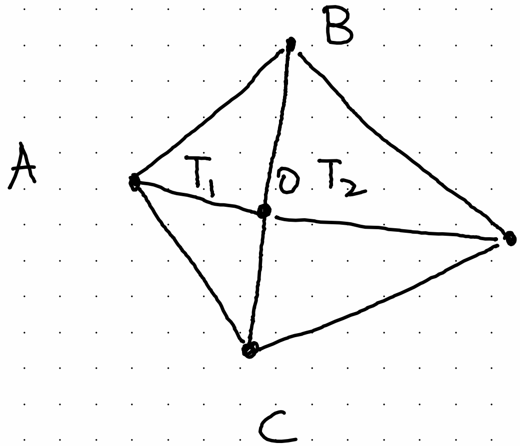
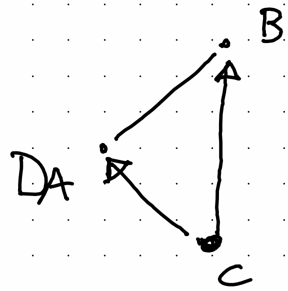
Commento sull'es. 6 della settimana 7:



Area poligono = Area (T_1) + Area (T_2)

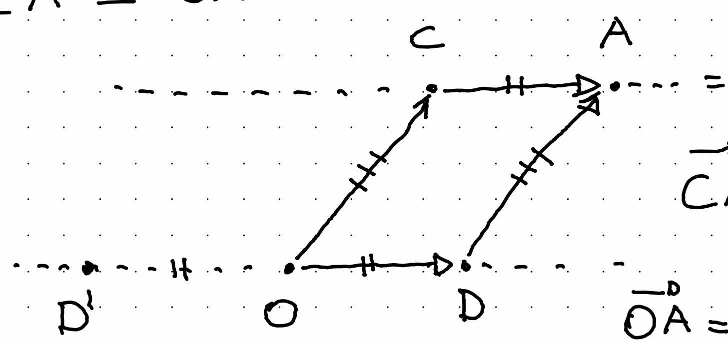


Area (T) =

$$\frac{1}{2}\left|\det\left(F_{\mathcal{B}}(\overrightarrow{CA}), F_{\mathcal{B}}(\overrightarrow{CB})\right)\right|$$

? ?

$$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$$



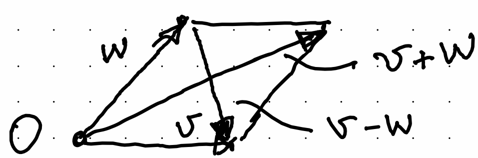
= vettore geometrico applicato a C. $\notin V_O^2$

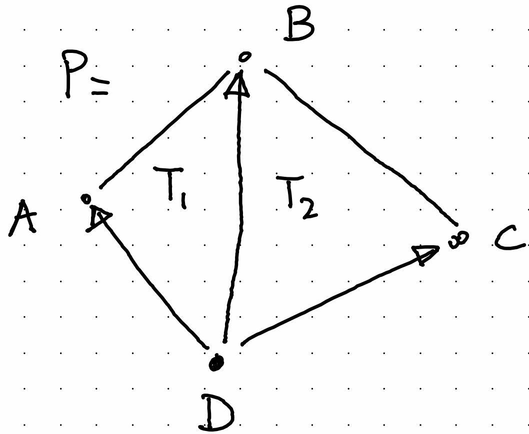
$\overrightarrow{CA} \equiv \overrightarrow{OD} \iff$

1) stessa lunghezza
2) stesso verso
3) stessa direzione.

$$\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{OD}$$

$$\Rightarrow \overrightarrow{OD} = \overrightarrow{OA} - \overrightarrow{OC}$$





$$F_{\mathcal{B}}(\vec{OA}) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad F_{\mathcal{B}}(\vec{OB}) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$F_{\mathcal{B}}(\vec{OC}) = \begin{pmatrix} 1 \\ -5 \end{pmatrix} \quad F_{\mathcal{B}}(\vec{OD}) = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 \\ -5 \end{pmatrix} \quad D = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

$$F_{\mathcal{B}}(\vec{OB} - \vec{OD}) = F_{\mathcal{B}}(\vec{OB}) - F_{\mathcal{B}}(\vec{OD})$$

$$\text{Area}(P) = \text{Area}(T_1) + \text{Area}(T_2)$$

$$= \frac{1}{2} \left| \det(A-D \mid B-D) \right| + \frac{1}{2} \left| \det(B-D \mid C-D) \right|$$

$$= \frac{1}{2} \left| \det\begin{pmatrix} -5 & 6 \\ -3 & 2 \end{pmatrix} \right| + \frac{1}{2} \left| \det\begin{pmatrix} -3 & 2 \\ -6 & -2 \end{pmatrix} \right| =$$

$$= \frac{1}{2}\, 8 + \frac{1}{2}\, 18 = 13$$

Richiami: Per calcolare $\det(A)$

- operiamo sulle righe o sulle colonne di A in maniera da rendere la prima colonna di A piena di zero oppure in maniera da ridurre A a scala

- Applichiamo la formula

$$\det A = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det(A_{i1})$$

- $\det(\text{a scala})$ = prodotto degli elementi diagonali.

- $\det(A^t) = \det A$.

Ricordiamo:

$$\det(P_{ij}A) = -\det A = \det(AP_{ij})$$

$$\det(D_i(\lambda)A) = \lambda \det A = \det(A D_i(\lambda))$$

$$\det(F_{ij}(c)A) = \det A = \det(A F_{ij}(c))$$

Sviluppi di Laplace

Teorema: Sia $A \in Mat_{n \times n}(\mathbb{K})$.

1) Dato un indice di riga i

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

" Sviluppo del det lungo la riga i "

2) Dat un indice di colonna j

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

" Sviluppo del det lungo la colonna j "

dove A_{ij} è la matrice ottenuta da A rimuovendo la riga i-esima e la colonna j-esima.

Es di A_{ij} :

•) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

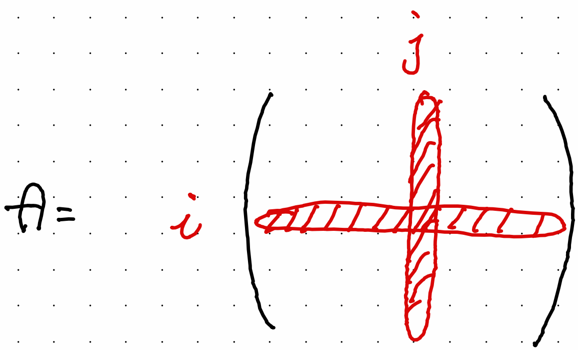
$A_{11} = (d)$ $\quad$ $A_{12} = (c)$

$A_{21} = (b)$ $\quad$ $A_{22} = (a)$

•) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$A_{22} = \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}$ $\quad$ $A_{23} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}$

$A_{31} = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$ $\quad$ $A_{32} = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$



Es. di sviluppo:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Sviluppiamo lungo la seconda riga: $(i=2)$

$$\det A = \sum_{j=1}^{2} (-1)^{2+j} a_{2j} \det(A_{2j})$$

$$= (-1)^{2+1} a_{21} \det(A_{21}) + (-1)^{2+2} a_{22} \det A_{22}$$

$$= -cb + da = ad - bc$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$ Sviluppiamo lungo la 3ª colonna

$$\det A = \sum_{i=1}^{3} (-1)^{i+3} a_{i3} \det A_{i3} = a_{13} \det(A_{13}) - a_{23} \det(A_{23}) +$$

$$+ a_{33} \det(A_{33}) = 3 \det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} - 6 \det\begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} + 9 \det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$$

dim: 1) Dimostriamo $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$

Consideriamo: $\hat{e}_1 = e_1^t = (1 0 \dots 0)$, $\hat{e}_2 = e_2^t = (0 1 0 \dots 0)$, ..

.., $\hat{e}_i = e_i^t = (0 \dots 0 \overset{i}{1} 0 \dots 0)$, .., $\hat{e}_n = e_n^t \in Mat_{1\times m}(\mathbb{K})$.

La i-esima riga di A è

$$A_i = (a_{i1}, a_{i2}, \dots, a_{im}) = \sum_{j=1}^{m} a_{ij}\, \hat{e}_j \qquad (*)$$

Es: $(2,-1,3) = 2(1,0,0) - (0,1,0) + 3(0,0,1)$.

$$\det A = \det(A_1, \dots, A_i, \dots, A_m) =$$

$$\overset{(*)}{=} \det(A_1, \dots, \sum_{j=1}^{m} a_{ij}\, \hat{e}_j, \dots, A_m)$$

$$\underset{\text{det è multilineare}}{=} \sum_{j=1}^{m} a_{ij} \det(A_1, \dots, A_{i-1}, \hat{e}_j, A_{i+1}, \dots, A_n)$$

Rimane da dimostrare

$$\det(A_1, \dots, A_{i-1}, \hat{e}_j, A_{i+1}, \dots, A_n) = (-1)^{i+j} \det A_{ij}.$$

$$\det(A_1, \dots, A_{i-1}, \hat{e}_j, A_{i+1}, \dots, A_n) =$$

$$= \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\,i-1} & a_{1i} & a_{1\,i+1} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,i-1} & a_{i-1\,i} & a_{i-1\,i+1} & \cdots & a_{i-1\,j} & \cdots & a_{i-1\,n} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ a_{i+1\,1} & a_{i+1\,2} & \cdots & a_{i+1\,i-1} & a_{i+1,i} & a_{i+1\,i+1} & \cdots & a_{i+1\,j} & \cdots & a_{i+1\,n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\,i-1} & a_{ni} & a_{n\,i+1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

$$= \det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\,i-1} & a_{1i} & a_{1\,i+1} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,i-1} & a_{i-1\,i} & a_{i-1\,i+1} & \cdots & a_{i-1\,j} & \cdots & a_{i-1\,n} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ a_{i+1\,1} & a_{i+1\,2} & \cdots & a_{i+1\,i-1} & a_{i+1,i} & a_{i+1\,i+1} & \cdots & a_{i+1\,j} & \cdots & a_{i+1\,n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\,i-1} & a_{ni} & a_{n\,i+1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

$$= \det\begin{array}{cc} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & 0 & \cdots & a_{1n} \\ \vdots & & & & & & & \\ a_{i-1\,1} & a_{i-1\,2} & \cdots & a_{i-1\,i} & \cdots & 0 & \cdots & a_{i-1\,n} \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ a_{i+1\,1} & a_{i+1\,2} & \cdots & a_{i+1\,i} & \cdots & 0 & \cdots & a_{i+1\,n} \\ \vdots & & & & & & & \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & 0 & \cdots & a_{nn} \end{pmatrix} & \begin{matrix} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{matrix} \\ \uparrow j & \end{array}$$

$$
= \det \left(\begin{array}{cccccccc}
a_{11} & a_{12} & \cdots & a_{1i} & \cdots & 0 & \cdots & a_{1n} \\
\vdots & & & & & & & \\
a_{i-1\,1} & a_{i-1\,2} & \cdots & a_{i-1\,i} & \cdots & 0 & \cdots & a_{i-1\,n} \\
0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\
a_{i+1\,1} & a_{i+1\,2} & \cdots & a_{i+1\,i} & \cdots & 0 & \cdots & a_{i+1\,n} \\
\vdots & & & & & & & \\
a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & 0 & \cdots & a_{nn}
\end{array}\right)
\begin{array}{l}
\\ \\ \\ \leftarrow i \\ \\ \\ \\
\end{array}
$$

$$
\begin{array}{cccccccc}
 & & & & & j & &
\end{array}
$$

i-1 scambi di riga

$$
\overset{!}{=} (-1)^{i-1} \det
\begin{array}{c}
\begin{array}{ccccccc}
 & & & & j & &
\end{array}\\
\left(\begin{array}{ccccccc}
0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\
a_{11} & \cdots & a_{1i} & \cdots & 0 & \cdots & a_{1n} \\
a_{21} & \cdots & a_{2i} & \cdots & 0 & \cdots & a_{2n} \\
\vdots & & & & & & \\
a_{i-1\,1} & \cdots & a_{i-1\,i} & \cdots & 0 & \cdots & a_{i-1\,n} \\
a_{i+1\,1} & \cdots & a_{i+1\,i} & \cdots & 0 & \cdots & a_{i+1\,n} \\
\vdots & & & & & & \\
a_{n1} & \cdots & a_{ni} & \cdots & 0 & \cdots & a_{nn}
\end{array}\right)
\end{array}
\begin{array}{l}
\\ \\ \\ \\ \\ \leftarrow i \\ \\ \\ \\
\end{array}
$$

$$
= (-1)^{i-1} \det
\begin{array}{c}
\begin{array}{ccccccc} & & & j & & & \\ & & & \downarrow & & & \end{array} \\
\begin{pmatrix}
0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\
a_{11} & \cdots & a_{1i} & \cdots & 0 & \cdots & a_{1n} \\
a_{21} & \cdots & a_{2i} & \cdots & 0 & \cdots & a_{2n} \\
\vdots & & & & & & \\
a_{i-1\,1} & \cdots & a_{i-1\,i} & \cdots & 0 & \cdots & a_{i-1\,n} \\
a_{i+1\,1} & \cdots & a_{i+1\,i} & \cdots & 0 & \cdots & a_{i+1\,n} \\
\vdots & & & & & & \\
a_{n1} & \cdots & a_{ni} & \cdots & 0 & \cdots & a_{nn}
\end{pmatrix}
\end{array}
\leftarrow i
$$

j-1 scambi di colonna

$$
\overset{!}{=} (-1)^{j-1}(-1)^{i-1} \det
\begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & & \\
\vdots & a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\
\vdots & a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\
\vdots & \vdots & & & \\
0 & a_{n1} & a_{n2} & \cdots & a_{n,n}
\end{pmatrix}
= (-1)^{i+1} \det A_{ij}
$$

A_{ij}.

Riepilogo: per calcolare det

1) Usare operazioni elementari sulle righe e sulle colonne in modo da creare una riga o una colonna piena di zeri

2) Sviluppare lungo quella riga o colonna

·) $\det A^t = \det A$

·) det (a scala) = prodotto degli elementi diagonali.

Determinante di matrici triangolari a blocchi

$$\det \begin{array}{cc} & \begin{array}{cc} k & t \end{array} \\ \begin{array}{c} k \\ t \end{array} & \left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right) \end{array}$$

$A \in Mat_{k \times k}$, $B \in Mat_{t \times t}$

$C \in Mat_{k \times t}$

$0 = 0_{t \times k}$.

<u>Teorema</u>: $\det \left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right) = \det A \det B.$ $\blacksquare$

$$\det \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 6 \\ \hline 0 & 0 & 11 \end{array}\right) = 11 \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = -22$$

Es:

$$\det\begin{pmatrix} 2 & 1 & 2 & 1 & 1 \\ 4 & 2 & 2 & 4 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 3 & 1 & 2 \end{pmatrix} = \det\begin{pmatrix} 2 & 1 & 2 & 1 & 1 \\ 4 & 2 & 2 & 4 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 3 & 1 & 2 \end{pmatrix}$$

sviluppo lungo la 3ª riga

$$= (-1)\det\begin{pmatrix} 2 & 1 & 1 & 1 \\ 4 & 2 & 4 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{pmatrix}$$

$$= (-1)\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 2 & 1 & 2 \end{pmatrix} = (-1)\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 2 & 2 & 1 \\ 0 & 2 & 1 & 2 \end{pmatrix}$$

$$(-1)\det\begin{pmatrix}1&1&1&1\\0&0&2&-1\\0&2&2&1\\0&2&1&2\end{pmatrix}=(-1)\det\begin{pmatrix}0&2&-1\\2&2&1\\2&1&2\end{pmatrix}$$

$$=-\det\begin{pmatrix}0&2&-1\\0&1&-1\\2&1&2\end{pmatrix}$$

$$=-2\det\begin{pmatrix}2&-1\\1&-1\end{pmatrix}$$

$$=-2\det\begin{pmatrix}0&1\\1&-1\end{pmatrix}=$$

$$=2\det\begin{pmatrix}1&-1\\0&1\end{pmatrix}=2$$

a scala

Calcolo dell'inversa tramite il determinante

Def: Dato $n \geq 1$ e $A \in Mat_{n \times n}(\mathbb{K})$ e due indici $i, j \in \{1, \dots, n\}$, il <u>co-fattore (i,j) di A</u> è il numero

$$C_{ij} = C_{ij}(A) = (-1)^{i+j} \det A_{ij}$$

segno di C_{ij}:

$$\begin{pmatrix} + & - & + & - & \\ - & + & - & + & \cdots \\ + & - & + & - & \\ - & + & - & + & \\ & \cdots & & & \end{pmatrix}$$

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}.$$

Es:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$C_{11} = (-1)^{1+1} \det A_{11} = d$$
$$C_{12} = (-1)^{1+2} \det A_{12} = -c$$
$$C_{21} = (-1)^{2+1} \det A_{21} = -b$$
$$C_{22} = (-1)^{2+2} \det A_{22} = a$$

La matrice aggiunta di A è la matrice $n \times n$ che ha per componenti i co-fattori:

$$\mathrm{Agg}(A) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

$$\mathrm{Agg}(A)_i^j := C_{ij}$$

Es: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \mathrm{Agg}(A) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

OSS: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ e $ad-bc \neq 0$ allora

$$\left.\begin{aligned} A^{-1} &= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ Agg(A) &= \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \end{aligned}\right] \Rightarrow A^{-1} = \frac{1}{ad-bc} Agg(A)^t$$

$$\Rightarrow \quad \mathbb{1}_2 = A A^{-1} = \frac{1}{ad-bc} A \, Agg(A)^t$$

$$\Rightarrow \quad A \, Agg(A)^t = (ad-bc)\mathbb{1}_2 = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$\Rightarrow \quad A \, Agg(A)^t = \det A \; \mathbb{1}_2 = \begin{pmatrix} \det A & 0 \\ 0 & \det A \end{pmatrix}.$$