$$\begin{tikzcd} \mathbb{R}^8 \arrow[r, "G"] \arrow[rr, bend right, "S_A = F \circ G"'] & W \arrow[r, "F"] & \mathbb{R}^5 \end{tikzcd}$$

$w_1 = 1$
$w_2 = \sin x$

$$F(f(x)) = \begin{pmatrix} f(0) \\ f(\pi/6) \\ f(\pi/4) \\ f(\pi/3) \\ f(\pi/2) \end{pmatrix}$$

$1 \quad \sin x \quad \cos x$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1/2 \\ 1 & \sqrt{2}/2 \\ 1 & \sqrt{3}/2 \\ 1 & 1 \end{pmatrix}$$

$$A^i = S_A(e_i) = F(G(e_i)) = F(w_i)$$

$w_1 = 1 \quad w_2 = \sin x \quad w_3 = \cos x \quad w_4 = \cos^2 x \quad w_5 = \sin^2 x$

$w_6 = \cos x \sin x \quad w_7 = \cos 2x \quad w_8 = \sin 2x$

$$w_1 = w_4 + w_5$$
$$w_7 = w_4 - w_5$$
$$w_8 = 2 w_6$$

$$A \rightsquigarrow S \Rightarrow \operatorname{rg} A = 5$$

Es 2 : $\mathcal{B} = (v_1, \dots, v_n) \overset{\text{base}}{\subset} V$

$F_{\mathcal{B}} : V \longrightarrow \mathbb{K}^n$

$$F_{\mathcal{B}}(x_1 v_1 + \dots + x_n v_n) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{aligned} v_1 &\longmapsto e_1 \\ v_2 &\longmapsto e_2 \\ &\vdots \\ v_n &\longmapsto e_n \end{aligned}$$

$V = \mathbb{K}^n$ $\quad F_{\mathcal{B}} : \mathbb{K}^n \longrightarrow \mathbb{K}^n$

$$\begin{aligned} v_1 &\longmapsto e_1 \\ v_2 &\longmapsto e_2 \\ &\vdots \\ v_n &\longmapsto e_n \end{aligned}$$

$F_{\mathcal{B}} = S_B$

$B^i = S_B(e_i) = F_{\mathcal{B}}(e_i)$ $\qquad F_{\mathcal{B}}^{-1} = S_C$

$F_{\mathcal{B}}^{-1} : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ $\qquad C = (v_1 | v_2 | \dots | v_n)$

$$\begin{aligned} e_1 &\longmapsto v_1 \\ e_2 &\longmapsto v_2 \\ &\vdots \\ e_n &\longmapsto v_n \end{aligned}$$

$C = B^{-1}$

$\Rightarrow B = C^{-1}$

$$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right)$$

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}^{-1}$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\mathcal{B} = (1+x,\ 1-x,\ 2-3x) \subset \mathbb{R}[x]_{\leq 1} = \langle 1, x \rangle$

$\mathcal{e} = (1, x)$

$F_e(1+x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $F_e(1-x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $F_e(2-3x) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$F_e : W \to \mathbb{R}^2$ è un isom. lineare.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \end{pmatrix}$$

Le colonne dominanti sono A^1 e A^2
e quindi una base di W è

$$\left(F_e^{-1}(A^1),\ F_e^{-1}(A^2) \right) = (1+x,\ 1-x).$$

Es1:

$$\left(\begin{array}{cccc|cccc} 2 & -2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 2 & 0 & 1 & 0 & 0 \\ -2 & -1 & 0 & 2 & 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 \\ 2 & -3 & 0 & 2 & 0 & 1 & 0 & 0 \\ -2 & -1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & -2 & 4 & 0 & 1 & 0 & 2 \\ 0 & -3 & 2 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 & 1 & 0 & 0 & 2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & -4 & 0 & -1 & 0 & -2 \\ 0 & -3 & 2 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 & 1 & 0 & 0 & 2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & -4 & 0 & -1 & 0 & -2 \\ 0 & 0 & 8 & -12 & 0 & -3 & 1 & -8 \\ 0 & 0 & -1 & 2 & 1 & 0 & 0 & 2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & -4 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -2 & -1 & 0 & 0 & -2 \\ 0 & 0 & 8 & -12 & 0 & -3 & 1 & -8 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & -4 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -2 & -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 4 & 8 & -3 & 1 & 8 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & -4 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -2 & -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 4 & 8 & -3 & 1 & 8 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & -4 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -2 & -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 & -\frac{3}{4} & \frac{1}{4} & 2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & -1 & 1 & 0 & 2 & -3/4 & 1/4 & 1 \\ 0 & 1 & 2 & 0 & 8 & -4 & 1 & 6 \\ 0 & 0 & 1 & 0 & 3 & -3/2 & 1/2 & 2 \\ 0 & 0 & 0 & 1 & 2 & -\frac{3}{4} & \frac{1}{4} & 2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & -1 & 3/4 & -1/4 & -1 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 & -3/2 & 1/2 & 2 \\ 0 & 0 & 0 & 1 & 2 & -3/4 & 1/4 & 2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1/4 & -1/4 & 1 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 & -3/2 & 1/2 & 2 \\ 0 & 0 & 0 & 1 & 2 & -3/4 & 1/4 & 2 \end{array}\right)$$

Es 2 : $\mathcal{B} = (v_1, \dots, v_n) \overset{\text{base}}{\subset} V$

$$F_{\mathcal{B}} : V \longrightarrow \mathbb{K}^n$$
$$v_1 \longmapsto e_1$$
$$v_2 \longmapsto e_2$$
$$\vdots$$
$$v_n \longmapsto e_n$$

$V = \mathbb{K}^n$

$$F_{\mathcal{B}} : \mathbb{K}^n \longrightarrow \mathbb{K}^n$$
$$v_1 \longmapsto e_1$$
$$v_2 \longmapsto e_2$$
$$\vdots$$
$$v_n \longmapsto e_n$$

$F_{\mathcal{B}} = S_B$, $B^i = F_{\mathcal{B}}(e_i)$

$$F_{\mathcal{B}}^{-1} : \mathbb{K}^n \longrightarrow \mathbb{K}^n$$
$$e_1 \longmapsto v_1$$
$$e_2 \longmapsto v_2$$
$$\vdots$$
$$e_n \longmapsto v_n$$

$F_{\mathcal{B}}^{-1} = S_C$ dove $C = (v_1 | \dots | v_n) = B^{-1}$

$B = C^{-1}$

__Es 5:__ $W = \langle w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8 \rangle$

$w_1 = 1$, $w_2 = \sin x$, $w_3 = \cos x$, $w_4 = \cos^2 x$,

$w_5 = \sin^2 x$, $w_6 = \sin x \cos x$, $w_7 = \cos 2x$, $w_8 = \sin 2x$

$$w_1 = w_4 + w_5$$

$$w_7 = w_4 - w_5$$

$$w_8 = 2 w_6$$

$$\mathbb{R}^8 \xrightarrow{G} W \xrightarrow{F} \mathbb{R}^5 \qquad F(f(x)) = \begin{pmatrix} f(0) \\ f(\pi/6) \\ f(\pi/4) \\ f(\pi/3) \\ f(\pi/2) \end{pmatrix}$$

$$e_1 \longmapsto w_1$$
$$e_2 \longmapsto w_2$$
$$\vdots$$
$$e_i \longmapsto w_i$$

$$F \circ G = S_A \qquad A^1 = F(w_1) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad A^2 = F(w_2) = \begin{pmatrix} 0 \\ \sqrt{3}/2 \\ \sqrt{2}/2 \\ 1/2 \\ 1 \end{pmatrix} \dots$$

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & \sqrt{3}/2 & 1/2 & 1/4 & 3/4 & \sqrt{3}/4 & -1/2 & \sqrt{3}/2 \\ 1 & \sqrt{2}/2 & \sqrt{2}/2 & 1/2 & 1/2 & 1/2 & 0 & 1 \\ 1 & 1/2 & \sqrt{3}/2 & 3/4 & 1/4 & \sqrt{3}/4 & 1/2 & \sqrt{3}/2 \\ 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

$$\begin{tikzcd} \mathbb{R}^8 \arrow[r, "G"] \arrow[rr, bend right, "S_A"'] & W \arrow[r, "F"] & \mathbb{R}^5 \end{tikzcd}$$

$\operatorname{rg} A = 5$

$\Rightarrow S_A$ suriettiva

$\Rightarrow F \circ G$ suriettiva

$\Rightarrow F$ suriettiva