Es (n°4, Gen 2022)

$V = \mathbb{R}[x]_{\leq 2}$. $\mathcal{E} = (1, x, x^2)$.

$\mathcal{B} = (1+x, 1+2x, 1+x+x^2)$.

1) Dimostrare che $\mathcal{B}$ è una base di V.

2) $T: V \longrightarrow V$ lineare t.c.

$$T(1+x) = 1$$
$$T(1+2x) = 2$$
$$T(1+x+x^2) = 1+2x^2$$

Scrivere la matrice A che rappresenta T nella base $\mathcal{B}$ in partenza e nella base $\mathcal{E}$ in arrivo.

3) Scrivere la matrice C che rappresenta T nella base $\mathcal{E}$.

4) Trovare $\mathcal{B}_{\mathrm{Ker}(T)}$

5) Trovare $\mathcal{B}_{\mathrm{Im}(T)}$

6) Calcolare $\dim(\mathrm{Im}\,T + \mathrm{Ker}\,T)$ e $\dim(\mathrm{Im}\,T \cap \mathrm{Ker}\,T)$.

Sol.: 1) $\dim V = 3 = |\mathcal{B}|$. Per cui è sufficiente dimostrare che $\mathcal{B}$ è lin. Ind.
$F_e : V \to \mathbb{R}^3$ è un isomorfismo lineare e quindi manda basi in basi. $\mathcal{B}$ è lin. Ind.
$\Leftrightarrow$ $F_e(\mathcal{B}) \subset \mathbb{R}^3$ è lin. Ind.

$$B = \left(F_e(1+X) \,|\, F_e(1+2X) \,|\, F_e(1+X+X^2) \right)$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ $\operatorname{rg} B = 3$ $\Rightarrow$ $\mathcal{B}$ è lin. Ind.

2)
$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "F_{\mathcal{B}}"'] & V \arrow[d, "F_e"] \\ \mathbb{R}^3 \arrow[r, "S_A"] & \mathbb{R}^3 \end{tikzcd}$$

$$F_e \circ T = S_A \circ F_{\mathcal{B}}$$
$$S_A = F_e \circ T \circ F_{\mathcal{B}}^{-1}$$

$$A = \left(F_e T(1+X) \,|\, F_e T(1+2X) \,|\, F_e T(1+X+X^2) \right)$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

3)

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "F_e"'] & V \arrow[d, "F_e"] \\ \mathbb{R}^3 \arrow[r, "S_c"] & \mathbb{R}^3 \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, equal] \arrow[d, "F_e"'] & V \arrow[r, "T"] \arrow[d, "F_{\mathcal{B}}"] & V \arrow[d, "F_e"] \\ \mathbb{R}^3 \arrow[rr, bend right, "S_c"'] & \mathbb{R}^3 \arrow[l, "S_B"] \arrow[r, "S_A"'] & \mathbb{R}^3 \end{tikzcd}$$

$S_C = S_A \circ S_B^{-1}$

$$B = \left(F_e(1+x) \,\middle|\, F_e(1+2x) \,\middle|\, F_e(1+x+x^2) \right)$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C = AB^{-1}$$

<u>NB</u>: $(A|B) \rightsquigarrow (\cancel{|A} B^{-1})$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(B \mid \mathbb{1}_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$B^{-1} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C = AB^{-1} =$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Questo ci dice

$$T(1) = 0$$

$$T(x) = 1$$

$$T(x^2) = 2x^2$$

4) $$\begin{tikzcd}
\operatorname{Ker} T \subset V \arrow[r, "T"] \arrow[d, "F_e"'] \arrow[d, "\simeq"] & V \supset \operatorname{Im} T \arrow[d, "F_e"] \arrow[d, "\simeq"'] \\
\operatorname{Ker} C \subset \mathbb{R}^3 \arrow[r, "S_c"] & \mathbb{R}^3 \supset \operatorname{Im} S_c = \operatorname{Col} C
\end{tikzcd}$$

$$\mathcal{B}_{\operatorname{Ker} T} = F_e^{-1}(\operatorname{Ker} C)$$

$$\operatorname{Ker}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$$\mathcal{B}_{\operatorname{Ker} T} = \left(F_e^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \right)$$

5) $$\mathcal{B}_{\operatorname{Im} T} = F_e^{-1}(\mathcal{B}_{\operatorname{Im} S_c})$$

$$= (1, 2x^2)$$

6) $\operatorname{Ker} T \cap \operatorname{Im} T \supseteq \langle 1 \rangle$, $\dim \operatorname{Ker} T = 1$

$$\langle 1 \rangle \subseteq \operatorname{Ker} T \cap \operatorname{Im} T \subseteq \operatorname{Ker} T = \langle 1 \rangle$$

$$\Rightarrow \quad \operatorname{Ker} T \cap \operatorname{Im} T = \operatorname{Ker} T = \langle 1 \rangle$$

$$\Rightarrow \quad \operatorname{Ker} T \subset \operatorname{Im} T$$

Grassmann

$$\Rightarrow \dim(\operatorname{Im} T + \operatorname{Ker} T) =$$

$$= \dim \operatorname{Im} T + \dim \operatorname{Ker} T - \dim \operatorname{Ker} T \cap \operatorname{Im} T$$

$$= \dim \operatorname{Im} T = 2$$

$$\operatorname{Im} T + \operatorname{Ker} T = \operatorname{Im} T = \langle 1, x^2 \rangle.$$

$\blacksquare$

$$\begin{tikzcd} V \arrow[r, equal] \arrow[d, "F_{\mathcal{B}_1}"'] & V \arrow[d, "F_{\mathcal{B}_2}"] \\ \mathbb{K}^n \arrow[r, "S_B"'] & \mathbb{K}^n \end{tikzcd}$$

$$S_B = F_{\mathcal{B}_2} \circ F_{\mathcal{B}_1}^{-1}$$

"B è la matrice di cambiamento di base da $\mathcal{B}_2$ a $\mathcal{B}_1$"

= "B è la matrice che rappresenta l'identità nella base $\mathcal{B}_1$ in partenza e $\mathcal{B}_2$ in arrivo"

Es (es. 4, Giu 2022)

$V = \mathbb{R}[x]_{\leq 2}$, $W = \mathbb{R}[x]_{\leq 4}$

$F : V \longrightarrow W$

$F(p) = p(x^2) - p(x)$

1) Dimostrare che F è lineare.

2) Calcolare $F(x^2 - 1)$

3) Scrivere la matrice associata ad F nelle basi canoniche $e = (1, x, x^2) \subset V$ e $e = (1, x, x^2, x^3, x^4) \subset W$.

4) $\mathcal{B}_{\operatorname{Ker} F}$, $\mathcal{B}_{\operatorname{Im} F}$.

Sol.: 1) $F = Val_{x^2} - Val_x$ è quindi combinazione lineare di funzioni lineari e quindi F è lineare.

2) $F(x^2-1) = (x^4-1) - (x^2-1)$

$= x^4 - x^2$

3)

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d, "F_e"'] & W \arrow[d, "F_e"] \\ \mathbb{R}^3 \arrow[r, "S_A"'] & \mathbb{R}^5 \end{tikzcd}$$

$A = \left(F_e\, F(1) \mid F_e\, F(x) \mid F_e\, F(x^2) \right).$

$F(1) = 0$

$F(x) = x^2 - x$

$F(x^2) = x^4 - x^2$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4)
$$\begin{tikzcd}
\mathrm{Ker}\,F \subset V \arrow[r, "F"] \arrow[d, "\simeq"] & W \supset \mathrm{Im}\,F \arrow[d, "\simeq"] \\
\mathrm{Ker}\,A \subset \mathbb{R}^3 \arrow[r, "S_A"] & \mathbb{R}^5 \supset \mathrm{Im}\,A
\end{tikzcd}$$

$$\mathcal{B}_{\mathrm{Ker}\,F} = F_e^{-1}\left(\mathcal{B}_{\mathrm{Ker}\,A}\right)$$

$$\mathcal{B}_{\mathrm{Im}\,F} = F_e^{-1}\left(\mathcal{B}_{\mathrm{Col}\,A}\right)$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{Ker}\,A = \mathrm{Ker}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \mathrm{Ker}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$$\operatorname{Ker}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x_2 = 0 \\ x_3 = 0 \end{matrix} \right\}$$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$$\mathcal{B}_{\operatorname{Ker} A} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$\Rightarrow \mathcal{B}_{\operatorname{Ker} F} = \left(F_e^{-1}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = 1 \right)$$

$$\mathcal{B}_{\operatorname{Col} A} = \left(A^2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, A^3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\Rightarrow \mathcal{B}_{\operatorname{Im} F} = \left(-x + x^2, \; -x^2 + x^4 \right)$$

$\blacksquare$

Proiezioni:

$$V = U \oplus W$$

$$pr_U^W : V \longrightarrow V$$
$$u + w \longmapsto u$$

è lineare.

Sia $\mathcal{B}_U = (u_1, \dots, u_r) \overset{\text{base}}{\subset} U$

sia $\mathcal{B}_W = (w_{r+1}, \dots, w_n) \overset{\text{base}}{\subset} W$.

$$\mathcal{B}_V = \mathcal{B}_U \cup \mathcal{B}_W$$
$$= (u_1, \dots, u_r, w_{r+1}, \dots, w_n)$$

è una base di V.

<u>OSS</u>: $\operatorname{Im} pr_U^W = U$, $\operatorname{Ker} pr_U^W = W$.

Scriviamo la matrice associata a pr_U^W nella base $\mathcal{B}_V$.

$$
\begin{aligned}
pr_U^W : V &\longrightarrow V \\
u_1 &\longmapsto u_1 \\
u_2 &\longmapsto u_2 \\
&\vdots \\
u_r &\longmapsto u_r \\
w_{r+1} &\longmapsto 0_V \\
&\vdots \\
w_n &\longmapsto 0_V
\end{aligned}
$$

$$
\begin{tikzcd}
V \arrow[r, "pr_U^W"] \arrow[d, "F_{\mathcal{B}_V}"'] & V \arrow[d, "F_{\mathcal{B}_V}"] \\
\mathbb{K}^n \arrow[r, "S_A"] & \mathbb{K}^n
\end{tikzcd}
$$

$$A = (e_1 | e_2 | \ldots | e_r | 0 | \ldots | 0)$$

$r = 2 \qquad n = 4$

$$A = (e_1 \mid e_2 \mid 0 \mid 0)$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} \mathbb{1}_2 & 0 \\ \hline 0 & 0 \end{array}\right)$$

In generale

$$A = \left(\begin{array}{c|c} \mathbb{1}_r & O_{n-r \times n-r} \\ \hline 0_{n-r \times n-r} & O_{n-r \times n-r} \end{array}\right)$$

Prop: Sia $f: V \to V$.

f è una proiezione (su $\operatorname{Im} f$ lungo $\operatorname{Ker} f$) se e solo se

$$f^2 = f$$

dim:

Se $f = pr_U^W$ allora $V = U \oplus W$

$\forall v = u + w \in V$

$$f^2(v) = f(f(v)) = f(u) = u = f(v)$$

e quindi $f^2 = f$.

Viceversa, se $f^2 = f$, poniamo

$U = \operatorname{Im} f$ e $W := \operatorname{Ker} f$ e dimostriamo

1) $V = U \oplus W$

2) $f(u+w) = u$, $\forall u \in U$ $\forall w \in W$.

Sia $v \in \mathrm{Ker}\, f \cap \mathrm{Im} f$

$\exists\, v' \in V$ t.c. $v = f(v')$.

e inoltre $f(v) = 0_V$

$$v = f(v') \underset{f=f^2}{=} f^2(v') = f(f(v')) = f(v) = 0_V$$

e quindi $\mathrm{Ker}\, f \cap \mathrm{Im} f = \{0_V\}$

$V = \mathrm{Im} f + \mathrm{Ker}\, f$

$$\forall v \in V \quad f(v - f(v)) = f(v) - f^2(v)$$
$$= f(v) - f(v) = 0_V$$

$\Rightarrow \quad v - f(v) \in \mathrm{Ker}\, f \quad \forall v \in V$

$\Rightarrow \quad \exists\, w \in \mathrm{Ker}\, f$ t.c.

$v - f(v) = w$

$\Rightarrow \quad v = f(v) + w \in \mathrm{Im} f + \mathrm{Ker}\, f$

$\forall v \in V \quad \exists!\ v' \in V$ e $w \in \operatorname{Ker} f$ t.c.

$$v = f(v') + w$$

$$f(v) = f^2(v') + f(w)$$

$$= f(v')$$

$$\Rightarrow \quad f = pr_{\operatorname{Im} f}^{\operatorname{Ker} f}.$$

$\square$