Matrici di proiezione (su un sottospazio vettoriale di $\mathbb{K}^n$ lungo un suo complementare)

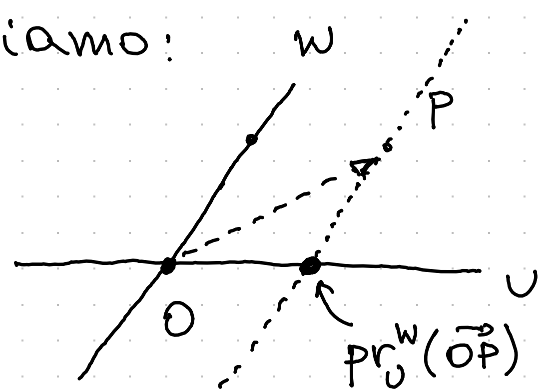
Sia $U \subseteq \mathbb{K}^n$ un s.sp. vettoriale. Sia $W \subseteq \mathbb{K}^n$ un suo complementare, i.e.

$$\mathbb{K}^n = U \oplus W$$

$$pr_U^W : \mathbb{K}^n \longrightarrow \mathbb{K}^n \quad \text{è lineare}$$

$$u + w \longmapsto u$$

Ricordiamo:



La matrice associata a pr_U^W nella base $\mathcal{E} = (e_1, \dots, e_n)$ (sia in partenza che in arrivo) si chiama <u>matrice di proiezione</u> (su U lungo W) e si denota $P = P_U^W$

$$\begin{tikzcd}
\mathbb{K}^n \arrow[r, "pr_U^W"] \arrow[d, "Id = F_e"'] & \mathbb{K}^n \arrow[d, "F_e = Id"] \\
\mathbb{K}^n \arrow[r, "S_P"'] & \mathbb{K}^n
\end{tikzcd}$$

$$S_P = F_e \circ pr_U^W \circ F_e^{-1} = pr_U^W .$$

$$\boxed{pr_U^W(v) = Pv}$$

Es: In $\mathbb{R}^3$ consideriamo i sottospazi vettoriali

$$U: \; x_1 + 2x_2 - x_3 = 0$$

$$W = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

1) Dimostrare che $\mathbb{R}^3 = U \oplus W$.

2) Trovare P_U^W (= matrice associata a pr_U^W nella base $\mathcal{E}$)

3) Calcolare $pr_U^W\left(\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}\right)$.

Sol.: 1) $U = \operatorname{Ker}(1, 2, -1) = \left\langle \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$\dim U = 2$. $\dim W = 1$.

$U \cap W \ni x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ x \\ x \end{pmatrix}$ e $x + 2x - x = 0$

$\Rightarrow \; U \cap W = \{ 0_{\mathbb{R}^3} \}$.

$\Rightarrow \; \dim(U+W) = \dim U + \dim W = 3 \Rightarrow U + W = \mathbb{R}^3$.

$$\mathcal{B}_U = \left(u_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\mathcal{B}_W = \left(w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

$$\mathcal{B} = \mathcal{B}_U \cup \mathcal{B}_W = (u_1, u_2, w) \overset{\text{base}}{\subset} \mathbb{R}^3.$$

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "pr_U^W"] \arrow[d, "F_{\mathcal{B}}"] & \mathbb{R}^3 \arrow[d, "F_{\mathcal{B}}"] \\
\mathbb{R}^3 \arrow[r, "S_A"] & \mathbb{R}^3
\end{tikzcd}$$

$$S_A := F_{\mathcal{B}} \circ pr_U^W \circ F_{\mathcal{B}}^{-1}$$

$$A^1 = S_A(e_1) = F_{\mathcal{B}} \circ pr_U^W \circ F_{\mathcal{B}}^{-1}(e_1)$$

$$= (F_{\mathcal{B}} \circ pr_U^W)(F_{\mathcal{B}}^{-1}(e_1))$$

$$= (F_{\mathcal{B}} \circ pr_U^W)(u_1) = F_{\mathcal{B}}(pr_U^W(u_1))$$

$$= F_{\mathcal{B}}(u_1) = e_1$$

$$\mathcal{B}_U = \left(u_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\mathcal{B}_W = \left(w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

$$\mathcal{B} = \mathcal{B}_U \cup \mathcal{B}_W = (u_1, u_2, w) \overset{\text{base}}{\subset} \mathbb{R}^3.$$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "pr_U^W"] \arrow[d, "F_{\mathcal{B}}"] & \mathbb{R}^3 \arrow[d, "F_{\mathcal{B}}"] \\ \mathbb{R}^3 \arrow[r, "S_A"] & \mathbb{R}^3 \end{tikzcd}$$

$$S_A := F_{\mathcal{B}} \circ pr_U^W \circ F_{\mathcal{B}}^{-1}$$

$$A^2 = S_A(e_2) = F_{\mathcal{B}} \circ pr_U^W \circ F_{\mathcal{B}}^{-1}(e_2)$$

$$= (F_{\mathcal{B}} \circ pr_U^W)(F_{\mathcal{B}}^{-1}(e_2))$$

$$= (F_{\mathcal{B}} \circ pr_U^W)(u_2) = F_{\mathcal{B}}(pr_U^W(u_2))$$

$$= F_{\mathcal{B}}(u_2) = e_2$$

$$\mathcal{B}_U = \left(u_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\mathcal{B}_W = \left(w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

$$\mathcal{B} = \mathcal{B}_U \cup \mathcal{B}_W = (u_1, u_2, w) \overset{\text{base}}{\subset} \mathbb{R}^3.$$

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{pr_U^W} & \mathbb{R}^3 \\ \downarrow F_{\mathcal{B}} & & \downarrow F_{\mathcal{B}} \\ \mathbb{R}^3 & \xrightarrow{S_A} & \mathbb{R}^3 \end{array}$$

$$S_A := F_{\mathcal{B}} \circ pr_U^W \circ F_{\mathcal{B}}^{-1}$$

$$A^3 = S_A(e_3) = F_{\mathcal{B}} \circ pr_U^W \circ F_{\mathcal{B}}^{-1}(e_3)$$

$$= (F_{\mathcal{B}} \circ pr_U^W)(F_{\mathcal{B}}^{-1}(e_3))$$

$$= (F_{\mathcal{B}} \circ pr_U^W)(w \quad) = F_{\mathcal{B}}(pr_U^W(w))$$

$$= F_{\mathcal{B}}(0_{\mathbb{R}^3}) = 0_{\mathbb{R}^3}$$

$$\mathcal{B}_U = \left(u_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\mathcal{B}_W = \left(w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

$$\mathcal{B} = \mathcal{B}_U \cup \mathcal{B}_W = (u_1, u_2, w) \overset{\text{base}}{\subset} \mathbb{R}^3.$$

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, equal] \arrow[d, "F_e"] & \mathbb{R}^3 \arrow[r, "pr_U^W"] \arrow[d, "F_{\mathcal{B}}"] & \mathbb{R}^3 \arrow[r, equal] \arrow[d, "F_{\mathcal{B}}"] & \mathbb{R}^3 \arrow[d, "F_e"] \\
\mathbb{R}^3 \arrow[rrr, bend right, "S_P"'] & \mathbb{R}^3 \arrow[l, "S_B"'] \arrow[r, "S_A"] & \mathbb{R}^3 \arrow[r, "S_B"] & \mathbb{R}^3
\end{tikzcd}$$

$$A = (e_1 \mid e_2 \mid 0_{\mathbb{R}^3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B = (F_e(u_1) \mid F_e(u_2) \mid F_e(w))$$

$$= \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\boxed{S_P = S_B \circ S_A \circ S_B^{-1}}$$

$$P = B A B^{-1}$$

$$B = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow[\text{Gauss Jordan}]{\text{Esercizio}} B^{-1} = \begin{pmatrix} -1/2 & 0 & 1/2 \\ -1/2 & -1 & 3/2 \\ 1/2 & 1 & -1/2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -1 & 0 & 1 \\ -1 & -2 & 3 \\ 1 & 2 & -1 \end{pmatrix}$$

$$P = B A B^{-1} = \dots = \frac{1}{2} \begin{pmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{pmatrix}$$

4) $$pr_U^W\left(\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}\right) = P \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \qquad \blacksquare$$

<u>OSS</u>:

$$P = \left(pr_U^W(e_1) \,\middle|\, pr_U^W(e_2) \,\middle|\, pr_U^W(e_3) \right)$$

$$
\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d] & \mathbb{K}^n \arrow[r, "L"] \arrow[d, "F_{B_1}"] & \mathbb{K}^m \arrow[r, equal] \arrow[d, "F_{B_2}"] & \cdot \arrow[d] \\
\cdot & \mathbb{K}^n \arrow[r, "S_A"] & \mathbb{K}^m & \cdot
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "F_{B_1}"'] & \cdot \arrow[d, "F_{B_2}"] \\
\cdot \arrow[r, "?", "S_B"'] & \cdot
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "F_{B_1}"] & \cdot \arrow[r, equal] \arrow[d, "F_e"] & \cdot \arrow[d, "F_{B_2}"] \\
\cdot \arrow[r, "S_{B_1}"'] & \cdot & \cdot \arrow[l, "S_{B_2}"]
\end{tikzcd}
$$

$$B = B_2^{-1} B_1$$

Es:

$$\mathbb{R}^3 \xrightarrow{L_1} \mathbb{R}^2 \qquad \mathbb{R}^2 \xrightarrow{L_2} \mathbb{R}^2$$

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\2\end{pmatrix} \qquad \begin{pmatrix}1\\-1\end{pmatrix} \longmapsto \begin{pmatrix}3\\2\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\1\end{pmatrix} \longmapsto \begin{pmatrix}1\\3\end{pmatrix} \qquad \begin{pmatrix}1\\1\end{pmatrix} \longmapsto \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\0\\1\end{pmatrix} \longmapsto \begin{pmatrix}1\\4\end{pmatrix}$$

Calcolare la matrice M che rappresenta $L_2 \circ L_1$ nelle basi canoniche di $\mathbb{R}^3$ e $\mathbb{R}^2$.

Sol.: $\mathcal{B}_1 = \left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right) \overset{\text{base}}{\subset} \mathbb{R}^3$, $\mathcal{B}_2 = \left(\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right) \overset{\text{base}}{\subset} \mathbb{R}^2$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "L_1"] \arrow[d, "F_{\mathcal{B}_1}"] & \mathbb{R}^2 \arrow[d, "F_e"] \\ \mathbb{R}^3 \arrow[r, "S_A"] & \mathbb{R}^2 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^2 \arrow[r, "L_2"] \arrow[d, "F_{\mathcal{B}_2}"] & \mathbb{R}^2 \arrow[d, "F_e"] \\ \mathbb{R}^2 \arrow[r, "S_C"] & \mathbb{R}^2 \end{tikzcd}$$

$$A = \begin{pmatrix}1 & 1 & 1\\2 & 3 & 4\end{pmatrix} \qquad C = \begin{pmatrix}3 & 1\\2 & 1\end{pmatrix}$$

$$\mathcal{B}_1 = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) \overset{\text{base}}{\subset} \mathbb{R}^3, \quad \mathcal{B}_2 = \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \overset{\text{base}}{\subset} \mathbb{R}^2$$

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, equal] \arrow[d, "F_e"] & \mathbb{R}^3 \arrow[r, "L_1"] \arrow[d, "F_{\mathcal{B}_1}"] & \mathbb{R}^2 \arrow[r, equal] \arrow[d, "F_e"] & \mathbb{R}^2 \arrow[r, "L_2"] \arrow[d, "F_{\mathcal{B}_2}"] & \mathbb{R}^2 \arrow[d, "F_e"] \\
\mathbb{R}^3 \arrow[rrrr, bend right, "S_M"'] & \mathbb{R}^3 \arrow[l, "S_{B_1}"'] \arrow[r, "S_A"] & \mathbb{R}^2 & \mathbb{R}^2 \arrow[l, "S_{B_2}"'] \arrow[r, "S_C"] & \mathbb{R}^2
\end{tikzcd}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$M = C B_2^{-1} A B_1^{-1} = \overset{\text{Esercizio}}{\text{----}} =$$

$$= \frac{1}{2} \begin{pmatrix} -1 & 1 & -3 \\ 0 & 1 & -1 \end{pmatrix}$$

<u>comunicazione</u>: Ricevimento di oggi di Azzurra è annullato.

Sistemi lineari

Un sistema in n variabili $x_1, \dots, x_n$ ed m equazioni a coefficienti in un campo $\mathbb{K}$ si dice lineare se è della forma

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

le equazioni sono della forma

$$P_i(x_1, \dots, x_n) = b_i$$

dove P_i è un polinomio omogeneo di grado 1 nelle n variabili $x_1, \dots, x_n$.

Una soluzione del sistema è un vettore

$$\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} \in \mathbb{K}^n$$

t.c.

$$P_i(\bar{x}_1, \dots, \bar{x}_n) = b_i \quad \forall i = 1, \dots, m.$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in Mat_{m \times n}(\mathbb{K})$$

= la matrice dei coefficienti del sistema

$$\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{K}^m$$ = matrice dei Termini noti.

OSS: $\bar{x} \in \mathbb{K}^n$ è soluzione

$$\iff A\bar{x} = b \iff S_A(\bar{x}) = b$$

Il sistema è equivalente all'equazione matriciale

$$AX = b$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{K}^n$$ variabile vettoriale.

$AX = b$ è risolubile

$\iff b \in \operatorname{Im} S_A$

$\iff b \in \operatorname{Col}(A)$

Soluzioni di $AX = b$

$$= S_A^{-1}(b) = \{X \in \mathbb{K}^n \mid S_A(X) = b\}$$

controimmagine

$(A|b)$ = <u>matrice completa</u> del sistema

Teorema (di Rouché-Capelli)

Il sistema $AX=b$ è risolubile

$$\Updownarrow$$

$$rg(A) = rg(A|b)$$

dim:

Il sistema ammette soluzione

$$\Updownarrow$$

$$b \in Col(A)$$

$$\Updownarrow$$

b **non** è una colonna dominante di $(A|b)$

$$\Updownarrow$$

colonne dominanti di $(A|b)$ = colonne dominanti di A

Teorema di struttura delle soluzioni

Se $AX=b$ è risolubile sia X_0 una sua soluzione. Allora l'insieme delle soluzioni è

$$S_A^{-1}(b) = X_0 + \operatorname{Ker} A$$
$$= \{ X_0 + v \mid v \in \operatorname{Ker} A \}$$

dim: Sia Y una soluzione di $AX=b$. Allora

$$AY = b = AX_0$$
$$\Rightarrow A(Y-X_0) = 0_{\mathbb{R}^m}$$
$$\Rightarrow Y - X_0 \in \operatorname{Ker} A$$
$$\Rightarrow Y = X_0 + \underbrace{(Y-X_0)}_{\in \operatorname{Ker} A} \in X_0 + \operatorname{Ker} A$$

viceversa se $Y = X_0 + v$
e $v \in \operatorname{Ker} A$ allora

$$AY = A(X_0 + v) = AX_0 + Av$$
$$= b + 0_{\mathbb{K}^m} = b$$
$$\Rightarrow Y \in S_A^{-1}(b). \quad \boxtimes$$

<u>Es</u>: Studiare il sistema in 3 variabili x_1, x_2, x_3:

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 2 \end{cases}$$

<u>Sol.</u>:

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 3 & -1 & 4 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$(A|b) = \left(\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 3 & -1 & 4 & 2 \end{array}\right)$$

$$(A|b) \rightsquigarrow \left(\begin{array}{ccc|c} 3 & -1 & 4 & 2 \\ 2 & 2 & 3 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 2 & 2 & 3 & 1 \end{array}\right)$$

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 3 & -1 & 4 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$(A|b) = \left(\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 3 & -1 & 4 & 2 \end{array}\right)$$

$$(A|b) \rightsquigarrow \left(\begin{array}{ccc|c} 3 & -1 & 4 & 2 \\ 2 & 2 & 3 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 2 & 2 & 3 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 8 & 1 & -1 \end{array}\right)$$

$\operatorname{rg} A = \operatorname{rg}(A|b) = 2.$

$\dim \operatorname{Ker} A = 3 - \operatorname{rg} A = 1$

Il sistema è risolubile e le soluzioni dipendono da $1 = \dim \operatorname{Ker} A$ parametro.

Troviamo $\operatorname{rref}(A|b)$:

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 8 & 1 & -1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 1 & \frac{1}{8} & -\frac{1}{8} \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{11}{8} & \frac{5}{8} \\ 0 & 1 & \frac{1}{8} & -\frac{1}{8} \end{array}\right) = \mathrm{rref}(A|b)$$

Il sistema iniziale è equivalente (= ha le stesse soluzioni) del sistema

$$\begin{cases} x_1 + \frac{11}{8}x_3 = \frac{5}{8} \\ x_2 + \frac{1}{8}x_3 = -\frac{1}{8} \end{cases}$$

Le soluzioni sono: $X_0 + \ker A$.

$$X_0 = \begin{pmatrix} 5/8 \\ -1/8 \\ 0 \end{pmatrix} \qquad S_A^{-1}(b) = \begin{pmatrix} 5/8 \\ -1/8 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -11 \\ -1 \\ 8 \end{pmatrix} \right\rangle$$

$$\ker A = \left\langle \begin{pmatrix} -11 \\ -1 \\ 8 \end{pmatrix} \right\rangle \qquad = \left\{ \begin{pmatrix} 5/8 - 11t \\ -1/8 - t \\ 8t \end{pmatrix} \middle| t \in \mathbb{R} \right\}$$