -2 & 2+2i \end{pmatrix}$

$= -(1+i)\,2\det\begin{pmatrix} i & 1 \\ -2+i & 2i+2 \end{pmatrix} +$

$-i\,(2-i)(2i+1) - (1+i)\det\begin{pmatrix} 1 & 1-i \\ 0 & 4 \end{pmatrix}$

$= -(1+i)\,2\det\begin{pmatrix} 0 & 1 \\ -i & 2i+2 \end{pmatrix} - i\,(2-i)(2i+1)$

$-4(1+i) = -2i(1+i) - i(2-i)(2i+1) - 4(1+i)$

$$2i(\underline{1+i}) - i(2-i)(2i+1) - 4(\underline{1+i})$$

$$= (1+i)(2i-4) - i(2-i)(2i+1)$$

$$= 2(1+i)(i-2) - i(2-i)(2i+1)$$

$$= (i-2)[2(1+i) + i(2i+1)]$$

$$= (i-2)[\underline{2} + 2i \underline{-2} + i]$$

$$= 3i(i-2)$$

$$= -3-6i \qquad \text{(SBAGLIATO)}$$

Es 3 :

$\mathcal{B} = (v_1, v_2, v_3, v_4) \subset \mathbb{R}^4$

$\mathcal{B}$ è lin. ind. $\Leftrightarrow$

$(F_e(v_1), F_e(v_2), F_e(v_3), F_e(v_4)) \subset \mathbb{R}^4$

è lin. ind. (dove $e = (e_1, e_2, e_3, e_4)$)

$\Leftrightarrow \quad \det(F_e(v_1) | F_e(v_2) | F_e(v_3) | F_e(v_4)) \neq 0$

$\Leftrightarrow \quad \det B \neq 0 \quad$ dove

$F_e = id$

$$B = (v_1 | v_2 | v_3 | v_4) = \begin{pmatrix} 1 & 4 & 4 & 1 \\ 2 & 2 & 3 & 2 \\ 4 & 2 & 3 & 4 \\ 3 & 1 & 4 & 4 \end{pmatrix}$$

$\det B = 8 \neq 0.$

2) $\boxed{F_{\mathcal{B}} = S_{\mathcal{B}}^{-1}}$ $\qquad F_{\mathcal{B}}(w) = B^{-1} w$

è l'unica soluzione del sistema

$$B X = w$$

$$X = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \qquad y_1 = \frac{1}{\det B} \det(w | v_2 | v_3 | v_4)$$

$$y_2 = \frac{1}{\det B} \det(v_1 | w | v_3 | v_4)$$

$$y_1 = \frac{1}{\det B} \det \left(w \mid v_2 \mid v_3 \mid v_4 \right) = -65$$

$$y_2 = \frac{1}{\det B} \det \left(v_1 \mid w \mid v_3 \mid v_4 \right) = 47$$

$$y_3 = \frac{1}{\det B} \det \left(v_1 \mid v_2 \mid w \mid v_4 \right) = -50$$

$$y_4 = \frac{1}{\det B} \det \left(v_1 \mid v_2 \mid v_3 \mid w \right) = 85$$

<u>OSS</u>: $w = 8 \begin{pmatrix} 1 \\ -16 \\ 3 \\ -1 \end{pmatrix} = \det B \begin{pmatrix} 1 \\ -16 \\ 3 \\ -1 \end{pmatrix}$

MATLAB:

$X = B \backslash w$

$F_{\mathcal{B}} = S_B^{-1}$ $\qquad \mathcal{B} \subset \mathbb{R}^n$ base.

$\mathcal{B} = (v_1, v_2, \dots, v_n)$

$F_{\mathcal{B}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ lineare

$$v = x_1 v_1 + \dots + x_n v_n \longmapsto \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$v_1 \longmapsto e_1$$
$$v_2 \longmapsto e_2$$
$$\vdots$$
$$v_n \longmapsto e_n$$

$$\mathbb{R}^n \longleftarrow \mathbb{R}^n : F_{\mathcal{B}}^{-1}$$

$$v_1 \longleftarrow\!\shortmid\ e_1$$
$$v_2 \longleftarrow\!\shortmid\ e_2$$
$$\vdots$$
$$v_n \longleftarrow\!\shortmid\ e_n$$

$F_{\mathcal{B}}^{-1} = S_B$ $\qquad$ dove $B = (v_1 | \cdots | v_n)$

$\Rightarrow \quad F_{\mathcal{B}} = S_B^{-1} = S_{B^{-1}}$.

$$\det \left(\begin{array}{cc|ccc|cc} x & x & x & x & x & x & x \\ x & x & x & x & x & x & x \\ \hline 0 & 0 & x & x & x & x & x \\ 0 & 0 & x & x & x & x & x \\ 0 & 0 & x & x & x & x & x \\ \hline 0 & 0 & 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x \end{array}\right) = \det A \det B \det C$$

Calcoliamo con il Teorema dei minori orlati

$$\mathrm{rg}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & -1 & -1 & -2 & -3 \\ 3 & 1 & 2 & 2 & 2 \end{pmatrix}$$

Sol: c'è un 2-minore diverso da zero:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & -1 & -1 & -2 & -3 \\ 3 & 1 & 2 & 2 & 2 \end{pmatrix}$$

orliamolo:

1) $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & -1 & -1 & -2 & -3 \\ 3 & 1 & 2 & 2 & 2 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 3 & 1 & 2 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -5 & -7 \end{pmatrix} = 0$$

2) $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & -1 & -1 & -2 & -3 \\ 3 & 1 & 2 & 2 & 2 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 2 & 4 \\ 2 & -1 & -2 \\ 3 & 1 & 2 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 4 \\ 0 & -5 & -10 \\ 0 & -5 & -10 \end{pmatrix} = 0$$

3) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & -1 & -1 & -2 & -3 \\ 3 & 1 & 2 & 2 & 2 \end{pmatrix}$

$$\det\begin{pmatrix} 1 & 2 & 5 \\ 2 & -1 & -3 \\ 3 & 1 & 2 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 5 \\ 0 & -5 & -13 \\ 0 & -5 & -13 \end{pmatrix} = 0$$

$\Rightarrow \quad rg\, A = 2.$

Es:

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \\ 7 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 3 \\ 7 \\ 80 \\ 100 \end{pmatrix}$$

Questo minore è $\neq 0$ quindi i due vettori sono lin. ind.