Proiezione ortogonale

Sia $U \subseteq \mathbb{R}^n$ un sottospazio vettoriale.

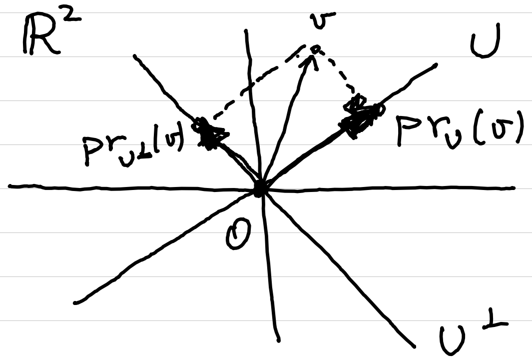
Abbiamo visto che $\mathbb{R}^n = U \oplus U^\perp$.

Def: La proiezione ortogonale è la funzione

$$pr_U^{U^\perp} : \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Notazione: $pr_U := pr_U^{U^\perp}$.

Es: $\mathbb{R}^2$



Sia $v \in \mathbb{R}^n$. Allora $\exists!\, u \in U$ e $u' \in U^\perp$ t.c.

$$v = u + u'$$

e $$pr_U(v) = u.$$

Oss: $v - pr_U(v) \in U^\perp$. Quindi,

$$(v - pr_U(v)) \cdot u = 0 \quad \forall u \in U.$$

Matrice di proiezione ortogonale

Sappiamo che $pr_U : \mathbb{R}^n \to \mathbb{R}^n$ è lineare. Quindi esiste una matrice $P_U \in Mat_{n \times n}(\mathbb{R})$ t.c. $pr_U = S_{P_U}$.

Com'è fatta P_U?

Per capirlo abbiamo bisogno del seguente interessante risultato:

<u>Prop.</u>: Sia $A \in Mat_{m \times n}(\mathbb{R})$. Allora

$$\operatorname{Ker} A^t A = \operatorname{Ker} A.$$

In particolare, $\operatorname{rg} A^t A = \operatorname{rg} A$.

<u>dim</u>: Sia $X \in \operatorname{Ker} A$ allora $A^t A X = 0_{\mathbb{R}^n}$

$\Rightarrow \operatorname{Ker} A \subseteq \operatorname{Ker} A^t A$. Viceversa se $A^t A X = 0_{\mathbb{R}^n}$

allora $(AX) \cdot (AX) = X^t A^t A X = X^t 0_{\mathbb{R}^n} = 0$

$$\Rightarrow AX = 0_{\mathbb{R}^m} \Rightarrow X \in \operatorname{Ker} A.$$

$$\Rightarrow \operatorname{Ker} A^t A \subseteq \operatorname{Ker} A.$$

$$\operatorname{rg} A^t A = n - \dim \operatorname{Ker} A^t A = n - \dim \operatorname{Ker} A = \operatorname{rg} A.$$

$\blacksquare$

Teorema: Sia $U \subseteq \mathbb{R}^n$ un s.sp. vettoriale. Sia $\mathcal{B}_U = (v_1, \dots, v_r)$ una base di U. Allora

$$P_U = A(A^tA)^{-1}A^t$$

dove $A = (v_1 | \dots | v_r) \in Mat_{n \times r}(\mathbb{R})$.

dim: Poichè $rg\, A = r$, allora anche $rg\, A^tA = r$ e quindi A^tA è invertibile. Quindi la matrice $A(A^tA)^{-1}A^t$ è ben definita.

Sia $X \in \mathbb{R}^n$. $P_U X \in U = \langle v_1, \dots, v_r \rangle = Col(A) \Rightarrow$

$\exists\, Y_0 \in \mathbb{R}^r$ t.c. $P_U X = AY_0$

Inoltre $X - P_U X \in U^\perp = Ker\, A^t$ (Teo di dec. ort.)

$$\Rightarrow \quad A^t(X - P_U X) = 0_{\mathbb{R}^r}$$

$$\Rightarrow \quad A^tX = A^tA.Y_0 \quad \Rightarrow \quad Y_0 = (A^tA)^{-1}A^tX$$

$$\Rightarrow \quad P_U(X) = AY_0 = A(A^tA)^{-1}A^tX.$$

$\blacksquare$

Es: $\quad U = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle \qquad A = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$$P_U = A\,(A^tA)^{-1}A^t =$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} \left((1\,1\,1) \begin{pmatrix}1\\1\\1\end{pmatrix} \right)^{-1} (1\,1\,1) = \frac{1}{3} \begin{pmatrix}1\\1\\1\end{pmatrix} (1\,1\,1)$$

$$= \frac{1}{3} \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}.$$

Notazione: $\quad U = \langle v \rangle, \quad pr_U(x) =: pr_v(x).$

Es: $\quad pr_{\binom{1}{2}} \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \left((1\,2) \begin{pmatrix}1\\2\end{pmatrix} \right)^{-1} (1\,2) \begin{pmatrix}2\\3\end{pmatrix}$

$$= \begin{pmatrix}1\\2\end{pmatrix} (5)^{-1} (1\,2) \begin{pmatrix}2\\3\end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix}1\\2\end{pmatrix} (1\,2) \begin{pmatrix}2\\3\end{pmatrix} = \frac{1}{5} \begin{pmatrix}1&2\\2&4\end{pmatrix} \begin{pmatrix}2\\3\end{pmatrix} =$$

$$= \frac{1}{5} \begin{pmatrix}8\\16\end{pmatrix} = \frac{8}{5} \begin{pmatrix}1\\2\end{pmatrix}$$

Calcolo di $pr_U(X)$ con una base ortogonale di U

Sia $U \leq \mathbb{R}^n$ un s.sp. vett.

Sia $\mathcal{B}_U = (F_1, F_2, \cdots, F_r)$ una base ortogonale di U. Allora $\forall X \in \mathbb{R}^n$,

$$pr_U(X) = \underbrace{\frac{X \cdot F_1}{F_1 \cdot F_1}} F_1 + \underbrace{\frac{X \cdot F_2}{F_2 \cdot F_2}} F_2 + \dots + \underbrace{\frac{X \cdot F_r}{F_r \cdot F_r}} F_r$$

coefficienti di Fourier

Infatti, poiché $\mathcal{B}_U$ è una base di U e $pr_U(X) \in U$:

$$pr_U(X) = \alpha_1 F_1 + \alpha_2 F_2 + \dots + \alpha_r F_r.$$

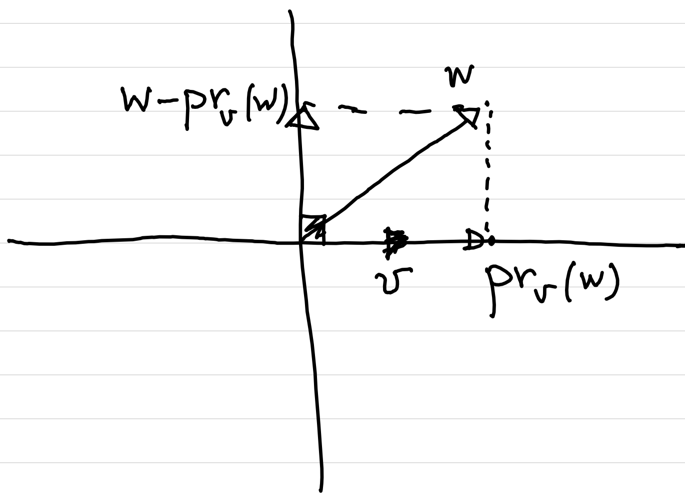
poiché $X - pr_U(X) \in U^\perp$, $\forall i = 1, \dots, r$:

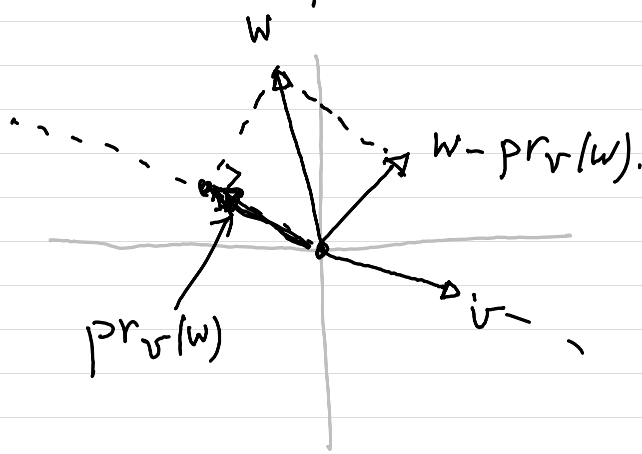
$$\begin{aligned} 0 &= (X - pr_U X) \cdot F_i \\ &= X \cdot F_i - (\alpha_1 F_1 + \dots + \alpha_r F_r) \cdot F_i \\ &= X \cdot F_i - \alpha_i F_i \cdot F_i \end{aligned}$$

$$\Rightarrow \alpha_i = \frac{X \cdot F_i}{F_i \cdot F_i} \qquad \blacksquare$$

Es: Se $v \neq 0_{\mathbb{R}^n}$,

$$pr_v(w) = \frac{w \cdot v}{v \cdot v} v$$





Algoritmo di Gram-Schmidt

Teorema: Sia $U \subseteq \mathbb{R}^n$ un s.sp. vett.
Sia $\mathcal{B}_U = (u_1, \dots, u_r)$ una base di U.
Allora esiste una base $\mathcal{C}_U = (F_1, \dots, F_r)$ di U con le seguenti proprietà:

1) $\mathcal{C}_U$ è ortogonale
2) $\langle u_1, \dots, u_i \rangle = \langle F_1, \dots, F_i \rangle \quad \forall i = 1, \dots, r$
3) $F_i \cdot u_i > 0 \quad (\cos \widehat{F_i u_i} > 0) \quad \forall i = 1, \dots, r.$

Una tale base si trova con la formula ricorsiva:

$$F_1 = u_1$$

$$F_i = u_i - pr_{\langle u_1, \dots, u_{i-1} \rangle}(u_i)$$

$$= u_i - \sum_{k=1}^{i-1} \frac{u_i \cdot F_k}{F_k \cdot F_k} F_k$$

dim: Poniamo $F_1 = u_1$ e

$$F_i = u_i - \sum_{k=1}^{i-1} \frac{u_i \cdot F_k}{F_k \cdot F_k} F_k \quad \forall i = 2, \dots, r.$$

Allora per il lemma di scambio,

$\langle u_1, \dots, u_i \rangle = \langle F_1, \dots, F_i \rangle \quad \forall i = 1, \dots, r.$

In particolare, $\mathcal{C}_V$ è una base

Se $i > j$.

$$F_i \cdot F_j = F_i \cdot \left(u_j - \sum_{k=1}^{j-1} \frac{u_j \cdot F_k}{F_k \cdot F_k} F_k \right)$$

$$= F_i \cdot u_j$$

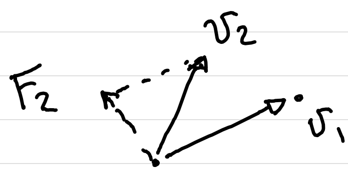
$u_j \in \langle u_1, \dots, u_i \rangle$

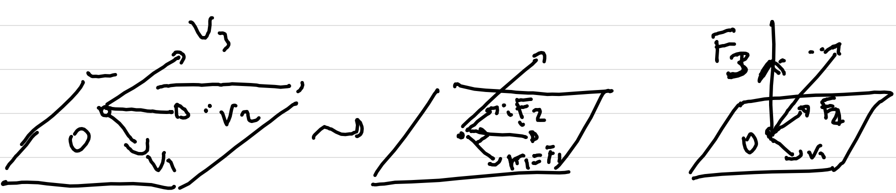
$$= \left(u_i - pr_{\langle u_1, \dots, u_i \rangle}(u_i) \right) \cdot u_j = 0$$

∎

<u>OSS</u>: Dividendo F_i per $\|F_i\|$ otteniamo una base ortonormale. Essa è l'unica base ortonormale di U con le proprietà 1), 2), 3).

<u>OSS</u>: L'algoritmo raddrizza i vettori





Es: Trovare una base ortonormale di

$$U = \left\langle \underset{v_1=}{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}, \underset{v_2=}{\begin{pmatrix}1\\2\\2\\1\end{pmatrix}}, \underset{v_3=}{\begin{pmatrix}1\\2\\-1\\3\end{pmatrix}} \right\rangle$$

Sol.:

$$F_1 = v_1$$

$$F_2' = v_2 - \frac{v_2 \cdot F_1}{F_1 \cdot F_1} F_1 =$$

$$= \begin{pmatrix}1\\2\\2\\1\end{pmatrix} - \frac{6}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}-2/4\\2/4\\2/4\\-2/4\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1\\1\\1\\-1\end{pmatrix}$$

Poniamo

$$F_2 = \begin{pmatrix}-1\\1\\1\\-1\end{pmatrix}.$$

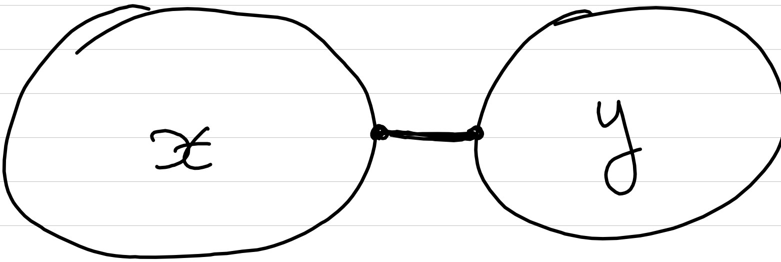
$$F_3 = v_3 - \frac{v_3 \cdot F_1}{F_1 \cdot F_1} F_1 - \frac{v_3 \cdot F_2}{F_2 \cdot F_2} F_2.$$

$$= \begin{pmatrix}1\\2\\-1\\3\end{pmatrix} - \frac{5}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{(-3)}{4}\begin{pmatrix}-1\\1\\1\\-1\end{pmatrix} = \begin{pmatrix}-1\\6/4\\-6/4\\1\end{pmatrix}$$

Distanze tra sottospazi affini

<u>Def</u>: Dati due sottoinsiemi $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$

$$\text{dist}(\mathcal{X}, \mathcal{Y}) = \min \{ \text{dist}(x, y) \mid x \in \mathcal{X}, y \in \mathcal{Y} \}.$$



distanza punto - spsp. affine

Sia $U = X_0 + U_0 \subseteq \mathbb{R}^n$ un s.p. affine.
con giacitura U_0. Sia $P \in \mathbb{R}^n$

$$\text{dist}(P, U) = \| P - X_0 - \text{pr}_{U_0}(P - X_0) \|$$

<u>dim</u>: Sia $u \in U$. $u = X_0 + u_0$

$$\text{dist}(P, u)^2 = \text{dist}(P, X_0 + u_0)^2$$

$$= \text{dist}(P - X_0, u_0)^2$$

$$= \| P - X_0 - u_0 \|^2 = \| \underbrace{(P - X_0 - \text{pr}_U(P - X_0))}_{\in U^\perp} + \underbrace{(\text{pr}_U(P - X_0) - u_0)}_{\in U} \|^2$$

Pitagora

$$= \| P - X_0 - \text{pr}_U(P - X_0) \|^2 + \| \text{pr}_U(P - X_0) - u_0 \|^2$$

$$\geq \| P - X_0 - \text{pr}_U(P - X_0) \|^2$$

$$\| P - X_0 - u_0 \|^2 = \| P - X_0 + pr_U(P-X_0) - pr_U(P-X_0) - u_0 \|^2$$

$$= \| \underbrace{(P - X_0 - pr_U(P-X_0))}_{\in U^\perp} + \underbrace{(pr_U(P-X_0) - u_0)}_{\in U} \|^2$$

$$= \| P - X_0 - pr_U(P-X_0) \|^2 + \| pr_U(P-X_0) - u_0 \|^2$$

pitogora

$$\geq \| P - X_0 - pr_U(P-X_0) \|^2$$

$\Rightarrow$ $\mathrm{dist}(P, u) \geq \mathrm{dist}(P, X_0 + pr_U(P-X_0))$

L'uguaglienze vale se e solo se

$$\| pr_U(P-X_0) - u_0 \| = 0$$

$\Leftrightarrow$ $u_0 = pr_U(P-X_0)$ $\Leftrightarrow$ $u = X_0 + pr_U(P-X_0)$.

$\Rightarrow$ $\mathrm{dist}(P, V) = \mathrm{dist}(P, X_0 + pr_U(P-X_0))$

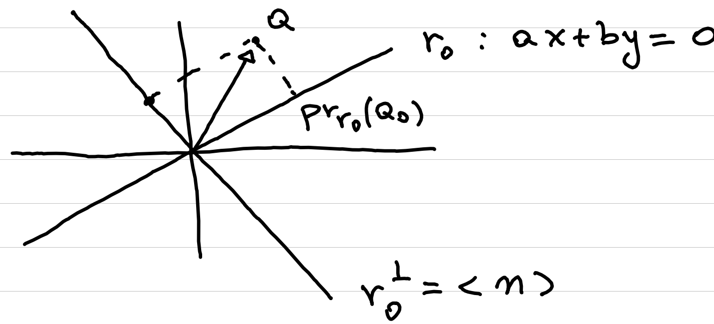
$$= \| P - X_0 - pr_U(P-X_0) \|.$$

Distanza punto-retta (in $\mathbb{R}^2$)

$P = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ $\quad r: ax+by=c$. Sia $X_0 \in r$.

$$\text{dist}(P,r) = \text{dist}(P-X_0, pr_{r_0}(P-X_0))$$

Poniamo $Q = P - X_0$, $n = \begin{pmatrix} a \\ b \end{pmatrix}$



osserviamo che

$$Q = \underbrace{Q - pr_{r_0}(Q)}_{\in\, r_0^\perp} + \underbrace{pr_{r_0}(Q)}_{\in\, r_0}$$

$$\Rightarrow \quad Q - pr_{r_0}(Q) = pr_n(Q) = \frac{Q \cdot n}{n \cdot n} n$$

$$\text{dist}(P,r) = \| Q - pr_{r_0}(Q) \|$$

$$= \left\| \frac{Q \cdot n}{n \cdot n} n \right\| = \frac{|Q \cdot n|}{n \cdot n} \|n\|$$

$$= \frac{|Q \cdot n|}{\|n\|} = \frac{|(P-X_0) \cdot n|}{\|n\|} =$$

$$= \frac{|P \cdot n - X_0 \cdot n|}{\|n\|} \overset{X_0 \cdot n = c}{=} \frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}$$

distanza punto-iperpiano

$U: \quad n \cdot X = b \qquad P \in \mathbb{R}^n$

$$\mathrm{dist}(P, U) = \frac{|n \cdot P - b|}{\|n\|}$$

Es: $\quad U: \quad 2x_1 - x_2 + 3x_3 = 4 \qquad P = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

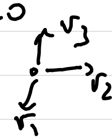
$$\mathrm{dist}(P, U) = \frac{|2 - 2 - 3 - 4|}{\sqrt{4+1+9}} = \frac{7}{\sqrt{14}}$$

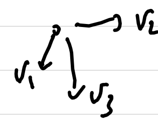
Determinante 3x3 come volume

Teorema: $|\det(v_1, v_2, v_3)| =$ volume del parallelepipedo di spigoli $v_1, v_2, v_3 =$ $\mathrm{vol}(\mathcal{P}(v_1, v_2, v_3))$

dim:

$$v(v_1, v_2, v_3) = \begin{cases} 0 & \text{se } \det(v_1, v_2, v_3) = 0 \\ \mathrm{vol}(\mathcal{P}(v_1, v_2, v_3)) & \text{se} \\ -\mathrm{vol}(\mathcal{P}(v_1, v_2, v_3)) & \text{se} \end{cases}$$





v è multilineare, alternante e $v(e_1, e_2, e_3) = 1$.

$\Rightarrow \quad v(v_1, v_2, v_3) = \det(v_1, v_2, v_3)$. $\blacksquare$

Il prodotto vettoriale

Siano $v=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, w=\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$ Consideriamo la funzione

$$L : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$L(x) := \det(v, w, X)$$

L è lineare. Quindi $\exists\, A = (a_1, a_2, a_3)$ t.c.

$$L = S_A.$$

Il <u>prodotto vettoriale</u> di v e w è il vettore

$$v \wedge w = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = A^t.$$

<u>Altre notazioni</u> : $v \times w$.

Com' è fatta A ?

$$a_1 = L(e_1) = \det(v, w, e_1) = \det\begin{pmatrix} v_1 & w_1 & 1 \\ v_2 & w_2 & 0 \\ v_3 & w_3 & 0 \end{pmatrix} = v_2 w_3 - w_2 v_3$$

$$a_2 = L(e_2) = \det(v, w, e_2) = \det\begin{pmatrix} v_1 & w_1 & 0 \\ v_2 & w_2 & 1 \\ v_3 & w_3 & 0 \end{pmatrix} = -v_1 w_3 + w_1 v_3$$

$$a_3 = L(e_3) = \det(v, w, e_3) = \det\begin{pmatrix} v_1 & w_1 & 0 \\ v_2 & w_2 & 0 \\ v_3 & w_3 & 1 \end{pmatrix} = v_1 w_2 - w_1 v_2$$

<u>Es</u>:

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -7 \end{pmatrix}$$

Proprietà:

•) $v \wedge w = - w \wedge v$

•) La funzione $\wedge : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$v, w \longmapsto v \wedge w$$

è bilineare.

•) $v \wedge w \cdot v = Av = L(v) = \det(v, w, v) = 0$

$v \wedge w \cdot w = Aw = L(w) = \det(v, w, w) = 0$

Quindi $v \wedge w \perp v$, $v \wedge w \perp w$.

Il prodotto misto di 3 vettori $v, w, u \in \mathbb{R}^3$ è

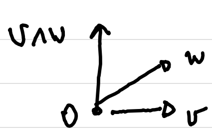
$$v \wedge w \cdot u = Au = L(u) = \det(v, w, u).$$

•) $v \wedge w \neq 0_{\mathbb{R}^3} \iff rg(v, w) = 2$

$\iff v$ e w sono lin. ind.

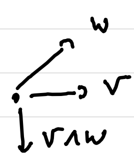
Infatti, $L = 0 \iff \det(v, w, X) = 0 \ \forall X \in \mathbb{R}^3$

Se $rg(v, w) = 2$, $(v, w, v \wedge w)$ è una base di $\mathbb{R}^3$. Se



(Equiversa)

Se $\det(v, w, v \wedge w) > 0$



(controversa)

se $\det(v, w, v \wedge w) < 0$.

Norma del prodotto vettoriale

$\|v \wedge w\| = \text{Area } \mathcal{P}(v,w).$

<u>dim</u>:

$\text{vol}\,(v, w, v\wedge w) = |\det(v, w, v\wedge w)|$

$\text{vol}\,(v, w, v\wedge w) = \|v\wedge w\| \text{ Area } (\mathcal{P}(v,w))$

$$\Rightarrow \quad \text{Area } (\mathcal{P}(v,w)) = \frac{|\det(v,w,v\wedge w)|}{\|v\wedge w\|} = \|v\wedge w\|.$$

$$\det(v, w, v\wedge w) = v\wedge w \cdot v\wedge w = \|v\wedge w\|^2$$

$\blacksquare$

Quindi $v\wedge w$ è un vettore:

- ·) Direzione: ortogonale al piano $\langle v, w\rangle$
- ·) Verso: con la regola della mano destra
- ·) norma: Area $\mathcal{P}(v,w)$.

Es: Calcolare l'area del Triangolo

$$P_1 = \begin{pmatrix}1\\2\\1\end{pmatrix} \quad P_2 = \begin{pmatrix}2\\-1\\3\end{pmatrix} \quad P_3 = \begin{pmatrix}4\\1\\2\end{pmatrix}$$

Sol.:

$$\text{Area } T = \frac{1}{2} \text{Area } \mathcal{P}(P_2 - P_1, P_3 - P_1)$$

$$= \frac{1}{2} \| (P_2 - P_1) \wedge (P_3 - P_1) \|$$

$$= \frac{1}{2} \left\| \begin{pmatrix}1\\-3\\2\end{pmatrix} \wedge \begin{pmatrix}3\\-1\\1\end{pmatrix} \right\|$$

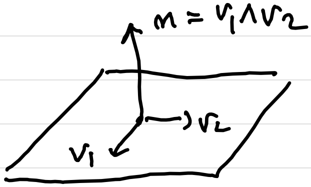
$$= \frac{1}{2} \left\| \begin{pmatrix}-1\\5\\8\end{pmatrix} \right\| = \frac{1}{2} \sqrt{1+25+64}$$

$$= \frac{1}{2} \sqrt{90} = \frac{3}{2} \sqrt{10}.$$

$\square$

OSS: $\pi = X_0 + \langle v_1, v_2 \rangle$. Allora

$$\pi : n \cdot X = n \cdot X_0 \quad \text{e} \quad n = v_1 \wedge v_2$$



Es: $\pi = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \left\langle \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\-1\\1\end{pmatrix} \right\rangle$

$$\begin{pmatrix}1\\2\\1\end{pmatrix} \wedge \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}3\\1\\-5\end{pmatrix} = n$$

$$n \cdot \begin{pmatrix}1\\2\\1\end{pmatrix} = 3 + 2 + 5 = 10$$

$$\pi : 3x + y - 5z = 10$$