$A \in Mat_{n\times n}(\mathbb{K})$. $\quad p_A(x) = \det(x\mathbb{1}_n - A)$ è un polinomio monico di grado n che si chiama il polinomio caratteristico di A.

•) $Sp(A)_{\mathbb{K}} = \{ \lambda \in \mathbb{K} \mid p_A(\lambda) = 0 \}$.

•) Se $\mathbb{K} = \mathbb{C}$, $\quad p_A(x) = (x-\lambda_1)^{m_1} \cdots (x-\lambda_k)^{m_k}$ per opportuni numeri complessi distinti $\lambda_1, \dots, \lambda_k \in \mathbb{C}$. $\quad Sp(A) = \{\lambda_1, \dots, \lambda_k\} = Sp(A)_{\mathbb{C}}$.

$$m_i = \text{molteplicità algebrica di } \lambda_i =: ma_A(\lambda_i).$$

•) Per $\lambda \in Sp(A)$, $\quad V_\lambda(A) = \ker(\lambda \mathbb{1}_n - A)$ = autospazio di autovalore λ.

$$\dim V_\lambda(A) := \text{molteplicità geometrica di } \lambda =: mg_A(\lambda).$$

•) $\mathcal{L}: V \to V$ è diagonalizzabile se $\exists\ \mathcal{B} = (v_1, \dots, v_n) \overset{\text{base}}{\subset} V$ t.c.

$$\begin{tikzcd} V \arrow[r, "\mathcal{L}"] \arrow[d, "F_{\mathcal{B}}"'] & V \arrow[d, "F_{\mathcal{B}}"] \\ \mathbb{K}^n \arrow[r, "S_D"] & \mathbb{K}^n \end{tikzcd}$$

$D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$.

$\mathcal{L}(v_i) = \lambda_i v_i$.

•) A è diagonalizzabile su $\mathbb{K}$ se $S_A : \mathbb{K}^n \to \mathbb{K}^n$ è diagonalizzabile.

$\overset{!}{\Longleftrightarrow}$ $\exists\ B$ invertibile e $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ t.c. $B^{-1}AB = D$.

$$AB^i = \lambda_i B^i.$$

$$\begin{tikzcd} \mathbb{K}^n \arrow[r, "S_A"] \arrow[d, "S_{B^{-1}} = F_{\mathcal{B}}"'] & \mathbb{K}^n \arrow[d, "F_{\mathcal{B}} = S_{B^{-1}}"] \\ \mathbb{K}^n \arrow[r, "S_C"] & \mathbb{K}^n \end{tikzcd}$$

$\mathcal{B} = (v_1, \dots, v_n) \overset{\text{base}}{\subset} \mathbb{K}^n$

$B = (v_1 | \dots | v_n)$

$C = B^{-1} A B$. $\quad$ A e C si dicono <u>simili</u>.

<u>Prop.</u>: Se A e C sono simili allora $P_A(x) = P_C(x)$.

<u>dim</u>: Se $C = B^{-1} A B$. Allora

$$P_C(x) = \det(x \mathbb{1}_n - B^{-1} A B) = \det(x B^{-1} B - B^{-1} A B)$$

$$= \det\left(B^{-1}(x \mathbb{1}_n - A) B\right) \overset{\text{Binet}}{=}$$

$$= \cancel{(\det B)^{-1}} \det(x \mathbb{1}_n - A) \cancel{\det B} = P_A(x). \quad \blacksquare$$

Es: ·) $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ $\quad P_A(x) = (x-2)(x-3)$

·) $D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_{nn} \end{pmatrix}$ $\quad P_D(x) = (x-d_{11})(x-d_{22})\cdots(x-d_{nn})$

$Sp(D) = \{d_{11}, d_{22}, \dots, d_{nn}\} \ni \lambda$

$ma_D(\lambda) = \#\{i \mid d_{ii} = \lambda\}$ = numero di volte che λ compare sulla diagonale.

·) $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $\quad P_D(x) = (x-2)^2(x-3)$

$ma_D(2) = 2 \quad , \quad ma_D(3) = 1$

$mg_D(2) = \dim \mathrm{Ker}(2\mathbb{1}_3 - D) = \dim \mathrm{Ker}\left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right) = 1$

·) $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Allora $De_i = d_{ii} e_i$.

$V_\lambda(D) = \langle e_i \mid d_{ii} = \lambda \rangle$

$mg_D(\lambda) = \#\{i \mid d_{ii} = \lambda\} = ma_D(\lambda)$.

•) $A = \begin{pmatrix} 2 & 4 \\ 0 & 3 \end{pmatrix}$ $\quad P_A(x) = (x-2)(x-3)$

$Sp(A) = \{2, 3\}$ $\quad ma_A(2) = 1 = ma_A(3)$

•) $A = \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix}$ $\quad P_A(x) = (x-2)^2$

$Sp(A) = \{2\}$ $\quad ma_A(2) = 2.$

$mg_A(2) = \dim \operatorname{Ker}(2 \mathbb{1}_2 - A) = \dim \operatorname{Ker} \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} =$

$= \dim \operatorname{Ker} (0 \ 1) = 1 \underset{\neq}{<} ma_A(2)$

•) $U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{nn} \end{pmatrix}$. $\quad P_U(x) = (x-u_{11})(x-u_{22})\cdots(x-u_{nn})$

$Sp(U) = \{u_{11}, u_{22}, \dots, u_{nn}\} \ni \lambda$

$ma_U(\lambda) = \#\{i \mid u_{ii} = \lambda\}.$

·) <u>Blocco di Jordan</u>

Sia $\lambda \in \mathbb{K}$, sia $n \geq 1$.

$$J_n(\lambda) := \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & & 0 & \lambda \end{pmatrix} \in Mat_{n \times n}(\mathbb{K})$$

$$J_2(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad J_3(\lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \quad J_4(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

$$P_{J_n(\lambda)} = (x-\lambda)^n \qquad Sp(J_n(\lambda)) = \{\lambda\} \qquad ma_{J_n(\lambda)}(\lambda) = n$$

$$mg_{J_n(\lambda)}(\lambda) = \dim \operatorname{Ker} \begin{pmatrix} 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & -1 & 0 & & \vdots \\ 0 & 0 & 0 & -1 & 0 & \vdots \\ \vdots & & & & \ddots & 0 \\ \vdots & & & \cdots & 0 & -1 \\ 0 & \cdots & & & \cdots & 0 \\ 0 & \cdots & & & & \end{pmatrix} = 1$$

$$\mathcal{B}_{V_\lambda(J_n(\lambda))} = (e_1).$$

$$mg_{J_n(\lambda)}(\lambda) = 1 < n = ma_{J_n(\lambda)}(\lambda).$$

Se $A \in Mat_{n \times n}(\mathbb{K})$ diagonalizzabile allora $\exists$ B invertibile e D diagonale t.c.

$$B^{-1}AB = D.$$

A e D sono simili $\Rightarrow$ ·) $\boxed{P_A(x) = P_D(x)}$

·) $Sp(A) = Sp(D) \subset \mathbb{K}$ ·) $\forall \lambda \in Sp(A)$.

$$ma_A(\lambda) = ma_D(\lambda) = mg_D(\lambda). \quad (*)$$

$$\begin{tikzcd}
F_B^{-1}(V_\lambda(D)) = V_\lambda(A) \arrow[r, phantom, "\subset"] & \mathbb{K}^n \arrow[r, "S_A"] \arrow[d, "F_B"] & \mathbb{K}^n \arrow[d, "F_B"] \\
V_\lambda(D) \arrow[u, "F_B^{-1}"', "\wr"] \arrow[r, phantom, "\subset"] & \mathbb{K}^n \arrow[r, "S_D"] & \mathbb{K}^n
\end{tikzcd}$$

$$\dim V_\lambda(D) = \dim V_\lambda(A) \Rightarrow mg_A(\lambda) = mg_D(\lambda)$$

$$\underset{(*)}{\Rightarrow} \quad \boxed{mg_A(\lambda) = ma_A(\lambda)}. \quad \forall \lambda \in Sp(A).$$

<u>Prop.</u>: Se $A \in Mat_{n \times n}(\mathbb{K})$ e $\lambda \in Sp(A)$, allora

$$mg_A(\lambda) \le ma_A(\lambda)$$

<u>dim</u>:

Sia $\lambda \in Sp(A) \subset \mathbb{K} \Rightarrow V_\lambda(A) \subset \mathbb{K}^n$ è non-nullo.

Sia $\mathcal{B}_\lambda = (v_1, \dots, v_k)$ una base di $V_\lambda(A)$.

Estendiamola ad una base $\mathcal{B} = (v_1, \dots, v_k, v_{k+1}, \dots, v_n)$ di $\mathbb{K}^n$. $A v_i = \lambda v_i \quad \forall i = 1, \dots, k$. Quindi la matrice che rappresenta S_A nella base $\mathcal{B}$ è

$$C = \left(\begin{array}{c|c} \lambda \mathbb{1}_k & T \\ \hline 0 & Z \end{array} \right) = \left(F_{\mathcal{B}}(Av_1) \middle| F_{\mathcal{B}}(Av_2) \middle| \dots \middle| F_{\mathcal{B}}(Av_n) \right).$$

A e C sono simili

$$\Rightarrow P_A(x) = P_C(x) = \det \left(\begin{array}{c|c} (x-\lambda)\mathbb{1}_K & -T \\ \hline 0 & x\mathbb{1}_{n-K} - Z \end{array} \right)$$

$$= \det\left((x-\lambda)\mathbb{1}_K\right) \det\left(x\mathbb{1}_{n-K} - Z\right)$$

$$= (x-\lambda)^K P_Z(x)$$

$$mg_A(\lambda) = K \leq ma_A(\lambda).$$

$\blacksquare$

Teorema: Sia $A \in Mat_{n \times n}(\mathbb{K})$.

$$S_A : \mathbb{K}^n \to \mathbb{K}^n \text{ è diagonalizzabile}$$

$$\Updownarrow$$

(1) $Sp(A) \subset \mathbb{K}$

(2) $mg_A(\lambda) = ma_A(\lambda) \quad \forall \lambda \in Sp(A)$.

dim: $\Downarrow$) Lucido precedente.

$\Uparrow$) $Sp(A) = \{\lambda_1, \dots, \lambda_k\} \underset{(1)}{\subset} \mathbb{K} \qquad \lambda_i \neq \lambda_j$.

$$W = V_{\lambda_1}(A) \oplus \cdots \oplus V_{\lambda_k}(A) \subseteq \mathbb{K}^n$$

$$\begin{aligned}
\dim W &= \dim V_{\lambda_1}(A) + \dots + \dim V_{\lambda_k}(A) \\
&= mg_A(\lambda_1) + \dots + mg_A(\lambda_k) \overset{(2)}{=} ma_A(\lambda_1) + \dots + ma_A(\lambda_k) \\
&= n \quad \Rightarrow \quad \mathbb{K}^n = V_{\lambda_1}(A) \oplus \dots \oplus V_{\lambda_k}(A).
\end{aligned}$$

Sia $\mathcal{B}_{\lambda_i}$ una base di $V_{\lambda_i}(A)$.

Allora

$$\mathcal{B} = \mathcal{B}_{\lambda_1} \cup \mathcal{B}_{\lambda_2} \cup \dots \cup \mathcal{B}_{\lambda_k}$$

è una base di $\mathbb{K}^n$ composta di autovettori di A. Quindi A è diagonalizzabile. (su $\mathbb{K}$).

$\blacksquare$

Es 4.1.1 : $A = \begin{pmatrix} 1 & 4 & 1 \\ -4 & -7 & 2 \\ 6 & 6 & 0 \end{pmatrix}$

Stabilire se A è diagonalizzabile su $\mathbb{R}$ e nel caso lo sia trovare una matrice invertibile B ed una matrice diagonale D t.c.

$$B^{-1}AB = D.$$

Sol.: $P_A(x) = x^3 - \mathrm{Tr}A\, x^2 + \frac{1}{2}\left(\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2)\right)x - \det A$

$$\det A = \det\begin{pmatrix} 1 & 4 & 1 \\ 0 & 9 & 6 \\ 0 & -18 & -6 \end{pmatrix} = 3\cdot(-6)\det\begin{pmatrix} 1 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 3 & 1 \end{pmatrix} =$$

$$= 3\cdot(-6)\det\begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix} = 3\cdot(-6)\det\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} = 9\cdot 6 = 54.$$

$\mathrm{Tr}A = -6 \qquad (\mathrm{Tr}A)^2 = 36$

$\mathrm{Tr}A^2 = -9 + 45 + 18 = 54$

$$P_A(x) = x^3 + 6x^2 + \frac{1}{2}(36-54)x - 54 =$$

$$= x^3 + 6x^2 - 9x - 54 = x^2(x+6) - 9(x+6) = (x+6)(x^2-9)$$

$$P_A(x) = (x+6)(x+3)(x-3). \qquad A = \begin{pmatrix} 1 & 4 & 1 \\ -4 & -7 & 2 \\ 6 & 6 & 0 \end{pmatrix}$$

$$Sp(A) = \{-6, -3, 3\} \subset \mathbb{R} \quad ma_A(-6) = ma_A(-3) = ma_A(3) = 1$$

$$\underset{\text{Prop.}}{\Rightarrow} \quad mg_A(-6) = mg_A(-3) = mg_A(3) = 1 \qquad A$$

<u>OSS</u> (generale): Se $\lambda \in Sp(A)$ e $ma_A(\lambda) = 1$,
allora se $\lambda \in \mathbb{K}$, $0 \neq \dim V_\lambda(A) = mg_A(\lambda) \underset{\text{Prop.}}{\leq} 1$

$$\Rightarrow \quad mg_A(\lambda) = 1 = ma_A(\lambda).$$

Per il Teorema A è diagonalizzabile.

$$V_{-6}(A) = \mathrm{Ker}\begin{pmatrix} -7 & -4 & -1 \\ 4 & 1 & -2 \\ -6 & -6 & -6 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 1 & 1 & 1 \\ -7 & -4 & -1 \\ 4 & 1 & -2 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & -3 & -6 \end{pmatrix}$$

$$= \mathrm{Ker}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\rangle$$

$$V_{-3}(A) = \mathrm{Ker}\begin{pmatrix} -4 & -4 & -1 \\ 4 & 4 & -2 \\ -6 & -6 & -3 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ -4 & -4 & -1 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 2 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \mathrm{Ker}\begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle \qquad A\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -3\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$V_3(A) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle$$

$$B = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} -6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$\blacksquare$

<u>COR</u>: Se $A \in Mat_{n\times n}(\mathbb{K})$, $Sp(A) \subset \mathbb{K}$. Se
Se A ha n autovalori distinti allora A è diagonalizzabile su $\mathbb{K}$.

<u>dim</u>: $|Sp(A)| = n \Rightarrow ma_A(\lambda) = 1 \quad \forall \lambda \in Sp(A)$

$$\Rightarrow mg_A(\lambda) = ma_A(\lambda) = 1 \,. \quad \blacksquare$$

OSS: $\mathcal{L}: V \to V$ lineare è diagonalizzabile se e solo se una matrice che rappresenta $\mathcal{L}$ è diagonalizzabile.

Es: $V = \mathbb{R}[x]_{\leq 2}$ $\quad \mathcal{L}: V \to V$

$$\mathcal{L}(p) = p'(x^2+1)$$

Stabilire se è diagonalizzabile.

Sol.: Sia A la matrice associata ad $\mathcal{L}$ nella base canonica $\mathcal{C} = (1, x, x^2)$.

$$\mathcal{L}(1) = 0$$

$$\mathcal{L}(x) = 1$$

$$\mathcal{L}(x^2) = 2(x^2+1) = 2x^2+2$$

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$Sp(\mathcal{L}) = Sp(A) = \{0, 2\}$. $\quad ma_A(0) = 2$, $\quad ma_A(2) = 1 = mg_A(2)$.

$V_0(A) = \mathrm{Ker}\, A \quad mg_A(0) = 3 - rg\, A = 3 - 2 = 1 \Rightarrow$ $\mathcal{L}$ NON è diag.

Se A è diagonalizzabile,

$$B^{-1} A B = D \quad ,$$

allora

$$A = B D B^{-1}$$

$$\Rightarrow \quad A^n = B D^n B^{-1}$$

$$\left(A^2 = \left(B D \cancel{B^{-1}}\right)\left(\cancel{B} D B^{-1}\right)\right).$$

$$D = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \quad \Rightarrow \quad D^n = \operatorname{diag}(\lambda_1^n, \dots, \lambda_n^n)$$

Es 4.1.4: $A = \begin{pmatrix} 5/2 & -1 \\ 3 & -1 \end{pmatrix}$

Calcolare A^n per ogni n.

Sol.: $P_A(X) = X^2 - \mathrm{Tr}A\, X + \det A$

$$= X^2 - \frac{3}{2} X + \frac{1}{2}$$

$$\lambda_{1,2} = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - 2}}{2} = \frac{\frac{3}{2} \pm \sqrt{\frac{1}{4}}}{2} = \frac{\frac{3}{2} \pm \frac{1}{2}}{2} = \frac{3}{4} \pm \frac{1}{4}$$

$$= \begin{cases} 1 \\ \frac{1}{2} \end{cases} \qquad Sp(A) = \left\{1, \frac{1}{2}\right\}.$$

$$V_1(A) = \mathrm{Ker}\,(\mathbb{1}_2 - A) = \mathrm{Ker}\begin{pmatrix} -3/2 & 1 \\ -3 & 2 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} -3 & 2 \end{pmatrix} = \left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\rangle$$

$$V_{\frac{1}{2}}(A) = \mathrm{Ker}\begin{pmatrix} -2 & 1 \\ -3 & \frac{3}{2} \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} -2 & 1 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle$$

$$B = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$B^{-1} A B = D \quad \Rightarrow \quad A = B D B^{-1}$$

$$\Rightarrow \quad A^n = B D^n B^{-1}$$

$$B^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \qquad D^n = \begin{pmatrix} 1 & 0 \\ 0 & \left(\frac{1}{2}\right)^n = 2^{-n} \end{pmatrix}$$

$$A^n = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^{-n} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2^{-n} \\ 3 & 2^{-n+1} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2^2 - 3 \cdot 2^{-n} & -2 + 2^{-n+1} \\ 6 - 3 \cdot 2^{-n+1} & -3 + 2^{-n+2} \end{pmatrix}$$