Geometria affine

Sottospazio affine di V di dimensione K

$=$

Traslato di un sottospazio vettoriale di dim. K

$=$

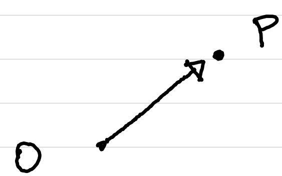
$U = X_0 + U_0 \subseteq V$, dove $X_0 \in V$, $U_0 \subseteq V$ s. sp. vett. $\dim U_0 = K$.

$=$

Se $V = \mathbb{K}^n$, l'insieme delle soluzioni di un sistema lineare di $n-K$ equazioni in n incognite.

punto $\Leftrightarrow$ $K = 0$

In $V = V_0^2$, $P \in \mathcal{E}^2 \Leftrightarrow \{\vec{OP}\}$



$P \leftrightarrow \vec{OP}$

punto — vettore-posizione di P.

retta $\Leftrightarrow$ $K = 1$

piano $\Leftrightarrow$ $K = 2$

iperpiano $\Leftrightarrow$ $K = \dim V - 1$.

$U = X_0 + U_0$ e $W = Y_0 + W_0$ sono paralleli

se $U_0 \subseteq W_0$ oppure $W_0 \subseteq U_0$.

Es: Se $U = X_0 + \{0_V\}$ è un p.to allora

U è parallelo ad ogni s.sp. affine.

Intersezione di sottospazi affini

$U = X_0 + U_0$, $W = Y_0 + W_0 \subset V$ s.sp. affini.

$$\boxed{U \cap W \neq \emptyset \iff X_0 - Y_0 \in U_0 + W_0}$$

(condizione di incidenza)

Infatti, se $U \cap W \neq \emptyset$ allora sia $P \in U \cap W$.

Quindi, $\exists\, u_0 \in U_0$ e $\exists\, w_0 \in W_0$ t.c.

$$P = X_0 + u_0 \quad \text{e} \quad P = Y_0 + w_0.$$

$$\Rightarrow X_0 + u_0 = Y_0 + w_0$$

$$\Rightarrow X_0 - Y_0 = w_0 - u_0 = \underbrace{w_0}_{\in W_0} + \underbrace{(-u_0)}_{\in U_0} \in W_0 + U_0$$

Viceversa, se $X_0 - Y_0 \in U_0 + W_0$, allora

$\exists\, u_0 \in U_0$ e $\exists\, w_0 \in W_0$ t.c. $X_0 - Y_0 = u_0 + w_0$

$$\Rightarrow \underbrace{X_0 - u_0}_{\in U} = \underbrace{Y_0 + w_0}_{\in W} \in U \cap W. \quad \blacksquare$$

Geometria affine del piano $(=\mathbb{R}^2)$

·) Posizione reciproca punto - retta

$P = \{P\} + \{0_{\mathbb{R}^2}\}$, $r = X_0 + r_0 \subset \mathbb{R}^2$ retta

$P \parallel r \iff 0_{\mathbb{R}^2} \in r_0$ vero sempre.

$P \cap r \neq \emptyset \iff P - X_0 \in \{0_{\mathbb{R}^2}\} + r_0 = r_0$

$\iff P \in X_0 + r_0$

·) $r: ax + by = c \qquad ((a,b) \neq (0,0))$

$P = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in r \iff a x_0 + b y_0 = c$

·) $r = Y_0 + \langle v \rangle \qquad (v \neq 0_{\mathbb{R}^2})$.

$P \in r \iff P - Y_0 \in \langle v \rangle$

$\iff rg(v \mid P - Y_0) = 1 = rg(v)$

Es: $P = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad r = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \langle \overbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}^{=v} \rangle$.

$P \in r \iff rg(v \mid P - Y_0) = 1$

$\iff \det(v \mid P - Y_0) = 0$

$\det(v \mid P - Y_0) = \det \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = 0$.

Posizione reciproca retta-retta

1) $r_1 = X_1 + \langle v_1 \rangle \qquad r_2 = X_2 + \langle v_2 \rangle \quad \subset \mathbb{R}^2$

$(v_1 \neq 0_{\mathbb{R}^2},\ v_2 \neq 0_{\mathbb{R}^2})$

$r_1 \parallel r_2 \iff \langle v_1 \rangle = \langle v_2 \rangle$

$\iff \operatorname{rg}(v_1\ v_2) = 1$

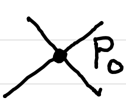
$r_1 \cap r_2 \neq \emptyset \iff X_1 - X_2 \in \langle v_1 \rangle + \langle v_2 \rangle = \langle v_1, v_2 \rangle$

$\iff \exists\ t_1, t_2 \in \mathbb{R}$ t.c.

$X_1 - X_2 = t_1 v_1 + t_2 v_2 \quad (*)$

$\iff \exists \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathbb{R}^2 : (v_1 v_2)\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = X_1 - X_2$

$\iff \operatorname{rg}(v_1\ v_2) = \operatorname{rg}(v_1\ v_2 \mid X_1 - X_2)$

	$\operatorname{rg}(v_1\ v_2)$	$\operatorname{rg}(v_1\ v_2 \mid X_1 - X_2)$
$r_1 \equiv r_2$	1	1
//	1	2
	2	2

Sia $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$ l'unica soluzione di $(v_1 v_2)\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = X_1 - X_2$

Allora, da $(*)$, $X_1 - t_1 v_1 = X_2 + t_2 v_2 = P_0$

NB

Es: Stabilire la posizione reciproca:

$r_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle$, $r_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\rangle$

(senza cambiare la loro forma).

Sol.:

$$rg(v_1\, v_2) = rg\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = rg\begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix} = 2$$

$\Rightarrow$ r_1 e r_2 sono due rette non parallele.

Risolviamo $(v_1\, v_2)\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = X_1 - X_2$

$$(v_1\, v_2 \mid X_1 - X_2) = \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & -1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & -1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1/5 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & -2/5 \\ 0 & 1 & 1/5 \end{array}\right)$$

L'unica soluzione è $\begin{pmatrix} -2/5 \\ 1/5 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$

$$t_1 v_1 + t_2 v_2 = X_1 - X_2$$

$$\Rightarrow \quad X_1 - t_1 v_1 = X_2 + t_2 v_2 = P_0$$

$$P_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 7/5 \\ 9/5 \end{pmatrix}$$

2) $r_1: ax+by=c$, $r_2 = X_2 + \langle v_2 \rangle \subset \mathbb{R}^2$

$((a,b) \neq (0,0), \ v_2 \neq 0_{\mathbb{R}^2})$

$r_1 \parallel r_2 \iff \operatorname{Ker}(a\ b) = \langle v_2 \rangle$

$\iff (a\ b)\, v_2 = 0_{\mathbb{R}}$

$r_1 \cap r_2 \neq \emptyset \iff X_1 - X_2 \in \operatorname{Ker}(a\ b) + \langle v_2 \rangle$

$\iff \exists\ u \in \operatorname{Ker}(a\ b)$ e $t \in \mathbb{R}$ t.c.

$$X_1 - X_2 = u + t v_2$$

$\implies (a\ b)(X_1 - X_2) = (a\ b)(u + t v_2)$

$\iff \boxed{c - (a\ b) X_2 = t\ (a\ b) v_2}$

Viceversa, se $\exists\ t \in \mathbb{R}$ t.c.

$$c - (a\ b) X_2 = t\ (a\ b) v_2$$

allora $(a\ b)(X_1 - X_2) = (a\ b)(t v_2)$

$\implies (a\ b)(X_1 - X_2 - t v_2) = 0_{\mathbb{R}}$

$\implies X_1 - X_2 - t v_2 \in \operatorname{Ker}(a\ b)$

$\implies \exists\ u \in \operatorname{Ker}(a\ b)$ t.c. $X_1 - X_2 = u + t v_2$.

In conclusione,

$r_1 \cap r_2 \neq \emptyset \iff c - (a\ b) X_2 \in \langle (a\ b) v_2 \rangle$

Ricapitolando,

$$r_1 \parallel r_2 \iff (a\ b)v_2 = 0 \iff \mathrm{rg}\big((a\ b)v_2\big) = 0$$

$$r_1 \cap r_2 \neq \emptyset \iff c - (a\ b)X_2 \in \langle (a\ b)v_2 \rangle$$

$$\underset{\text{Rouché Capelli}}{\iff} \mathrm{rg}\big((a\ b)v_2\big) = \mathrm{rg}\big((a\ b)v_2 \mid c - (a\ b)X_2\big)$$

	$\mathrm{rg}\big((a\ b)v_2\big)$	$\mathrm{rg}\big((a\ b)v_2 \mid c-(a\ b)X_2\big)$
$r_1 \equiv r_2$	0	0
//	0	1
P_0	1	1

Sia t l'unica soluzione di

$$t(a\ b)v_2 = c - (a\ b)X_2$$

Allora $(a\ b)(X_2 + t v_2) = c$

e quindi

$$P_0 = X_2 + t v_2.$$

3) $r_1 : ax+by=c$, $r_2 : a'x+b'y=c' \subset \mathbb{R}^2$

$\left((a,b) \neq (0\ 0) \right.$, $\left. (a'\ b') \neq (0\ 0) \right)$.

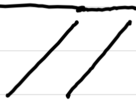
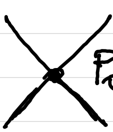
$r_1 \parallel r_2 \iff \operatorname{Ker}(a\ b) = \operatorname{Ker}(a'\ b')$

$\iff \operatorname{Ker}\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = \operatorname{Ker}(a\ b) \cap \operatorname{Ker}(a'\ b') \neq \{0_{\mathbb{R}^2}\}$.

$\iff \operatorname{rg}\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = 1$

$r_1 \cap r_2 \neq \emptyset \iff$ Il sistema $\begin{cases} ax+by=c \\ a'x+b'y=c' \end{cases}$

è risolubile $\underset{R-C}{\iff} \operatorname{rg}\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = \operatorname{rg}\left(\begin{array}{cc|c} a & b & c \\ a' & b' & c' \end{array}\right)$

	$\operatorname{rg}\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$	$\operatorname{rg}\left(\begin{array}{cc\|c} a & b & c \\ a' & b' & c' \end{array}\right)$
$r_1 \equiv r_2$	1	1
	1	2
 P_0	2	2

P_0 è l'unica soluzione del sistema

$\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} X = \begin{pmatrix} c \\ c' \end{pmatrix}$.

Geometria affine dello spazio $(=\mathbb{R}^3)$

Posizione reciproca retta-retta

1) $r_1 = X_1 + \langle v_1 \rangle$, $r_2 = X_2 + \langle v_2 \rangle \subset \mathbb{R}^3$

$(v_1 \neq 0_{\mathbb{R}^3},\ v_2 \neq 0_{\mathbb{R}^3})$.

$$r_1 \parallel r_2 \iff \langle v_1 \rangle = \langle v_2 \rangle$$

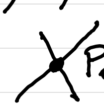
$$\iff \operatorname{rg}(v_1\ v_2) = 1$$

$$r_1 \cap r_2 \neq \emptyset \overset{\text{cond.}}{\underset{\text{incidenza}}{\iff}} X_1 - X_2 \in \langle v_1, v_2 \rangle$$

$$\iff \exists\ t_1, t_2 \in \mathbb{R} \text{ t.c. } t_1 v_1 + t_2 v_2 = X_1 - X_2$$

$\iff$ Il sistema $(v_1\ v_2)\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = X_1 - X_2$ è risolubile

$$\underset{\text{R-C}}{\iff} \operatorname{rg}(v_1\ v_2) = \operatorname{rg}(v_1\ v_2 \mid X_1 - X_2)$$

	$\operatorname{rg}(v_1\ v_2)$	$\operatorname{rg}(v_1\ v_2 \mid X_1 - X_2)$
$r_1 \equiv r_2$	1	1
//	1	2
	2	2
 "sghembe"	2	3

__Es:__

$$r_1 = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle, \quad r_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\rangle$$

Stabilire la pos. reciproca di r_1 e r_2 senza cambiare la loro forma.

__Sol.__:

$$rg \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 4 \end{pmatrix} = 2 \quad \text{perché} \quad \det \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = -1 \neq 0.$$

$$rg \begin{array}{c} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 4 & 5 \end{array}\right) \\ \begin{array}{ccc} v_1 & v_2 & X_1 - X_2 \end{array} \end{array} = rg \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{array}\right) = rg \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$$= 2$$

$$\Rightarrow r_1 \cap r_2 = \{ P_0 \}.$$

Determiniamo P_0:

$$(v_1 \; v_2 \mid X_1 - X_2) \underset{R}{\rightsquigarrow} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

L'unica soluzione è $t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$t_1 v_1 + t_2 v_2 = X_1 - X_2 \Rightarrow X_1 - t_1 v_1 = X_2 + t_2 v_2 = P_0$$

$$P_0 = X_2 + v_2 = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} = X_1 - v_1$$

2) $r_1 : AX = b$ e $r_2 = X_2 + \langle v_2 \rangle \subset \mathbb{R}^3$

$\left(A = \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}, \ \mathrm{rg}\, A = 2, \ v_2 \neq 0_{\mathbb{R}^3} \right).$

$$r_1 \parallel r_2 \iff (r_1)_0 = (r_2)_0$$

$$\iff \mathrm{Ker}\, A = \langle v_2 \rangle$$

$$\iff v_2 \in \mathrm{Ker}\, A$$

$$\iff A v_2 = 0_{\mathbb{R}^2}$$

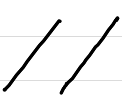
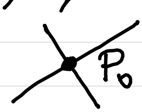
$$\iff \mathrm{rg}\, A v_2 = 0$$

$$r_1 \cap r_2 \neq \emptyset \underset{\text{incidenza}}{\overset{\text{cond.}}{\iff}} X_1 - X_2 \in \mathrm{Ker}\, A + \langle v_2 \rangle$$

dove $X_1 \in r_1$.

$$\iff b - AX_2 \in \langle A v_2 \rangle$$

$$\underset{\text{R-C}}{\iff} \mathrm{rg}\,(A v_2) = \mathrm{rg}\,(A v_2 \,|\, b - AX_2)$$

	$\mathrm{rg}\,(A v_2)$	$\mathrm{rg}\,(A v_2 \,\vert\, b - AX_2)$
$r_1 \equiv r_2$	0	0
	0	1
	1	1
 "sghembe"	1	2

$$P_0 \in r_1 \cap r_2 \iff \begin{array}{l} P_0 = X_2 + t v_2 \\ A P_0 = b \end{array}$$

$$\Rightarrow \quad A(X_2 + t v_2) = b$$

$$\Rightarrow \quad t A v_2 = b - A X_2$$

$P_0 \in r_1 \cap r_2 \iff \exists\, t \in \mathbb{R}$ t.c.

$$t A v_2 = b - A X_2$$

$$\overset{R-C.}{\iff} rg(A v_2) = rg(A v_2 \mid b - A X_2)$$

Per trovare P_0, sia t l'unica soluzione di $(A v_2) t = b - A X_2$

allora $A(X_2 + t v_2) = b$

e quindi

$$P_0 = X_2 + t v_2.$$

Es:

$$r: \begin{cases} 2x+3y-z=5 \\ x+2y+z=6 \end{cases} \qquad s = \begin{pmatrix}1\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle.$$

Posizione reciproca?

Sol.:
$$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix} \qquad b = \begin{pmatrix}5\\6\end{pmatrix}$$

$$X_2 = \begin{pmatrix}1\\1\\1\end{pmatrix} \qquad v_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} \neq 0_{\mathbb{R}^3}.$$

$\operatorname{rg} A = 2$ perché $\det \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = 1 \neq 0$.

$\Rightarrow$ r è una retta.

$r \parallel s \iff A v_2 = 0_{\mathbb{R}^2}$

$$A v_2 = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$$

$\Rightarrow$ r ed s <u>non</u> sono parallele.

$(A v_2) t = b - A X_2$. :

$$A X_2 = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix}$$

Dobbiamo studiare il sistema

$$\begin{pmatrix}1\\2\end{pmatrix} t = \begin{pmatrix}5\\6\end{pmatrix} - \begin{pmatrix}4\\4\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \qquad \begin{pmatrix}2\\1\\2\end{pmatrix}$$

ha soluzione $t = 1$. $r \cap s = \{P_0\}$. $P_0 = X_2 + v_2 = \begin{pmatrix}2\\1\\2\end{pmatrix}$

retta - piano

1) $r = X_0 + \langle w \rangle \qquad \pi = Y_0 + \langle v_1, v_2 \rangle \subset \mathbb{R}^3$

$\left(w \neq 0_{\mathbb{R}^3}, \quad rg(v_1\ v_2) = 2 \right)$

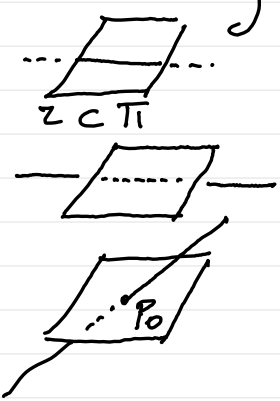
$r \parallel \pi \iff \langle w \rangle \subset \langle v_1, v_2 \rangle$

$\iff w \in \langle v_1, v_2 \rangle$

$\underset{R-C}{\iff} rg(v_1\ v_2) = rg(v_1\ v_2 \mid w) = 2$

$r \cap \pi \neq \emptyset \underset{\text{indidenza}}{\overset{\text{cond.}}{\iff}} X_0 - Y_0 \in \langle v_1, v_2, w \rangle$

$\underset{R-C}{\iff} rg(v_1\ v_2\ w) = rg(v_1\ v_2\ w \mid X_0 - Y_0)$



$rg(v_1\ v_2\ w)$	$rg(v_1\ v_2\ w \mid X_0 - Y_0)$
2	2
2	3
3	3

$P_0 = X_0 - t_3 w = Y_0 + t_1 v_1 + t_2 v_2 \ : \ (v_1\ v_2\ w)\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = X_0 - Y_0$

.) $r : AX = b$, $\pi = Y_0 + \langle w_1, w_2 \rangle$.
($A = \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$, $rg\, A = 2$, $rg\,(w_1\ w_2) = 2$)

$r \parallel \pi \iff \mathrm{Ker}\, A \subseteq \langle w_1, w_2 \rangle$

$$A\,(w_1\ w_2) = (Aw_1\ \ Aw_2) \text{ è } 2\times 2$$

e non è nulla (se lo fosse,
$w_1, w_2 \in \mathrm{Ker}\, A$ che però ha dim 1)

$\mathrm{Ker}\, A \subseteq \langle w_1, w_2 \rangle$ vuol dire $\exists\, t_1, t_2$ t.c.

$$A\,(t_1 w_1 + t_2 w_2) = 0_{\mathbb{R}^2}$$

$$t_1\, Aw_1 + t_2\, Aw_2 = 0_{\mathbb{R}^2}$$

Se $\mathrm{Ker}\, A \subseteq \langle w_1, w_2 \rangle$ allora $rg\,(Aw_1\ Aw_2) = 1$
Viceversa se $rg\,(Aw_1\ Aw_2) = 1$ allora
$\exists\, t_1, t_2 \in \mathbb{R}$ t.c. $t_1\, Aw_1 + t_2\, Aw_2 = 0_{\mathbb{R}^2}$

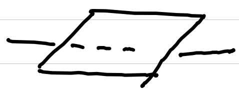
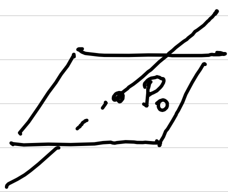
$$\Rightarrow A\,(t_1 w_1 + t_2 w_2) = 0_{\mathbb{R}^2}$$

$$\Rightarrow \mathrm{Ker}\, A = \langle t_1 w_1 + t_2 w_2 \rangle.$$

$r \parallel \pi \iff rg\,(Aw_1\ Aw_2) = 1$

$$r \cap \pi \neq \emptyset \iff Y_0 - X_0 \in \operatorname{Ker} A + \langle w_1, w_2 \rangle$$

$$\iff b - AX_0 \in \langle Aw_1, Aw_2 \rangle$$

	$\operatorname{rg}(Aw_1\ Aw_2)$	$\operatorname{rg}(Aw_1\ Aw_2\ b-AX_0)$
$r \subset \pi$	1	1
	1	2
	2	2