Matrice di proiezione ortogonale

$U \subset \mathbb{R}^n$, $\mathcal{B}_U = \{v_1, \dots, v_r\}$ base di U,

$A = (v_1 | \dots | v_r) \in Mat_{n\times r}(\mathbb{R})$.

$X \cdot Y = X^t Y = x_1y_1 + \dots + x_ny_n$: prodotto scalare standard di $\mathbb{R}^n$.

$\mathbb{R}^n = U \oplus U^\perp$, $U^\perp = \{X \in \mathbb{R}^n \mid X \cdot u = 0 \ \forall u \in U\}$

$= \{X \in \mathbb{R}^n \mid X \cdot v_1 = 0, X \cdot v_2 = 0, \dots, X \cdot v_r = 0\}$.

$= \{X \in \mathbb{R}^n \mid A^t X = 0\} = \mathrm{Ker}\, A^t$.

$pr_U : U \oplus U^\perp \longrightarrow U \oplus U^\perp$

$v = u_1 + u_2 \longmapsto u_1 \in U$, $(u_2 \in U^\perp)$.

$pr_U = S_C$. C = matrice di proiezione ortogonale su U.

$v \in \mathbb{R}^n$, $pr_U(v) \in U = \mathrm{Im}\, A \Rightarrow pr_U(v) = AY$
per qualche $Y \in \mathbb{R}^r$.

$$v - pr_U(v) \in U^{\perp} = \operatorname{Ker} A^t$$

$$A^t(v - AY) = 0$$

$$A^t v = A^t A\, Y$$

$$\Rightarrow \qquad Y = (A^t A)^{-1} A^t v$$

$rg(A^t A) = rg(A)$

$\Rightarrow$ $A^t A$ è invertibile

$$\Rightarrow \qquad pr_U(v) = AY = A(A^t A)^{-1} A^t v$$

$$\Rightarrow \qquad \boxed{C = A\,(A^t A)^{-1} A^t}$$

$$pr_U(v) = Cv.$$

Es:

$U: \quad 2x+3y-z=1 \quad \subset \mathbb{R}^3$ s.sp. affine

$U_0: \quad 2x+3y-z=0 \quad \subset \mathbb{R}^3$ s.sp. vett. = giacitura di U.

Calcolare la distanza di $P=\begin{pmatrix}1\\1\\1\end{pmatrix}$ da U.

Sol.: Calcoliamo la matrice di proiezione ortogonale su U_0.

$\mathcal{B}=\left\{\begin{pmatrix}-3\\2\\0\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right\}$ base di U_0.

$$A=\begin{pmatrix}-3&1\\2&0\\0&2\end{pmatrix} \Rightarrow P_{U_0}=A\,(A^tA)^{-1}A^t$$

$$=\begin{pmatrix}-3&1\\2&0\\0&2\end{pmatrix}\left(\begin{pmatrix}-3&2&0\\1&0&2\end{pmatrix}\begin{pmatrix}-3&1\\2&0\\0&2\end{pmatrix}\right)^{-1}\begin{pmatrix}-3&2&0\\1&0&2\end{pmatrix}$$

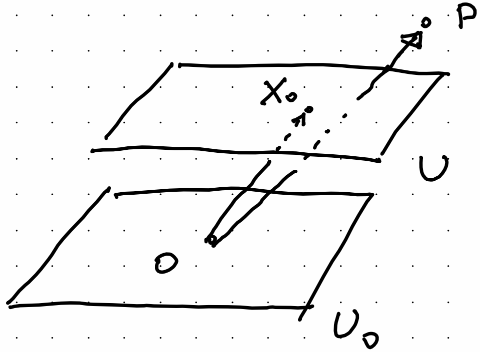
$$=\begin{pmatrix}-3&1\\2&0\\0&2\end{pmatrix}\begin{pmatrix}13&-3\\-3&5\end{pmatrix}^{-1}\begin{pmatrix}-3&2&0\\1&0&2\end{pmatrix}$$

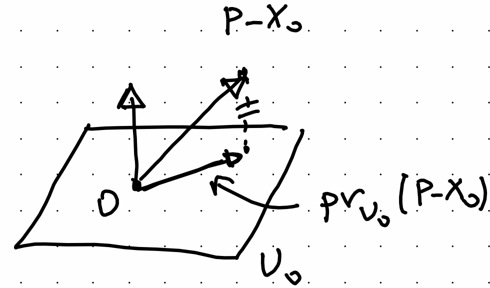
$$= \begin{pmatrix} -3 & 1 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 13 & -3 \\ -3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} -3 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 1 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \frac{1}{56} \begin{pmatrix} 5 & 3 \\ 3 & 13 \end{pmatrix} \begin{pmatrix} -3 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{56} \begin{pmatrix} -3 & 1 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -12 & 10 & 6 \\ 4 & 6 & 26 \end{pmatrix}$$

$$= \frac{1}{56} \begin{pmatrix} 40 & -24 & 8 \\ -24 & 20 & 12 \\ 8 & 12 & 52 \end{pmatrix}$$





$$\operatorname{dist}(P, U) = \operatorname{dist}(P - X_0, U_0)$$

$$= \| P - X_0 - pr_{U_0}(P - X_0) \|$$

$$C e_i = C^i$$

$$pr_{U_0}(P - X_0) = C\,(P - X_0)$$

$$U : 2x + 3y - z = 1 \quad , \quad X_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad P - X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\frac{1}{14}\begin{pmatrix} -6 \\ 5 \\ 3 \end{pmatrix}$$

$$\|$$

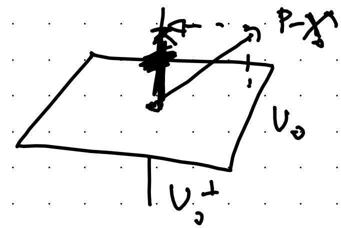
$$C^2 = \frac{1}{56}\begin{pmatrix} -24 \\ 20 \\ 12 \end{pmatrix}$$

$$\operatorname{dist}(P, U) = \left\| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - C \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{e_2} \right\| = \left\| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - C^2 \right\| = \left\| \begin{pmatrix} -6/14 \\ 9/14 \\ -3/14 \end{pmatrix} \right\| =$$

$$= \frac{3}{14} \left\| \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} \right\| = \frac{3}{\sqrt{14}}$$

$U_0: \; 2x+3y-z=0$ $\qquad U_0^{\perp} = \left\langle \underbrace{\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}}_{v} \right\rangle$

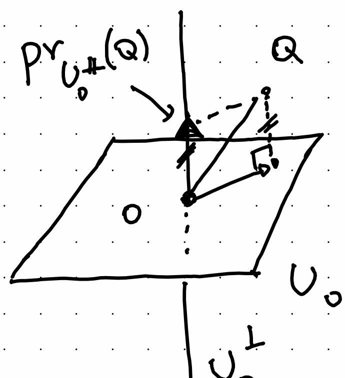
$$= \left\{ \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{U_0} \;\middle|\; \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$$



$$\operatorname{dist}(P, U) = \operatorname{dist}(P-X_0, U_0) = \left\| pr_{U_0^{\perp}}(P-X_0) \right\|$$

$$= \left\| \frac{(P-X_0)\cdot v}{v \cdot v} v \right\| = \frac{|(P-X_0)\cdot v|}{\|v\|}$$

$$= \frac{\left| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right|}{\sqrt{4+9+1}} = \frac{3}{\sqrt{14}}$$

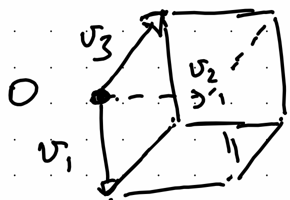


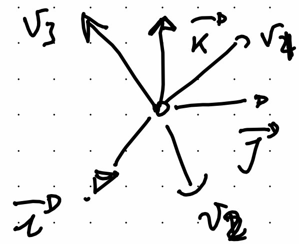
$$\operatorname{dist}(Q, U_0) = \| pr_{U_0}(Q) \|$$

$$\left|\det(v_1|v_2|v_3)\right| = \mathrm{Vol}\left(\text{parallelepipedo } v_1 v_2 v_3 O\right)$$

$$\parallel$$

$$|v_1 \cdot v_2 \wedge v_3| \qquad \mathcal{V}_0^3 \xrightarrow{F_O} \mathbb{R}^3$$





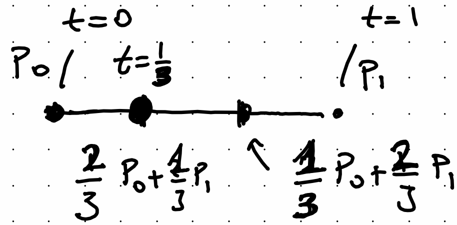
$B = \{\vec{i}, \vec{j}, \vec{k}\}$.

$B = (v_1, v_2, v_3, \dots, v_n) \subset \mathbb{R}^n$ si dice equiversa se

$$\det(v_1|\dots|v_n) > 0$$

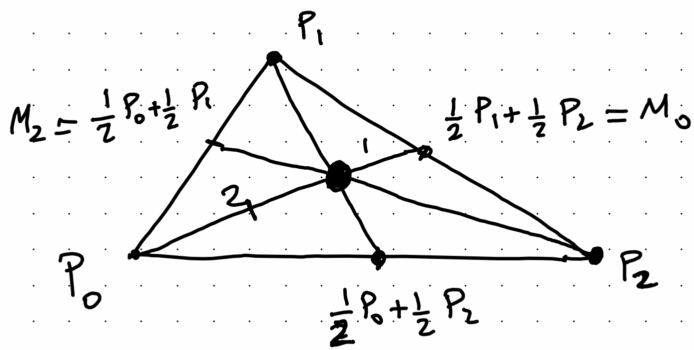
altrimenti B si dice contra-versa.

Combinazioni convesse



$$P_t = P_0 + t\,(P_1 - P_0) = (1-t)P_0 + t P_1$$

Es: P

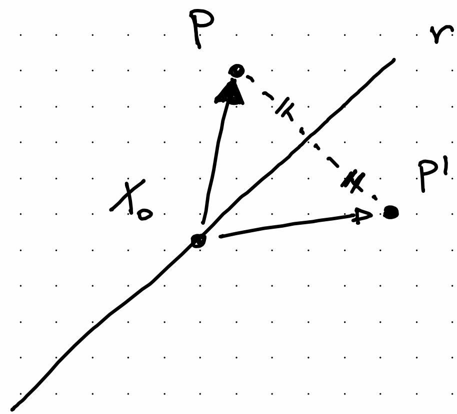


$$\frac{1}{3}P_0 + \frac{2}{3}M_0 = \frac{2}{3}P_0 + \frac{1}{6}P_1 + \frac{1}{6}P_2$$

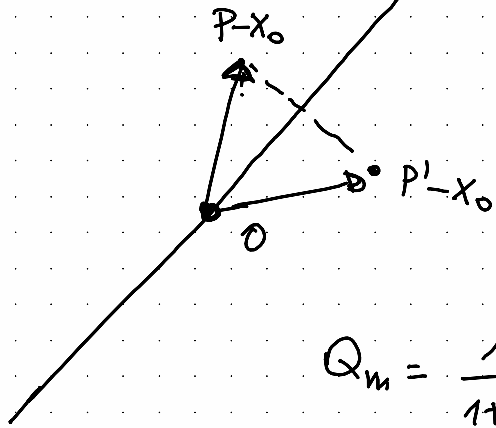
$$= \frac{1}{3}P_2 + \frac{2}{3}M_2$$

$$= \frac{1}{3}P_3 + \frac{2}{3}M_3$$

Riflessione ortogonale



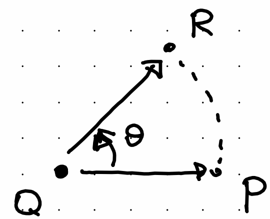
$r_0 : y = mx$



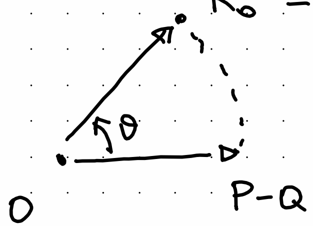
$$P' - X_0 = Q_m (P - X_0)$$

$$P' = X_0 + Q_m (P - X_0)$$

$$Q_m = \frac{1}{1+m^2} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$$



$$R_o = R_\theta (P-Q)$$

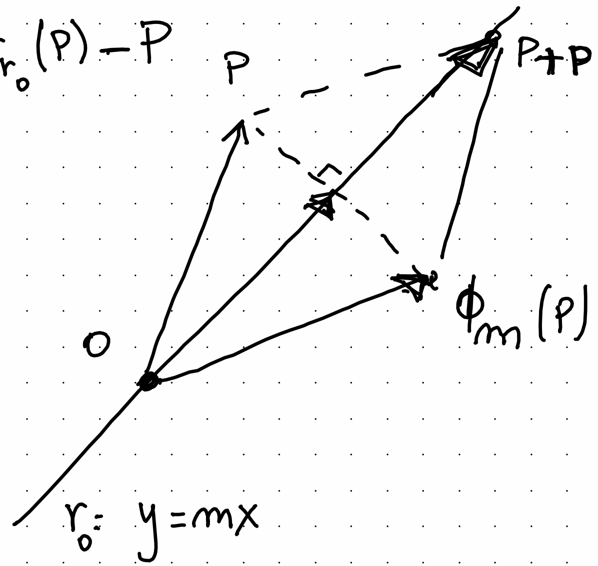


$$R - Q = R_o = R_\theta (P-Q)$$

$$R = Q + R_\theta (P-Q)$$

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\phi_m(P) = 2\,pr_{r_0}(P) - P$$



Es 1 (Sett. 10)

$$b(X,Y) = (x_1+x_2+x_3)(y_1+y_2+y_3) - 2(x_1+x_3)(y_1+y_3)$$

$$= \underline{x_1y_1} + x_1y_2 + x_1y_3 - \underline{2x_1y_1} - 2x_1y_3 +$$
$$+ x_2y_1 + x_2y_2 + x_2y_3$$
$$+ \underline{x_3}y_1 + x_3y_2 + \underline{x_3y_3} - 2\underline{x_3}y_1 - 2\underline{x_3}y_3$$

$$= -x_1y_1 + x_1y_2 - x_1y_3 +$$
$$+ x_2y_1 + x_2y_2 + x_2y_3 +$$
$$- x_3y_1 + x_3y_2 - x_3y_3 = X^t A Y \quad \text{dove}$$

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \qquad b = b_A.$$

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \qquad b = b_A.$$

$Sg(b) = ?$: Troviamo una base ortogonale di $(\mathbb{R}^3, b_A)$

$$\operatorname{Ker} b_A = \operatorname{Ker} A = \operatorname{Ker}\begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & 0 \end{pmatrix}$$

$$= \operatorname{Ker}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \langle \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{= v_3} \rangle$$

$\{e_1, e_2, v_3\}$ è una base di $\mathbb{R}^3$, $U = \langle e_1, e_2 \rangle \oplus \operatorname{Ker} b = \mathbb{R}^3$.

$e_1^2 = a_{11} = -1 < 0$. $\quad e_2 - \dfrac{b_A(e_1, e_2)}{e_1^2} e_1 \in \langle e_1 \rangle^\perp$

$\langle e_1 \rangle^\perp = \{ X \in \mathbb{R}^3 \mid e_1^t A X = 0 \} = \{ X \in \mathbb{R}^3 \mid (-1, 1, -1) X = 0 \} : x - y + z = 0$

$$= \langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \rangle$$

$v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2^2 = (1,1,0)\begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = (1,1,0)\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2 > 0$

$\Rightarrow \{e_1, v_2, v_3\}$ è una base ortogonale di $(\mathbb{R}^3, b_A)$.

$sg(b) = (1, 1)$

$$b(x,y) = (x_1+x_2+x_3)(y_1+y_2+y_3) - 2(x_1+x_3)(y_1+y_3)$$

$$0 = b(x,x) = (x_1+x_2+x_3)^2 - 2(x_1+x_3)^2$$

$$\Leftrightarrow \quad (x_1+x_2+x_3)^2 = 2(x_1+x_3)^2$$

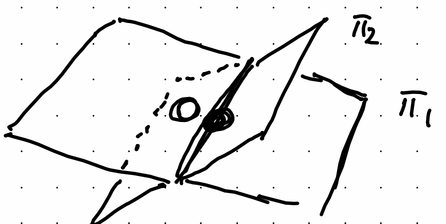
$$\Leftrightarrow \quad x_1+x_2+x_3 = \pm\sqrt{2}(x_1+x_3)$$

$+$: $\quad x_1 - \sqrt{2}x_1 + x_2 + x_3 - \sqrt{2}x_3 = 0 \qquad \pi_1$

$-$: $\quad x_1 + \sqrt{2}x_1 + x_2 + x_3 + \sqrt{2}x_3 = 0 \qquad \pi_2$

π_1: $\quad (1-\sqrt{2})x_1 + x_2 + (1-\sqrt{2})x_3 = 0 \quad = \left\langle \begin{pmatrix} 1-\sqrt{2} \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$

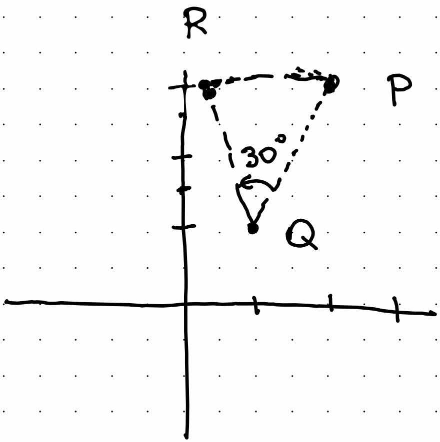
π_2: $\quad (1+\sqrt{2})x_1 + x_2 + (1+\sqrt{2})x_3 = 0 \quad = \left\langle \begin{pmatrix} 1+\sqrt{2} \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$



$U = \langle u \rangle$ $\qquad U \cap \operatorname{Ker} b = \{0\} \Rightarrow b|_U$ è non-degenere

$$u^2 = b(u,u) = 0 \iff u \in \operatorname{Ker} b \iff u = 0$$

Es: $P = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Trovare il punto R ottenuto ruotando P di 30° in senso anti-orario attorno al punto $Q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



$$R - Q = R_{30°}(P-Q)$$

$$R = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

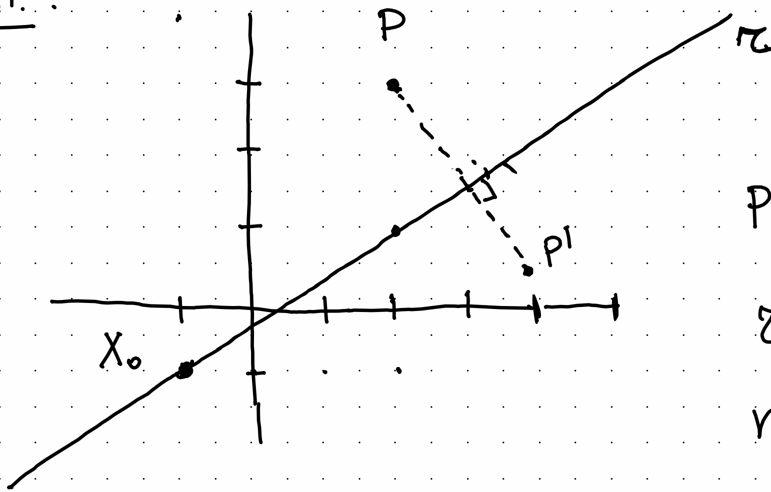
$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{3}-2}{2} \\ \frac{1+2\sqrt{3}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 \\ \frac{3+2\sqrt{3}}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 3+2\sqrt{3} \end{pmatrix}$$

Es: $P = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $r: \ 2x - 3y = 1$

Trovare il punto P' ottenuto riflettendo ortogonalmente P attraverso r.

Sol.:



$r = X_0 + \langle \begin{pmatrix} 3 \\ 2 \end{pmatrix} \rangle$

$X_0 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

$P' = X_0 + Q_m (P - X_0)$

$r_0 = \langle \begin{pmatrix} 3 \\ 2 \end{pmatrix} \rangle : \ 2x - 3y = 0$

$r_0 : \ y = \frac{2}{3} x$

$m = \frac{2}{3}$

$$P' = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \frac{1}{\frac{13}{9}} \begin{pmatrix} \frac{5}{9} & \frac{4}{3} \\ \frac{4}{3} & -\frac{5}{9} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$Q_m = \frac{1}{m^2+1} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \frac{1}{13} \begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \frac{1}{13} \begin{pmatrix} 63 \\ 16 \end{pmatrix} = \begin{pmatrix} 50/13 \\ 3/13 \end{pmatrix}$$

$\mathrm{Ker}\, b \oplus U$

$\mathcal{B}_U = \{ v_1, \dots, v_r \}$. $\quad v_1^2, v_2^2, \dots, v_r^2$

$b(v_i, v_j) \neq 0 \Rightarrow v_i + v_j$ non è isotropo

$\langle v_1 \rangle^\perp \ni v_2$

$\langle v_2 \rangle^\perp \ni v_3 \notin \langle v_1 \rangle^\perp$

$v_1^2 \neq 0 \Rightarrow \quad F_2 = v_2 - \frac{b(v_2, v_1)}{v_1^2} v_1 \in \langle v_1 \rangle^\perp$