Ieri "abbiamo visto" che per ogni $n \geq 1$ la funzione

$$d^{(n)} : Mat_{n\times n}(\mathbb{K}) \to \mathbb{K}$$

definita come

$$d^{(1)} = Id_{\mathbb{K}}$$

$$d^{(n)}(A) = \sum_{k=1}^{n} (-1)^{k+1} a_{k1} d^{(n-1)}(A_{k1})$$

è multilineare e alternante sulle righe

$$d^{(n)}(\mathbb{1}_n) = d^{(n)}(e_1^t, \dots, e_n^t) = 1$$

Equivalentemente,

$$d^{(n)}(P_{ij}A) = -d^{(n)}(A)$$

$$d^{(n)}(D_i(\lambda)A) = \lambda\, d^{(n)}(A)$$

$$d^{(n)}(F_{ij}(c)A) = d^{(n)}(A)$$

$\forall A \in Mat_{n\times n}(\mathbb{K})$, $\forall i \neq j$, $\forall \lambda, c \in \mathbb{K}$, $\lambda \neq 0$.

$$d^{(2)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$d^{(2)}\begin{pmatrix} a & b \\ a & b \end{pmatrix} = ab - ab = 0$$

$$d^{(2)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

Teorema (unicità del determinante)

Sia $f: \mathrm{Mat}_{n\times n}(\mathbb{K}) \to \mathbb{K}$ t.c.

$$f(P_{ij}A) = -f(A)$$

$$f(D_i(\lambda)A) = \lambda f(A)$$

$$f(F_{ij}(c)A) = f(A)$$

$\forall A \in \mathrm{Mat}_{n\times n}(\mathbb{K})$, $\forall i \neq j$,
$\forall \lambda, c \in \mathbb{K}$, $\lambda \neq 0$
ed inoltre

$$f(\mathbb{1}_n) = 1.$$

Allora $f = d^{(n)}$.

dim: Sia $A \in \mathrm{Mat}_{n\times n}(\mathbb{K})$ e dimostriamo che

$$f(A) = d^{(n)}(A).$$

$$A \underset{R}{\rightsquigarrow} R = \mathrm{rref}(A)$$

$R = E_k \cdots E_1 A$ per opportune matrici elementari $E_1, \dots, E_k$.

$$A = E_1^{-1} \cdots E_k^{-1} R$$

Poiché

$$P_{ij}^{-1} = P_{ij}$$

$$D_i(\lambda)^{-1} = D_i\left(\frac{1}{\lambda}\right)$$

$$F_{ij}(c)^{-1} = F_{ij}(-c)$$

l'inversa di una matrice el.

è una matrice elementare.

$$A = F_1 \cdots F_k R$$

dove $F_1, \dots, F_k$ sono matrici elementari.

$$f(A) = c\, f(R)$$

dove c è una costante che non dipende da f ma solo dalle operazioni elementari svolte per trasformare A in R.

$R \to$ ① L'ultima riga di R è nulla

$R \to$ ② $R = \mathbb{1}_n$

① $\iff \operatorname{rg} A < n$

② $\iff \operatorname{rg} A = n \iff A$ inv.

$$f(A) = c\, f(R) = \begin{cases} 0 & ① \\ c & ② \end{cases}$$

Similmente,

$$d^{(n)}(A) = \begin{cases} 0 & ① \\ c & ② \end{cases}$$

$$f(A) = d^{(n)}(A)$$

$\blacksquare$

COR: Sia V un $\mathbb{K}$-sp. vett.
Sia $\mathcal{B} = (v_1, \dots, v_n)$ una base di V.
Sia $f: \underbrace{V \times \dots \times V}_{n = \dim V} \longrightarrow \mathbb{K}$
t.c. f è multilineare, alternante e

$$f(v_1, \dots, v_n) = 1.$$

Allora tale f esiste ed è unica, più precisamente

$$f(w_1, \dots, w_n) = \det\left(F_{\mathcal{B}}(w_1)^t, \dots, F_{\mathcal{B}}(w_n)^t\right) \quad (*)$$

dim: Sia f definita da $(*)$.
Vediamo che f è

- multilineare:

$$f(-, \alpha u + \beta w, -) \overset{(*)}{=}$$

$$= \det(-, F_{\mathcal{B}}(\alpha u + \beta w)^t, -)$$

$F_{\mathcal{B}}$ e ^{t} sono lineari

$$= \det(-, \alpha F_{\mathcal{B}}(u)^t + \beta F_{\mathcal{B}}(w)^t, -)$$

det è multilineare

$$= \alpha \det(-, F_{\mathcal{B}}(u)^t, -) + \beta \det(-, F_{\mathcal{B}}(w)^t, -).$$

$$= \alpha f(-, u, -) + \beta f(-, w, -).$$

- alternante

$f(-, w, -, w, -) =$

$= \det(-, F_{\mathcal{B}}(w)^t, -, F_{\mathcal{B}}(w)^t, -)$

$= 0$

- $f(v_1, \ldots, v_n) =$

$= \det(F_{\mathcal{B}}(v_1)^t, \ldots, F_{\mathcal{B}}(v_n)^t)$

$= \det(e_1^t, \ldots, e_n^t) = \det(\mathbb{1}_n) = 1.$

Sia $g : V \times \ldots \times V \to \mathbb{K}$ multilineare, alternante $g(v_1, \ldots, v_n) = 1$.

Dimostriamo che

$g(w_1, \ldots, w_n) = \det(F_{\mathcal{B}}(w_1)^t, \ldots, F_{\mathcal{B}}(w_n)^t)$

Consideriamo la f.ne $g' : \mathrm{Mat}_{n\times n}(\mathbb{K}) \to \mathbb{K}$ definita come segue

$g'(X_1, \ldots, X_n) :=$

$= g(F_{\mathcal{B}}^{-1}(X_1^t), \ldots, F_{\mathcal{B}}^{-1}(X_n^t)).$

g' è multilineare e alternante sulle righe $g'(e_1^t, \ldots, e_n^t) = 1$. [Esercizio!]

$\Rightarrow g'(X_1, \dots, X_n) =$

$= \det(X_1, \dots, X_n)$

$g(w_1, \dots, w_n) =$

$= g'(F_{\mathcal{B}}(w_1)^t, \dots, F_{\mathcal{B}}(w_n)^t)$

$= \det(F_{\mathcal{B}}(w_1)^t, \dots, F_{\mathcal{B}}(w_n)^t).$

$\blacksquare$

Tale f.ne si chiama determinante

$\det : V \times \dots \times V \to \mathbb{K}.$

(dipende dalla scelta di una base di V).

Determinante 2x2 come area "orientata".

Sia $V = \mathcal{V}_O^2 = \{\overrightarrow{OP} \mid P \in \mathcal{E}^2\}$

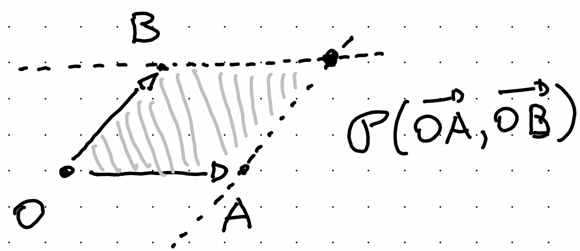
Una base $\mathcal{B} = (\overrightarrow{OA}, \overrightarrow{OB})$ di V è unitaria se

Area $\mathcal{P}(\overrightarrow{OA}, \overrightarrow{OB}) = 1$

dove

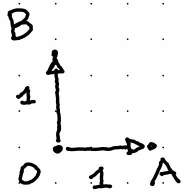
$\mathcal{P}(\overrightarrow{OA}, \overrightarrow{OB})$ = parallelogramma generato da $\overrightarrow{OA}$ e $\overrightarrow{OB}$ =

=



Es:

•)



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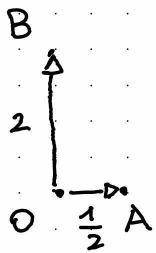

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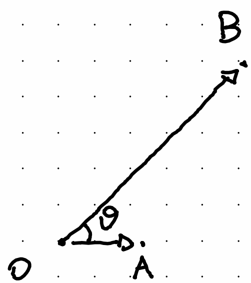
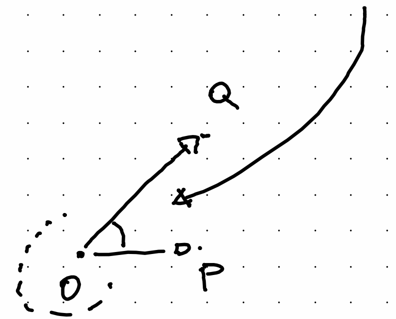


Area $\mathcal{P}(\vec{OP}, \vec{OQ}) =$

$= |\overline{OP}|\,|\overline{OQ}| \sin \widehat{POQ}$



Teorema:

Sia $\mathcal{B}$ una base <u>unitaria</u> di $\mathcal{V}_O^2$. Allora

$$\text{Area}\left(\mathcal{P}(\vec{OP},\vec{OQ})\right) = \left|\det\left(F_{\mathcal{B}}(\vec{OP})^t, F_{\mathcal{B}}(\vec{OQ})^t\right)\right|$$

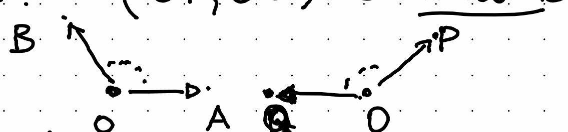
<u>dim</u>: Sia

$$A : \mathcal{V}_O^2 \times \mathcal{V}_O^2 \longrightarrow \mathbb{R}$$

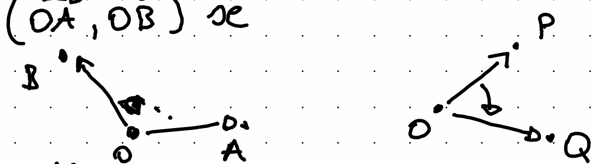
definita come

$$A(\vec{OP},\vec{OQ}) = \begin{cases} \text{Area}\left(\mathcal{P}(\vec{OP},\vec{OQ})\right) & \text{se } (\vec{OP},\vec{OQ}) \text{ è concorde con } \mathcal{B} \\ -\text{Area}\left(\mathcal{P}(\vec{OP},\vec{OQ})\right) & \text{se } (\vec{OP},\vec{OQ}) \text{ non è concorde con } \mathcal{B} \end{cases}$$

<u>dove</u>: $(\vec{OP},\vec{OQ})$ è <u>concorde</u> con $(\vec{OA},\vec{OB})$ se



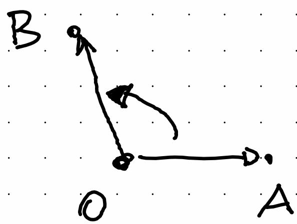
concordi



discordi

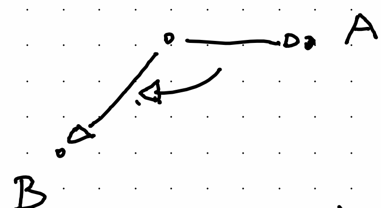
Coppie concordi di vettori geometrici:

Anti-orarie oraria



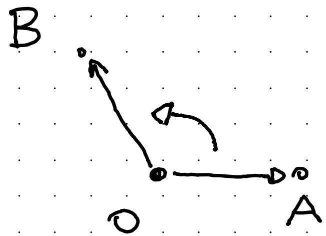
$(\vec{OA}, \vec{OB})$

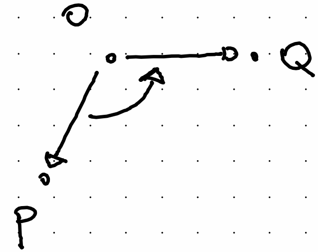
senso anti-orario



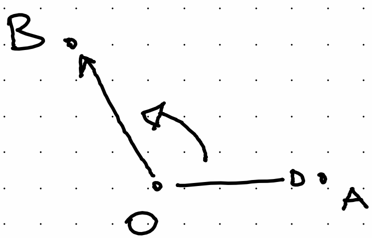
senso orario

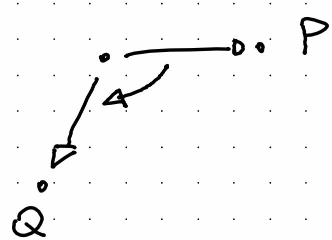
$(\vec{OA}, \vec{OB})$ e $(\vec{OP}, \vec{OQ})$ è <u>concorde</u> se l'angolo da $\vec{OA}$ a $\vec{OB}$ e da $\vec{OP}$ a $\vec{OQ}$ è per entrambi orario o per entrambi anti-orario.





$(\vec{OA}, \vec{OB})$ e $(\vec{OP}, \vec{OQ})$ sono <u>concordi</u>.





$(\vec{OA}, \vec{OB})$ e $(\vec{OP}, \vec{OQ})$ sono discordi

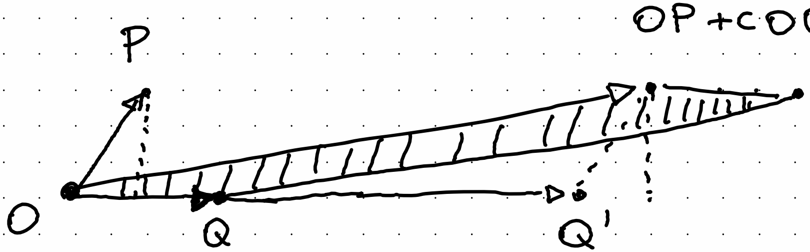
$(\overrightarrow{OA}, \overrightarrow{OB})$ e $(\overrightarrow{OP}, \overrightarrow{OQ})$ sono discordi.

Dimostriamo che A è multilineare, alternante e vale 1 sugli elementi di $\mathcal{B}$.

- Sia $c>0$

$$A(\overrightarrow{OP}+c\overrightarrow{OQ},\overrightarrow{OQ}) \overset{?}{=} A(\overrightarrow{OP},\overrightarrow{OQ})$$

$$\text{Area}(\mathcal{P}(\overrightarrow{OP}+c\overrightarrow{OQ},\overrightarrow{OQ})) = \text{Area}(\mathcal{P}(\overrightarrow{OP},\overrightarrow{OQ}))$$



$\overrightarrow{OQ'} = c\,\overrightarrow{OQ}$

•) $A(c\overrightarrow{OP},\overrightarrow{OQ})$
$= c\,A(\overrightarrow{OP},\overrightarrow{OQ})$

•) $\mathcal{B} = (\overrightarrow{OA},\overrightarrow{OB})$
$A(\overrightarrow{OA},\overrightarrow{OB}) = 1$

$$\text{Area}(\mathcal{P}(\overrightarrow{OP},\overrightarrow{OQ})) = \text{Area}(\mathcal{P}(\overrightarrow{OQ},\overrightarrow{OP}))$$

•) $A(\overrightarrow{OQ},\overrightarrow{OP}) = -A(\overrightarrow{OP},\overrightarrow{OQ})$

Per il corollario

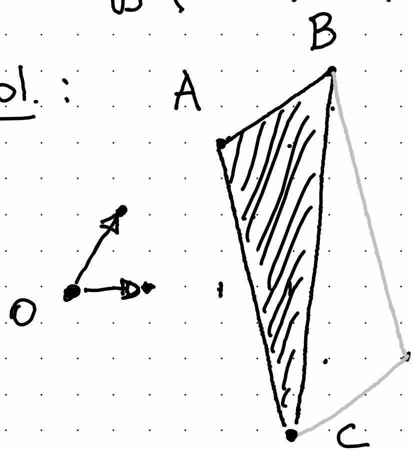
$$A(\vec{OP}, \vec{OQ}) = \det\left(F_{\mathcal{B}}(\vec{OP})^t, F_{\mathcal{B}}(\vec{OQ})^t\right)$$

$\boxtimes$

Es: Sia $\mathcal{B}$ una base unitaria di V_0^2.
Calcolare l'area del triangolo di vertici $A, B, C \in \mathcal{E}^2$ dove

$$F_{\mathcal{B}}(\vec{OA}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad F_{\mathcal{B}}(\vec{OB}) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad F_{\mathcal{B}}(\vec{OC}) = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

Sol.:



$$\text{Area}(T) = \frac{1}{2}\,\text{Area}\,\mathcal{P}(\vec{AC}, \vec{AB})$$

$\vec{AC} = \vec{OC} - \vec{OA}$, $\vec{AB} = \vec{OB} - \vec{OA}$

$$= \frac{1}{2}\left|\det\left(F_{\mathcal{B}}(\vec{OC} - \vec{OA})^t, F_{\mathcal{B}}(\vec{OB} - \vec{OA})^t\right)\right|$$

$$= \frac{1}{2}\left|\det\begin{pmatrix} 3 & -4 \\ 1 & 1 \end{pmatrix}\right| = \frac{1}{2}|3+4| = \frac{7}{2}.$$

$\boxtimes$

Tecniche di calcolo

$\det(A) = ?$

$A \rightsquigarrow S \rightsquigarrow \mathrm{rref}(A)$

Prop.: Sia

$$S = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ 0 & s_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & s_{nn} \end{pmatrix}$$

una matrice $n \times n$ a scale.
Allora

$$\det(S) = s_{11} s_{22} \cdots s_{nn}$$

$\blacksquare$

(Lunedì)

Es:

$$\det \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} =$$

$$= 2 \det \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}$$

$$= 2 \det \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \\ 0 & -\frac{5}{2} & -\frac{1}{2} \end{pmatrix}$$

def di $d^{(3)}$.

$$\overset{!}{=} 2 \det \begin{pmatrix} 0 & -1 \\ -\frac{5}{2} & -\frac{1}{2} \end{pmatrix}$$

$$= -2 \det \begin{pmatrix} -\frac{5}{2} & -\frac{1}{2} \\ 0 & -1 \end{pmatrix} \overset{\text{Prop}}{=}$$

$$= -2 \left(-\frac{5}{2}\right)(-1) = -5. \quad \blacksquare$$