Base di Ker$(\mathcal{L})$ $\mathcal{L}: V \to W$ $\dim V = n$, $\dim W = m$

1) Fissare basi $\mathcal{B}_V \subset V$ e $\mathcal{B}_W \subset W$, $\mathcal{B}_V = \{v_1, .., v_n\}$

2) Trovare la matrice che rappresenta $\mathcal{L}$ in queste basi

$$\begin{tikzcd} V \arrow[r, "\mathcal{L}"] \arrow[d, "F_{\mathcal{B}_W}"'] & W \arrow[d, "F_{\mathcal{B}_W}"] \\ \mathbb{K}^n \arrow[r, "A"] & \mathbb{K}^m \end{tikzcd}$$

$$A = \left(F_{\mathcal{B}_W}(\mathcal{L}(v_1)) \,\middle|\, F_{\mathcal{B}_W}(\mathcal{L}(v_2)) \,\middle|\, .. \,\middle|\, F_{\mathcal{B}_W}(\mathcal{L}(v_n)) \right).$$

3) Trovare $R = \text{rref}(A)$ e le soluzioni base $\{X_1, .., X_k\}$ di R.

Una base di Ker $\mathcal{L}$ è $\{F_{\mathcal{B}_V}^{-1}(X_1), .., F_{\mathcal{B}_V}^{-1}(X_k)\}$.

Es: $\mathcal{B}_V = \{v_1, v_2, w_3\}$ $\qquad$ $\mathcal{B}_W = \{w_1, w_2, w_3\}$

$$\mathcal{L}(v_1) = 2w_1 - w_2 + w_3$$
$$\mathcal{L}(v_2) = w_1 + w_2 - w_3$$
$$\mathcal{L}(v_3) = w_1 - 2w_2 + 2w_3$$

$$\begin{tikzcd} V \arrow[r, "\mathcal{L}"] \arrow[d, "F_{\mathcal{B}_V}"'] & W \arrow[d, "F_{\mathcal{B}_W}"] \\ \mathbb{K}^3 \arrow[r, "A"] & \mathbb{K}^3 \end{tikzcd}$$

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 1 & -2 \\ 1 & -1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 1 & -2 \\ 1 & -1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

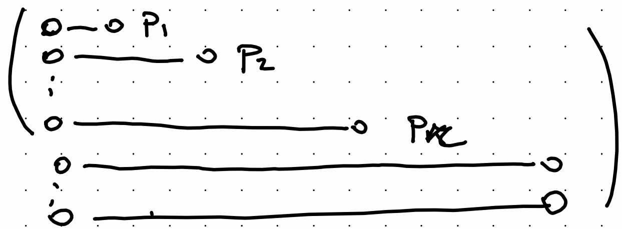
$$\sim \begin{array}{ccc} \downarrow & \downarrow & \\ \end{array}$$
$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}(A) \qquad \operatorname{Ker} A = \left\langle \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$$\Rightarrow \quad \operatorname{Ker} \mathcal{L} = \left\langle F_{\mathcal{B}_V}^{-1}\left(\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right) \right\rangle = \langle -v_1 + v_2 + v_3 \rangle$$

Immagine: $\mathcal{L} : V \to W$ lineare, $\dim V = n$, $\dim W = m$

Per trovare una base di $\operatorname{Im} \mathcal{L}$

1) Fissare basi $\mathcal{B}_V \subset V$, $\mathcal{B}_W \subset W$.
2) Trovare la matrice associata A ad $\mathcal{L}$ in queste basi.
3) Trovare una forma a scala di A:

$A \sim S =$ 

Allora $\operatorname{rg} A = \operatorname{rg} S = r = \#$ dei pivot.

Le colonne dominanti di A (quindi una base per $\operatorname{Im} A$) sono $\{A^{j_1}, \dots, A^{j_r}\}$

dove $j_1 \dots j_r$ sono gli indici delle colonne che contengono i pivot.

<u>Es</u>: Trovare una base di $\mathrm{Im}\, A$, dove

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 2 & 3 & 2 & -1 \\ 0 & 1 & 2 & 3 & 4 & 1 & -2 \\ 0 & 1 & 2 & 4 & 5 & 1 & -3 \end{pmatrix}$$

<u>Sol.</u>: Cerchiamo le colonne dominanti di A.

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 2 & 3 & 2 & -1 \\ 0 & 1 & 2 & 3 & 4 & 1 & -2 \\ 0 & 1 & 2 & 4 & 5 & 1 & -3 \end{pmatrix} \sim \begin{array}{c} \quad P_1 \\ \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 2 & 2 & 2 & -2 \\ 0 & 0 & 0 & 3 & 3 & 2 & -3 \end{pmatrix} \end{array} \; !$$

$$\sim \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

P_1 P_2 P_3

$$R_4 \mapsto R_4 - \frac{(-7)}{(-4)} R_3$$

$$\mathcal{B}_{\mathrm{Im}A} = \left\{ A^2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, A^4 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, A^6 = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \right\}$$ è una base di $\mathrm{Im}A$.

$V = \mathbb{R}[x]_{\leq 2} \qquad \mathcal{L}: V \longrightarrow V$

$$\mathcal{L}(p(x)) = p(x+1) - p(x)$$

$\mathcal{L}(x^2) =$

$$p(x) = a_0 + a_1 x + a_2 x^2$$

$$\mathcal{L}(p(x)) = a_0 + a_1 (x+1) + a_2 (x+1)^2 - a_0 - a_1 x - a_2 x^2$$

$$\mathcal{L}(x^2) = (x+1)^2 - x^2$$

$$F(p(x)) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix}$$

$$\begin{tikzcd} V \arrow[r] \arrow[d, "F"'] & V \arrow[d, "F"] \\ \mathbb{R}^3 & \mathbb{R}^3 \end{tikzcd}$$

F è un isomorfismo lineare $\Rightarrow \exists\ \mathcal{B} = \{b_1, b_2, b_3\}$ t.c.

$F = F_{\mathcal{B}}$: $\quad F(b_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad F(b_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad F(b_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$b_1(-1) = 1$, $b_1(0) = 0$, $b_1(1) = 0 \Rightarrow b_1(x) = \lambda x (x-1)$

$$1 = b_1(-1) = \lambda(-1)(-2) = 2\lambda \Rightarrow \lambda = \frac{1}{2}$$

$V = \mathbb{R}[x]_{\le 2}$ $\qquad \mathcal{L} : V \longrightarrow V$

$$\mathcal{L}(p(x)) = p(x+1) - p(x)$$

$$\begin{tikzcd}
V \arrow[r, "\mathcal{L}"] \arrow[d, "F"'] & V \arrow[d, "F"] \\
\mathbb{R}^3 & \mathbb{R}^3
\end{tikzcd}$$

$F(p(x)) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix}$ $\qquad F(x^2+1) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$F(p(x)) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow p(-1) = p(0) = p(1) = 0$

$p(x) = \lambda (x+1) x (x-1) \Leftrightarrow \lambda = 0$

F è un isomorfismo lineare $\Rightarrow \exists\ \mathcal{B} = \{b_1, b_2, b_3\}$ t.c.

$F = F_{\mathcal{B}}$: $\quad F(b_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad F(b_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad F(b_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$b_1(-1) = 1$, $b_1(0) = 0$, $b_1(1) = 0$ $\Rightarrow$ $\boxed{b_1(x) = \lambda x (x-1)}$

$1 = b_1(-1) = \lambda (-1)(-2) = 2\lambda \Rightarrow \lambda = \frac{1}{2}$

$$b_1(x) = \frac{1}{2} x (x-1)$$

$$b_2(x) = -\frac{(x+1)(x-1)}{1} = -(x+1)(x-1)$$

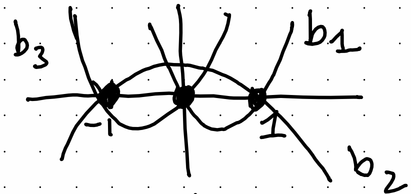
$$b_3(x) = \frac{(x+1)x}{2}$$

Polinomi di Lagrange.

$$\mathcal{L}: V \longrightarrow V : \quad \mathcal{L}(p(x)) = p(x+1) - p(x).$$

$$\begin{tikzcd} V \arrow[r, "\mathcal{L}"] \arrow[d, "F = F_{\mathcal{B}}"'] & V \arrow[d, "F_{\mathcal{B}} = F"] \\ \mathbb{R}^3 \arrow[r, "A"] & \mathbb{R}^3 \end{tikzcd}$$

$$\mathcal{B} = \{ b_1(x), b_2(x), b_3(x) \}.$$



$$A = \left(F_{\mathcal{B}}(\mathcal{L}(b_1)) \;\middle|\; F(\mathcal{L}(b_2)) \;\middle|\; F(\mathcal{L}(b_3)) \right)$$

$$= \left(F(b_1(x+1) - b_1(x)) \;\middle|\; F(b_2(x+1) - b_2(x)) \;\middle|\; F(b_3(x+1) - b_3(x)) \right)$$

$$= \left(\begin{array}{c|c|c} \cancel{b_1(0)} - b_1(-1) & b_2(0) - \cancel{b_2(-1)} & \cancel{b_3(0)} - \cancel{b_3(-1)} \\ \cancel{b_1(1)} - \cancel{b_1(0)} & \cancel{b_2(1)} - b_2(0) & b_3(1) - \cancel{b_3(0)} \\ b_1(2) - \cancel{b_1(1)} & b_2(2) - \cancel{b_2(1)} & b_3(2) - b_3(1) \end{array} \right)$$

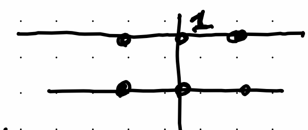
$$= \left(\begin{array}{c|c|c} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{array} \right)$$

$$\begin{tikzcd} V \arrow[r, "\mathcal{L}"] \arrow[d, "F"'] & V \arrow[d, "F"] \\ \mathbb{R}^3 \arrow[r, "A"] & \mathbb{R}^3 \end{tikzcd}$$

$$A = \left(\begin{array}{c|c|c} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{array}\right)$$

$$A \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = R = \text{rref}(A)$$



$$\text{Ker}\, A = \text{Ker}\, R = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$p(-1) = 1$
$p(0) = 1$
$p(1) = 1$

$$\Rightarrow \quad \text{Ker}\, \mathcal{L} = \left\langle F^{-1}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \right\rangle = \langle p(x) \rangle = \langle 1 \rangle = \{\text{polinomi costanti}\}.$$

<u>OSS</u>: $1 = b_1 + b_2 + b_3$.

$$R = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{Ker} R = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x_1 = x_3 \\ x_2 = x_3 \end{matrix} \right\} = \left\{ \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_3 \in \mathbb{R} \right\}$$

$$= \left\{ x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;\middle|\; x_3 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

<u>soluzione-base</u> di $\operatorname{Ker} R$

$$\mathcal{e} = \{1, x, x^2\}$$

$$\begin{tikzcd}
V \arrow[r, equal] \arrow[d, "F_{\mathcal{e}}"'] & V \arrow[r, "\mathcal{L}"] \arrow[d, "F"] & V \arrow[r, equal] \arrow[d, "F"] & V \arrow[d, "F_e"] \\
\mathbb{R}^3 \arrow[r, "B"] & \mathbb{R}^3 \arrow[r, "A"] & \mathbb{R}^3 & \mathbb{R}^3 \arrow[l, "B"']
\end{tikzcd}$$

Se $p(x) = 1 + 0x + 0x^2$

$p(-1) = 1 + 0(-1) + 0(-1)^2 = 1$

$p(x) = a_0 + a_1 x + a_2 x^2$

$p(-1) = a_0 - a_1 + a_2$

$$B^1 = B e_1 = F(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$B^2 = B e_2 = F(x) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$B^3 = B e_3 = F(x^2) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

la matrice che rappresenta $\mathcal{L}$ nella base canonica è $C = B^{-1} A B$.

$$A = \left(\begin{array}{c|c|c} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{array}\right) \qquad B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Calcoliamo $C = B^{-1}AB$

$$B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}$$

"B^-1"
MATLAB

Calcoliamo B^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right)$$

$$B^{-1} =$$

$$A = \left(\begin{array}{c|c|c} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{array} \right) \qquad B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}$$

$$C = B^{-1} A B = \begin{pmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \checkmark$$

$$\left.\begin{aligned} \mathcal{L}(1) &= 0 \\ \mathcal{L}(x) &= x+1-x = 1 \\ \mathcal{L}(x^2) &= (x+1)^2 - x^2 = x^2+2x+1-x^2 = 2x+1 \end{aligned}\right] \text{verifica.}$$

$$S = \begin{pmatrix} 0 & \cdots & 0 & p_1 & * & \cdots & \cdots & * & \cdots & \cdots & * & \cdots & * \\ & & & 0 & 0 & \cdots & 0 & p_2 & * & \cdots & \cdots & \cdots & * \\ & & & \vdots & & & & 0 & 0 & \cdots & 0 & p_3 & * \\ & & & \vdots & & & & \vdots & & & & 0 & 0 \cdots 0 \; p_4 \\ & & & \vdots & & & & \vdots & & & & \vdots & 0 \\ & & & 0 & & & & 0 & & & & 0 & \vdots \end{pmatrix}$$

- rref(A) è unica
- S non è unica.
- A è invertibile $\Leftrightarrow$ rref$(A) = \mathbb{1}_n$.

$$\mathrm{Sym}('a', [m, n])$$

$$\mathrm{Sym}('a', [2, 2]) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{pmatrix} 0 & \boxed{1} \\ 0 & \boxed{0} \end{pmatrix}$$

e_1

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

<u>non</u>
é
a
scala
ridotta

$$\operatorname{rref}(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2$$

$$\begin{pmatrix} \boxed{1} & 2 \\ \boxed{0} & 0 \end{pmatrix}$$

e_1

é a
scala ridotta

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & & & \\ 0 & \underset{A^2}{\boxed{}} & \cancel{x}A^2 & \underset{A^4}{\boxed{}} \\ 0 & & & \end{pmatrix}$$

$$\begin{pmatrix} 0 & \boxed{1} & \boxed{1} & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

$$\mathcal{L}(p) = p(x+1) - p(x)$$

$$\mathcal{L}(\alpha p+\beta q) = (\alpha p+\beta q)(x+1) - (\alpha p+\beta q)(x)$$

$$= \alpha \underline{p(x+1)} + \beta q\underline{\underline{(x+1)}} - \alpha \underline{p(x)} - \underline{\underline{\beta q(x)}}$$

$$= \alpha\left(\mathcal{L}(p)\right) + \beta\, \mathcal{L}(q)$$

$$F(p) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix} \qquad F(\alpha p+\beta q) = \begin{pmatrix} (\alpha p+\beta q)(-1) \\ (\alpha p+\beta q)(0) \\ (\alpha p+\beta q)(1) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha\, p(-1) + \beta\, q(-1) \\ \alpha\, p(0) + \beta\, q(0) \\ \alpha\, p(1) + \beta\, q(1) \end{pmatrix} = \alpha \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix} + \beta \begin{pmatrix} q(-1) \\ q(0) \\ q(1) \end{pmatrix}$$

$$= \alpha\, F(p) + \beta\, F(q).$$

$$C = \begin{pmatrix} -3 & 3 & -6 & -2 & -4 & 5 & 3 & 8 \\ -2 & 2 & -4 & -3 & -1 & 0 & -2 & -2 \\ -3 & 3 & -6 & 1 & -7 & 11 & 0 & 11 \\ -2 & 2 & -4 & 3 & -7 & 12 & -1 & 11 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & -2 & 1 & -3 & 5 & 5 & 10 \\ -2 & 2 & -4 & -3 & -1 & 0 & -2 & -2 \\ -3 & 3 & -6 & 1 & -7 & 11 & 0 & 11 \\ -2 & 2 & -4 & 3 & -7 & 12 & -1 & 11 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & -1 & 3 & -5 & -5 & -10 \\ -2 & 2 & -4 & -3 & -1 & 0 & -2 & -2 \\ -3 & 3 & -6 & 1 & -7 & 11 & 0 & 11 \\ -2 & 2 & -4 & 3 & -7 & 12 & -1 & 11 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & -1 & 3 & -5 & -5 & -10 \\ 0 & 0 & 0 & -5 & 5 & -10 & -12 & -22 \\ 0 & 0 & 0 & -2 & 2 & -4 & -15 & -19 \\ 0 & 0 & 0 & 1 & -1 & 2 & -11 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 & -1 & 3 & -5 & -5 & -10 \\ 0 & 0 & 0 & 1 & -1 & 2 & -11 & -9 \\ & & & & & & & \\ & & & & & & & \end{pmatrix}$$