$$\pi = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle \subset \mathbb{R}^3$$

Eq. cartesiane? Determinare C : $\operatorname{Ker} C = \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$

$$\left(\begin{array}{cc|c} 1 & 1 & x_1 \\ 2 & -1 & x_2 \\ 1 & 1 & x_3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & x_1 \\ 0 & -3 & x_2 - 2x_1 \\ \boxed{0 \quad 0} & & x_3 - x_1 \end{array}\right)$$

$$\Rightarrow \pi_0 = \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle : x_3 - x_1 = 0$$

Poiché $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \pi_0$, $\pi = \pi_0 : x_3 - x_1 = 0$.

Da cartesiana a parametrica: risolvere il sistema.

$$r: \begin{cases} 2x_1 + x_2 - x_3 = 1 \\ 3x_1 - x_2 + 2x_3 = 2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 3 & -1 & 2 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & -2 & 3 & 1 \end{array}\right)$$

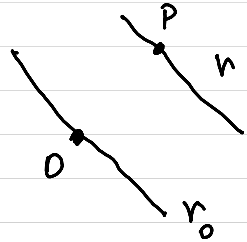
$$\rightsquigarrow \left(\begin{array}{ccc|c} 0 & 5 & -7 & -1 \\ 1 & -2 & 3 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & -1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -\frac{7}{5} & -\frac{1}{5} \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1/5 & 3/5 \\ 0 & 1 & -\frac{7}{5} & -\frac{1}{5} \end{array}\right)$$

Il sistema è equivalente al sistema a scala ridotta

$$\begin{cases} x_1 + \frac{1}{5}x_3 = 3/5 \quad 0 \\ x_2 - \frac{7}{5}x_3 = -\frac{1}{5} \quad 0 \end{cases} \qquad \begin{pmatrix} 3/5 \\ -1/5 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ 7 \\ 5 \end{pmatrix} \right\rangle \checkmark$$

$$r = \underbrace{\begin{pmatrix}1\\2\\1\end{pmatrix}}_{P=} + \left\langle \underbrace{\begin{pmatrix}1\\1\\1\end{pmatrix}}_{=r_0} \right\rangle \subset \mathbb{R}^3$$



$$\begin{cases} 2x_1 - x_2 - x_3 = 0 \\ x_1 - x_2 = 0 \end{cases} \qquad \begin{cases} x_2 - x_1 = 0 \\ x_3 - x_1 = 0 \end{cases}$$

$$\left(\begin{array}{c|c} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{array}\right) \rightsquigarrow \left(\begin{array}{c|c} 1 & x_1 \\ \boxed{0} & \boxed{x_2 - x_1} \\ \boxed{0} & \boxed{x_3 - x_1} \end{array}\right) \rightsquigarrow \boxed{r_0 : \begin{cases} x_2 - x_1 = 0 \\ x_3 - x_1 = 0 \end{cases}}$$

$$\left(\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right)$$

C

$$C = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad \operatorname{Ker} C = \operatorname{Col}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)$$

$$\operatorname{Ker} C = \left\{ X = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \;\middle|\; CX = \begin{pmatrix}0\\0\end{pmatrix} \right\} = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} -x_1 + x_2 = 0 \\ -x_1 + x_3 = 0 \end{matrix} \right\}$$

$$: \begin{cases} -x_1 + x_2 = 0 \\ -x_1 + x_3 = 0 \end{cases}$$

$$\begin{matrix} 2 - 1 = 1 \\ 1 - 1 = 0 \end{matrix} \qquad r :\; CX = CP$$

$$r : \begin{cases} x_2 - x_1 = 1 \\ x_3 - x_1 = 0 \end{cases}$$

Eq. cart. di:

$$U = \left\langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle = \mathrm{Col} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 1 & x_1 \\ 1 & 2 & x_2 \\ 2 & 1 & x_3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & x_1 \\ 0 & 1 & x_2 - x_1 \\ 0 & -1 & x_3 - 2x_1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} & & \\ \hline 0 & 0 & x_3 - 2x_1 + x_2 - x_1 \end{array}\right)$$

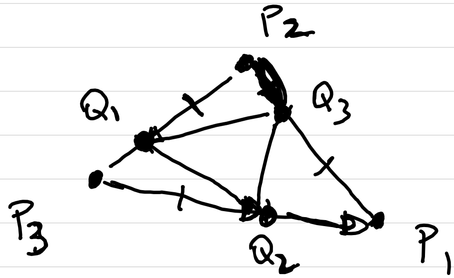
$$U : \quad -3x_1 + x_2 + x_3 = 0$$

$$\left(\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & -2 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} & & \\ 0 & 0 & \underbrace{\boxed{-3 \quad 1 \quad 1}}_{C} \end{array}\right) \qquad C = (-3 \quad 1 \quad 1)$$

$\boxed{U = \mathrm{Ker}\, C}$ Eq. cart.

$$U : \quad -3x_1 + x_2 + x_3 = 0$$



$Q_1 \in \overline{P_2 P_3}$

$\vec{P_2 Q_1} = 2\, \vec{Q_1 P_3}$

$\vec{AB} = B - A$

$$\det(Q_2 - Q_1 \mid Q_3 - Q_1) = \frac{1}{3} \det(P_1 - P_3 \mid P_2 - P_3)$$

$\Rightarrow$ Area $\overset{\triangle}{Q_1 Q_2 Q_3} = \frac{1}{2} |\det(Q_2 - Q_1 \mid Q_3 - Q_1)|$
$= \frac{1}{3}$

$\vec{P_2 Q_1} = 2\, \vec{Q_1 P_3}$

$Q_1 - P_2 = 2(P_3 - Q_1) \Rightarrow Q_1 = P_2 + 2P_3 - 2Q_1$

$\Rightarrow 3Q_1 = P_2 + 2P_3 \Rightarrow \boxed{Q_1 = \frac{1}{3} P_2 + \frac{2}{3} P_3.}$

Similmente, $Q_2 = \frac{1}{3} P_3 + \frac{2}{3} P_1$

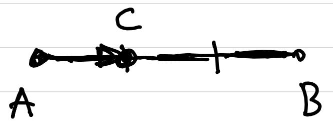
$Q_3 = \frac{1}{3} P_1 + \frac{2}{3} P_2.$

$\det(Q_2 - Q_1 \mid Q_3 - Q_1) =$

$$= \det(Q_2 - P_3 + P_3 - Q_1 \mid Q_3 - P_2 + P_2 - Q_1)$$

$$= \det\left(\frac{2}{3}(P_1 - P_3) + \frac{1}{3}(P_3 - P_2) \,\middle|\, \frac{1}{3}(P_1 - P_2) + \frac{2}{3}(P_2 - P_3)\right)$$

$$= \det\left(\frac{2}{3} P_1 - \frac{1}{3} P_3 - \frac{1}{3} P_2 \,\middle|\, \frac{1}{3} P_1 + \frac{1}{3} P_2 - \frac{2}{3} P_3\right)$$



$$\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB}$$

$$= \det\left(\frac{2}{3}P_1 - \frac{1}{3}P_3 - \frac{1}{3}P_2 \;\middle|\; \frac{1}{3}P_1 + \frac{1}{3}P_2 - \frac{2}{3}P_3\right)$$

$$= \det\left(\frac{1}{3}(P_1 - P_3) + \frac{1}{3}(P_1 - P_2) \;\middle|\; \frac{1}{3}(P_1 - P_3) + \frac{1}{3}(P_2 - P_3)\right)$$

$$= \frac{1}{9}\Big[\det(P_1 - P_3 \mid P_2 - P_3) + \det(P_1 - P_2 \mid P_1 - P_3) +$$
$$+ \det(P_1 - P_2 \mid P_2 - P_3)\Big]$$

$$= \frac{1}{9}\Big[\det(P_1 - P_3 \mid P_2 - P_3) + \det(P_3 - P_2 \mid P_1 - P_3) +$$
$$+ \det(P_1 - P_3 \mid P_2 - P_3)\Big] =$$

$$= \frac{1}{9}\Big[3 \det(P_1 - P_3 \mid P_2 - P_3)\Big] = \frac{1}{3}\det(P_1 - P_3 \mid P_2 - P_3)$$

$P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad r: \ 2x + 3y = 5$

$r_0 = \mathrm{Ker}\,(2\ 3) = \left\langle \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\rangle$

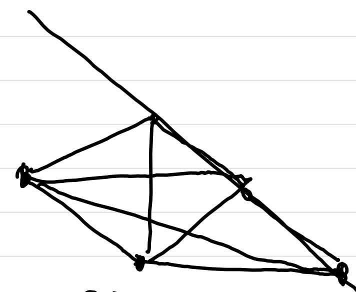
$S = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\rangle$ (retta cercata)

$S_0: \ 2x + 3y = 0$

$2 + 3 = 5$

$S: \ 2x + 3y = 5. \quad r \equiv s.$

$r: \ 2x + 3y = 4$



$P_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$P_1 - P_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \neq \left\langle \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\rangle = r_0$

$R \in r \qquad r: \ x + \frac{3}{2} y = 2$

$R = \begin{pmatrix} 2 - \frac{3}{2} y \\ y \end{pmatrix}$

$\det(R - P_2, P_1 - P_2) = \det \begin{pmatrix} 1 - \frac{3}{2} y & 0 \\ y & 1 \end{pmatrix} = 1 - \frac{3}{2} y$

$\{ y \in \mathbb{R} \mid |1 - \frac{3}{2} y| = 4 \}$

$$\left|1-\frac{3}{2}y\right| = 4$$

$1-\frac{3}{2}y=4$

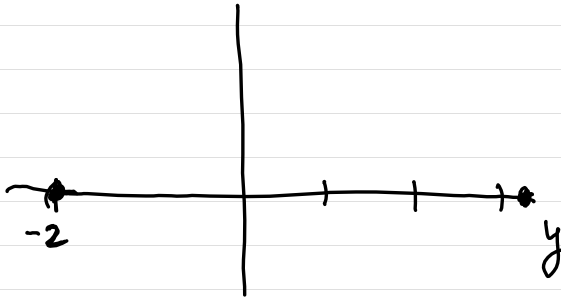
$\frac{3}{2}y=-3$

$y=-2$

$\frac{3}{2}y-1=4$

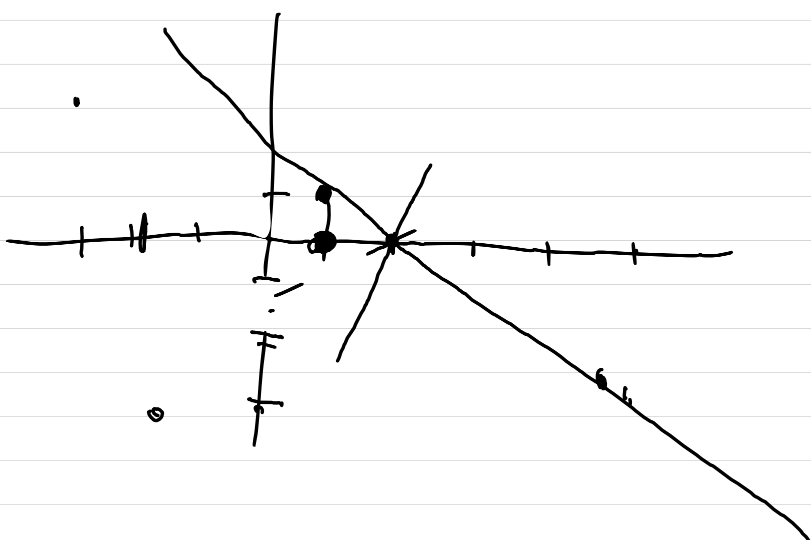
$\frac{3}{2}y=5$

$y=\frac{10}{3}$

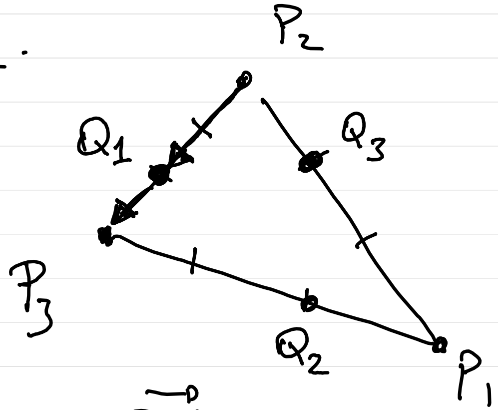


$R_1 = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ $\uparrow$ $y=-2$

$R_2 = \begin{pmatrix} -2 \\ \frac{10}{3} \end{pmatrix}$ $\uparrow$ $y=\frac{10}{3}$



Es 3.



$Q_1 \in \overline{P_2 P_3}$

$\overrightarrow{P_2 Q_1} = 2\,\overrightarrow{Q_1 P_3}$

$$\overrightarrow{P_2 Q_1} = \frac{2}{3}\,\overrightarrow{P_2 P_3}$$

$$Q_1 - P_2 = \frac{2}{3}(P_3 - P_2) \Rightarrow Q_1 = \frac{1}{3}P_2 + \frac{2}{3}P_3$$

Similmente,

$$Q_2 = \frac{1}{3}P_3 + \frac{2}{3}P_1$$

$$Q_3 = \frac{1}{3}P_1 + \frac{2}{3}P_2.$$

$$r_{P_1P_2} : \quad P_1 + \langle \overrightarrow{P_1P_2} \rangle = P_1 + \langle P_2 - P_1 \rangle$$

Es:

$$r = \underbrace{\begin{pmatrix}4\\3\\7\end{pmatrix}}_{= x_1} + \left\langle \underbrace{\begin{pmatrix}1\\2\\1\end{pmatrix}}_{= v_1} \right\rangle \qquad s = \underbrace{\begin{pmatrix}1\\-2\\2\end{pmatrix}}_{= x_2} + \left\langle \underbrace{\begin{pmatrix}2\\3\\4\end{pmatrix}}_{= v_2} \right\rangle$$

Sol.:

Sono parallele? $\left\langle \begin{pmatrix}1\\2\\1\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}2\\3\\4\end{pmatrix} \right\rangle$

$\Leftrightarrow \begin{pmatrix}1\\2\\1\end{pmatrix} \in \left\langle \begin{pmatrix}2\\3\\4\end{pmatrix} \right\rangle$ che è falso

Quindi non sono parallele.

Sono incidenti? Vuol dire che

$\exists\, t, s \in \mathbb{R}$ t.c.

$$\begin{pmatrix}4\\3\\7\end{pmatrix} + t\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\2\end{pmatrix} + s\begin{pmatrix}2\\3\\4\end{pmatrix}$$

$\Leftrightarrow \exists\, t, s \in \mathbb{R}$ t.c.

$$t\begin{pmatrix}1\\2\\1\end{pmatrix} - s\begin{pmatrix}2\\3\\4\end{pmatrix} = \begin{pmatrix}1\\-2\\2\end{pmatrix} - \begin{pmatrix}4\\3\\7\end{pmatrix} = \begin{pmatrix}-3\\-5\\-5\end{pmatrix}$$

$\Leftrightarrow \exists\, t, s \in \mathbb{R}$ t.c.

$$\begin{pmatrix}1 & 2\\2 & 3\\1 & 4\end{pmatrix}\begin{pmatrix}t\\-s\end{pmatrix} = \begin{pmatrix}-3\\-5\\-5\end{pmatrix}$$

$$(v_1\ v_2)\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = X_1 - X_2 \qquad \leftarrow !!$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 5 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 4 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

L'unica soluzione è

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Quindi

$$t_1 \underline{v_1} + t_2 v_2 = X_1 - X_2$$

$$\Rightarrow \quad X_1 - t_1 \underline{v_1} = X_2 + t_2 v_2 = P_0$$

$$P_0 = X_1 - \underline{v_1} = X_2 + v_2 = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

Es 4:

$$\pi_1 = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \left\langle \begin{pmatrix}2\\-1\\3\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix} \right\rangle$$

$$\pi_2 : 2x+3y+z=1$$

Sol.:

Paralleli? $(\pi_1)_0 = (\pi_2)_0$?

$\Leftrightarrow$ $\begin{pmatrix}2\\-1\\3\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix} \in (\pi_2)_0 : 2x+3y+z=0$ Falso

perché $\begin{pmatrix}2\\-1\\3\end{pmatrix} \notin (\pi_2)_0$.

$\Rightarrow$ Non sono paralleli

Incidenza?

$\pi_1 = X_0 + \langle v_1, v_2 \rangle \qquad \pi_2 : AX = b$

$X_0 + t_1 v_1 + t_2 v_2 \in \pi_2 \Leftrightarrow$

$A(X_0 + t_1 v_1 + t_2 v_2) = b \Leftrightarrow$

$t_1 A v_1 + t_2 A v_2 = b - AX_0 \Leftrightarrow$

$(Av_1 \mid Av_2) \begin{pmatrix}t_1\\t_2\end{pmatrix} = b - AX_0$ è risolubile

Risolviamo: $(Av_1 | Av_2)\, X = b - A X_0$

$$(Av_1 | Av_2) = \left((2\ 3\ 1)\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \middle| (2\ 3\ 1)\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right)$$

$$= (4 \quad 7)$$

$$b - AX_0 = 1 - (2\ 3\ 1)\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 1 - 4 = -3$$

$$4t_1 + 7t_2 = -3$$

$$\Leftrightarrow \quad t_1 + \frac{7}{4} t_2 = -\frac{3}{4} \qquad \text{(a scala ridotta)}$$

Soluzioni:

$$\begin{pmatrix} -3/4 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -7 \\ 4 \end{pmatrix} \right\rangle$$

soluzione-base di $AX = 0$

$$t_1 + \frac{7}{4} t_2 = 0$$

$$t_1 = -\frac{7}{4} t_2$$

$$\pi_1 \cap \pi_2 = \{ X_0 + t_1 v_1 + t_2 v_2 \mid 4t_1 + 7t_2 = -3 \}$$

$$= \{ X_0 + t_1 v_1 + t_2 v_2 \mid t_1 = -\frac{3}{4} - \frac{7}{4} t_2 \}$$

$$= \{ X_0 - \frac{3}{4} v_1 - \frac{7}{4} t_2 v_1 + t_2 v_2 \mid t_2 \in \mathbb{R} \} =$$

$$= \{ X_0 - \frac{3}{4} v_1 + \left(-\frac{7}{4} v_1 + v_2\right) t_2 \mid t_2 \in \mathbb{R} \}$$

$$= X_0 - \frac{3}{4} v_1 + \langle -\frac{7}{4} v_1 + v_2 \rangle$$

$$= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \frac{3}{4} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \langle -7 v_1 + 4 v_2 \rangle$$

$$= \begin{pmatrix} -2/4 \\ 11/4 \\ -13/4 \end{pmatrix} + \langle -7 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \rangle$$

$$= \frac{1}{4} \begin{pmatrix} -2 \\ 11 \\ -13 \end{pmatrix} + \langle \begin{pmatrix} -10 \\ 11 \\ -13 \end{pmatrix} \rangle : \begin{cases} X_2 + \frac{11}{10} X_1 = \frac{11 \cdot 22}{10} \\ X_3 - \frac{13}{10} X_1 = \frac{3}{10} \cdot \frac{13}{4} \end{cases}$$

$$\frac{11}{4} - \frac{11}{10} \cdot \frac{2}{4} = \frac{11 \cdot 88}{40}$$

$$-\frac{13}{4} + \frac{13}{10} \cdot \frac{13}{4}$$

$$\left(\begin{array}{c|c} -10 & x_1 \\ 11 & x_2 \\ -13 & x_3 \end{array}\right) \to \left(\begin{array}{c|c} 1 & -\frac{1}{10} x_1 \\ 11 & x_2 \\ -13 & x_3 \end{array}\right)$$

$$\to \left(\begin{array}{c|c} 1 & -\frac{1}{10} x_1 \\ 0 & x_2 + \frac{11}{10} x_1 \\ 0 & x_3 - \frac{13}{10} x_1 \end{array}\right) \quad \begin{cases} X_2 + \frac{11}{10} X_1 = 0 \\ X_3 - \frac{13}{10} X_1 = 0 \end{cases}$$

$U: AX = b \qquad U \neq \emptyset$

$\Rightarrow U = X_0 + \operatorname{Ker} A$

$\operatorname{Ker} A = U_0$ = sottospazio di giacitura di U.

$$r: \begin{cases} 3x + 2y - z = 1 \\ 2x + 3y + 2z = 2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 2 & 3 & 2 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & -3 & -1 \\ 2 & 3 & 2 & 2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & -3 & -1 \\ 0 & 5 & 8 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & -3 & -1 \\ 0 & 1 & \frac{8}{5} & \frac{4}{5} \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -7/5 & -1/5 \\ 0 & 1 & \frac{8}{5} & \frac{4}{5} \end{array}\right)$$

$$r: \begin{cases} x_1 \qquad - \frac{7}{5} x_3 = -\frac{1}{5} \\ \qquad x_2 + \frac{8}{5} x_3 = \frac{4}{5} \end{cases}$$

$$r = \begin{pmatrix} -1/5 \\ 4/5 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 7/5 \\ -8/5 \\ 1 \end{pmatrix} \right\rangle$$ Forma parametrica.