Esercizio sulle proiezioni

V ha base $\{v_1, v_2, v_3\}$

$U = \langle v_1 + 2v_2 \rangle \qquad W = \langle v_2, v_3 \rangle \quad \subset V$

1) $V = U \oplus W$. i.e. 1) $U + W = V$
2) $U \cap W = \{0_V\}$.

Sceno

1° modo: Y $\mathcal{B}_U = \{v_1 + 2v_2\}$ $\mathcal{B}_W = \{v_2, v_3\}$ basi di U e W rispettivamente. Dimostriamo che $\mathcal{B}_U \cup \mathcal{B}_W = \{v_1 + 2v_2, v_2, v_3\}$ è una base di V.

Per il lemma di scambio

$V = \langle v_1, v_2, v_3 \rangle = \langle v_1 + 2v_2, v_2, v_3 \rangle$

e quindi $\mathcal{B}_U \cup \mathcal{B}_W$ genera V. Dato che $|\mathcal{B}_U \cup \mathcal{B}_W| = \dim V$

$\mathcal{B}_U \cup \mathcal{B}_W$ è una base di V.

$\Rightarrow$ 1) $U + W = V$. Inoltre, $U \cap W = \{0_V\}$ perché:

$$0_V = U + W = \underbrace{\qquad\qquad}_{\in \langle B_U \rangle} + \underbrace{\qquad\qquad}_{\in \langle B_W \rangle}$$

$$= x_1 (v_1 + 2v_2) + x_2 (v_2) + x_3 v_3$$

$$\Leftrightarrow \quad \begin{aligned} 0 &= x_1 \\ 0 &= 2x_1 + x_2 \\ 0 &= x_3 \end{aligned} \quad \Leftrightarrow \quad x_1 = x_2 = x_3 = 0$$

$$U \cap W = \{0_V\}.$$

OSS : $U \cap W = \{0_V\} \iff 0_V = u + w$ se e solo se

$u = 0_V$ e $w = 0_V$

$v \in U \cap W \qquad v = u = w$

$u - w = 0_V$

$pr_U^W(v) \qquad v = 2v_1 + 3v_2 - v_3 \quad , \quad U = \langle v_1 + 2v_2 \rangle$
$W = \langle v_2, v_3 \rangle$

$v = u + w \qquad$ con $u \in U$ e $w \in W$

$pr_U^W(v) = u$

Def : $pr_U^W : U \oplus W \longrightarrow U \oplus W$

$u + w \longmapsto u$

$$v = 2v_1 + 3v_2 - v_3 \quad , \quad \begin{array}{l} U = \langle v_1 + 2v_2 \rangle \\ W = \langle v_2 , v_3 \rangle \end{array}$$

$$v = \underbrace{x_1 (v_1 + 2v_2)}_{u} + \underbrace{x_2 v_2 + x_3 v_3}_{w}$$

$$pr_U^W (v) = x_1 (v_1 + 2v_2) \quad ; \quad pr_W^U (v) = x_2 v_2 + x_3 v_3$$

$V = U \oplus W \ : \quad \forall v \in V \ \exists! \, u \in U \ \exists! \, w \in W$ t.c.

$$v = u + w$$

$$pr_U^W (v) = u$$

$$x_1 = 2 \quad \Rightarrow \quad pr_U^W (v) = 2 (v_1 + 2v_2) .$$

$$2 v_1 + 3 v_2 - v_3 = x_1 v_1 + (2x_1 + x_2) v_2 + x_3 v_3$$

$$pr_U^W(v) = 2\,(v_1 + 2v_2)$$

$$pr_W^U(v) = ?$$

$$\begin{aligned} x_1 &= 2 \\ 2x_1 + x_2 &= 3 \\ x_3 &= -1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 2 \\ x_2 &= 3 - 4 = -1 \\ x_3 &= -1 \end{aligned}$$

$$pr_W^U(v) = -v_2 - v_3$$

Base di Ker A e di Im A

$$(a+ib)^{-1} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\, i$$

$$A = \begin{pmatrix} 1 & -i \\ i & 1 \\ 1+i & 1-i \end{pmatrix} \in Mat_{3\times 2}(\mathbb{C})$$

$$-i(1+i) = -i - i^2 = 1-i$$

$$\mathrm{Im}\, A := \mathrm{Im}\, S_A = Col(A) = \left\langle \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}, \begin{pmatrix} -i \\ 1 \\ 1-i \end{pmatrix} \right\rangle$$

$$x_1 \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} + x_2 \begin{pmatrix} -i \\ 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x_1 - i x_2 = 0 \\ i x_1 + x_2 = 0 \\ (1+i) x_1 + (1-i) x_2 = 0 \end{cases}$$

$$x_1 = i x_2$$

$$i(i x_2) + x_2 = 0 \quad V$$

$$(1+i)^{-1} = \frac{1}{2} - \frac{1}{2} i$$

$$x_1 = -(1+i)^{-1}(1-i)\, x_2 = -\left(\frac{1}{2} - \frac{1}{2} i\right)(1-i)\, x_2 = \frac{1}{2}(1-i)^2 x_2$$

$$=$$

$$\begin{cases} x_1 = i\, x_2 \\ i(i x_2) + x_2 = 0 \quad \checkmark \\ x_1 = (1+i)^{-1}(1-i)\, x_2 = -\frac{1}{2}(1-i)^2 x_2 \end{cases}$$

$$= -\frac{1}{2}(1 - 2i - 1)\, x_2 = +i\, x_2$$

$$x_1 = i\, x_2 \qquad\qquad i\begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} + \begin{pmatrix} -i \\ 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{é}$$

una rel. di dip. lineare.

$\Rightarrow$ Una base di $\operatorname{Im} A$ è $\left\{ \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} \right\}$.

$$\operatorname{Ker} A = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; x_1 \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} + x_2 \begin{pmatrix} -i \\ 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$x_2 = -i x_1$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; x_1 = i x_2 \right\}$$

Appena fatto

$A_{m \times n}$

$S_A : \mathbb{K}^n \longrightarrow \mathbb{K}^m$

$$= \left\langle \begin{pmatrix} i \\ 1 \end{pmatrix} \right\rangle$$

Una base di Ker A è $\left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$

$$\rightsquigarrow \quad \mathcal{B}_{\mathbb{C}^2} = \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix}, \overset{e_1}{\overset{\shortparallel}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}} \right\}$$

$$\mathcal{B}_{\operatorname{Im} S_A} = \left\{ S_A(e_1) \right\} = \left\{ A^1 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} \right\}$$

$$A = \left(A^1 \mid \dots \mid A^n\right) \in \mathrm{Mat}_{m \times n}(\mathbb{K})$$

$$S_A(X) := x_1 A^1 + x_2 A^2 + \dots + x_n A^n$$

$$S_A(e_1) = A^1 \quad , \quad S_A(e_2) = A^2,$$

$$S_A(e_i) = A^i$$

$$V = \langle v_1, \dots, v_n \rangle \qquad \mathcal{L}: V \to W \quad \text{lineare.}$$

$$\operatorname{Im} \mathcal{L} = \{ \mathcal{L}(v) \mid v \in V \}$$

$\{v_1, \dots, v_n\}$ genera V

$$= \{ \mathcal{L}(x_1 v_1 + \dots + x_n v_n) \mid x_1, x_2, \dots, x_n \in \mathbb{K} \}$$

$$= \{ \mathcal{L}(x_1 v_1) + \mathcal{L}(x_2 v_2) + \dots + \mathcal{L}(x_n v_n) \mid x_1, \dots, x_n \in \mathbb{K} \}$$

$\mathcal{L}$ è lineare

$$= \{ x_1 \mathcal{L}(v_1) + x_2 \mathcal{L}(v_2) + \dots + x_n \mathcal{L}(v_n) \mid x_1, \dots, x_n \in \mathbb{K} \}$$

$$= \langle \mathcal{L}(v_1), \dots, \mathcal{L}(v_n) \rangle .$$

Ripasso: $\mathcal{L}: \mathbb{K}^n \longrightarrow \mathbb{K}^m$ lineare

$$\mathcal{L}\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = x_1 \mathcal{L}(e_1) + \dots + x_n \mathcal{L}(e_n)$$

$$A = \left(\mathcal{L}(e_1) \middle| \dots \middle| \mathcal{L}(e_n)\right) \in \mathrm{Mat}_{m \times n}(\mathbb{K})$$

Es (Matrice associata a $\mathcal{L}: \mathbb{K}^n \to \mathbb{K}^m$).

Sia $\mathcal{L}: \mathbb{K}^3 \to \mathbb{K}^3$ l'unica f.ne lineare t.c.

$$\mathcal{L}(e_1+e_2-e_3) = e_1+e_2$$

$$\mathcal{L}(e_2+e_3) = e_1+e_3$$

$$\mathcal{L}(e_2-e_3) = e_2-e_3$$

$\{e_1+e_2-e_3, e_2+e_3, e_2-e_3\}$ è una base di $\mathbb{K}^3$:

1) $\langle e_1, e_2, e_3 \rangle = \langle e_1+e_2-e_3, e_2, e_3 \rangle = \langle e_1+e_2-e_3, e_2+e_3, e_3 \rangle$

$= \langle e_1+e_2-e_3, e_2+e_3, e_2-e_3 \rangle. \Rightarrow$ genera.

oppure

2) $x_1(e_1+e_2-e_3) + x_2(e_2+e_3) + x_3(e_2-e_3) = 0$

$$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_1+x_2+x_3 = 0 \\ -x_1+x_2-x_3 = 0 \end{cases}$$

Es (Matrice associata a $\mathcal{L}: \mathbb{K}^n \longrightarrow \mathbb{K}^m$).

Sia $\mathcal{L}: \mathbb{K}^3 \longrightarrow \mathbb{K}^3$ l'unica f.ne lineare t.c.

$$\mathcal{L}(e_1+e_2-e_3) = e_1+e_2$$

$$\mathcal{L}(e_2+e_3) = e_1+e_3$$

$$\mathcal{L}(e_2-e_3) = e_2-e_3$$

Trovare A t.c. $\mathcal{L} = S_A$, base di $\operatorname{Ker}\mathcal{L} = \operatorname{Ker} A$ e di $\operatorname{Im}\mathcal{L} = \operatorname{Im} A$.

Sol.: $A = \left(\mathcal{L}(e_1) \,|\, \mathcal{L}(e_2) \,|\, \mathcal{L}(e_3)\right) \in Mat_{3\times 3}(\mathbb{K})$

$e_1 = ?$ $\qquad e_1 = (e_1+e_2-e_3) - (e_2-e_3)$

$$\mathcal{L}(e_1) = \mathcal{L}(e_1+e_2-e_3) - \mathcal{L}(e_2-e_3) = \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\+1\end{pmatrix}$$

Es (Matrice associata a $\mathcal{L}: \mathbb{K}^n \to \mathbb{K}^m$).

Sia $\mathcal{L}: \mathbb{K}^3 \to \mathbb{K}^3$ l'unica f.ne lineare t.c.

$$\mathcal{L}(e_1+e_2-e_3) = e_1+e_2$$

$$\mathcal{L}(e_2+e_3) = e_1+e_3$$

$$\mathcal{L}(e_2-e_3) = e_2-e_3$$

Trovare A t.c. $\mathcal{L} = S_A$, base di $\operatorname{Ker} \mathcal{L} = \operatorname{Ker} A$ e di $\operatorname{Im} \mathcal{L} = \operatorname{Im} A$.

Sol.: $A = \left(\mathcal{L}(e_1) \,|\, \mathcal{L}(e_2) \,|\, \mathcal{L}(e_3) \right) \in Mat_{3\times 3}(\mathbb{K})$

$e_2 = ?$ $\quad 2e_2 = (e_2+e_3)+(e_2-e_3) \Rightarrow e_2 = \frac{1}{2}(e_2+e_3)+\frac{1}{2}(e_2-e_3)$

$$\mathcal{L}(e_2) = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

Es (Matrice associata a $\mathcal{L}: \mathbb{K}^n \longrightarrow \mathbb{K}^m$).

Sia $\mathcal{L}: \mathbb{K}^3 \longrightarrow \mathbb{K}^3$ l'unica f.ne lineare t.c.

$$\mathcal{L}(e_1+e_2-e_3) = e_1+e_2$$

$$\mathcal{L}(e_2+e_3) = e_1+e_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\mathcal{L}(e_2-e_3) = e_2-e_3$$

Trovare A t.c. $\mathcal{L}=S_A$, base di $\operatorname{Ker}\mathcal{L}=\operatorname{Ker}A$ e di $\operatorname{Im}\mathcal{L}=\operatorname{Im}A$.

Sol.: $A = \left(\mathcal{L}(e_1)\,|\,\mathcal{L}(e_2)\,|\,\mathcal{L}(e_3)\right) \in Mat_{3\times3}(\mathbb{K})$

$e_3 = ?$ $\qquad -(e_2-e_3)+(e_2+e_3) = 2e_3$

$$\Rightarrow e_3 = \frac{1}{2}(e_2+e_3)-\frac{1}{2}(e_2-e_3)$$

$$\Rightarrow \mathcal{L}(e_3) = \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}1/2\\-1/2\\1\end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\operatorname{Ker} \mathcal{L} = \operatorname{Ker} A$$

$$\operatorname{Im} \mathcal{L} = \operatorname{Im} A.$$

$$\operatorname{Ker} \mathcal{L} = \operatorname{Ker} A = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{K}^3 \;\middle|\; x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{K}^3 \;\middle|\; \begin{cases} x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_3 = 0 \\ \frac{1}{2} x_2 - \frac{1}{2} x_3 = 0 \\ x_1 + x_3 = 0 \end{cases} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{K}^3 \;\middle|\; \cancel{x_1} \quad \begin{cases} x_1 = -x_3 \\ x_2 = x_3 \\ \cancel{x_1 = -x_3} \end{cases} \right\} = \left\langle \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$$\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \Rightarrow \mathcal{B}_{\operatorname{Im} \mathcal{L}} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} \right\}$$

"Determinare la funzione" cosa vuol dire?

Un'applicazione lineare è univocamente determinata dai valori che assume su una base.

$\mathcal{L}: V \to W$ lineare, $\mathcal{B} = \{v_1, \dots, v_n\}$ base di $\mathcal{L}$.

$$\mathcal{L}(v_1) = w_1, \; \mathcal{L}(v_2) = w_2, \; \dots, \; \mathcal{L}(v_n) = w_n.$$

Determinare $\mathcal{L}$ vuol dire quanto fa $\mathcal{L}(x_1 v_1 + \dots + x_n v_n)$?

$$\mathcal{L}(x_1 v_1 + \dots + x_n v_n) = x_1 w_1 + \dots + x_n w_n.$$

Es: Sia $\mathcal{L}: \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}[x]_{\leq 1}$ lineare t.c.

$$\mathcal{L}(1+x) = 1$$
$$\mathcal{L}(1-x) = 3 \qquad\qquad 1+x+x^2-(1+x)=x^2$$
$$\mathcal{L}(1+x+x^2) = 2+2x$$

Determinare $\mathcal{L}$: $\qquad \mathcal{L}(1+2x) = 2-2+0+0x=0$

$$\mathcal{L}(a_0+a_1x+a_2x^2) = ? \qquad = 2a_0 - a_1 + a_2 + 2a_2 x$$

$$\boxed{\mathcal{L}\left(b_1(1+x)+b_2(1-x)+b_3(1+x+x^2)\right) = b_1+3b_2+2b_3+2b_3x}$$

$$1 = \frac{1}{2}(1+x)+\frac{1}{2}(1-x) \qquad \mathcal{L}(1) = \frac{1}{2}+\frac{3}{2} = 2$$

$$x = \frac{1}{2}(1+x)-\frac{1}{2}(1-x) \qquad \mathcal{L}(x) = \frac{1}{2}-\frac{3}{2} = -1$$

$$x^2 = -(1+x)+(1+x+x^2) \qquad \mathcal{L}(x^2) = -1+2+2x = 1+2x.$$

$$\mathcal{L}: \mathbb{R}[x]_{\le 2} \longrightarrow \mathbb{R}[x]_{\le 1}$$

$$\boxed{\mathcal{L}\left(b_1(1+x)+b_2(1-x)+b_3(1+x+x^2)\right) = b_1+3b_2+2b_3+2b_3x}$$

$$\operatorname{Ker}\mathcal{L} = \left\{ b_1(1+x)+b_2(1-x)+b_3(1+x+x^2) \;\middle|\; \mathcal{L}\left(b_1(1+x)+b_2(1-x)+b_3(1+x+x^2)\right)=0 \right\}$$

$$= \left\{ b_1(1+x)+b_2(1-x)+b_3(1+x+x^2) \;\middle|\; b_1+3b_2+2b_3+2b_3x = 0+0x \right\}$$

$$= \left\{ b_1(1+x)+b_2(1-x)+b_3(1+x+x^2) \;\middle|\; \begin{cases} b_1+3b_2+2b_3 = 0 \\ 2b_3 = 0 \end{cases} \right\}$$

$$= \left\{ b_1(1+x)+b_2(1-x) \;\middle|\; b_1 = -3b_2 \right\}$$

$$= \langle -3(1+x)+(1-x) \rangle = \langle -3+1-3x-x \rangle = \langle -2-4x \rangle$$

$$= \langle 1+2x \rangle$$ Verifichiamo: $\mathcal{L}(1+2x)=0$

Estendiamo $\{1+2x\}$ ad una base di $\mathbb{R}[x]_{\leq 2}$:

$$\{1+2x,\ 1,\ x^2\}$$

$\{1+2x,\ 1+x,\ 1+x+x^2\}$ è una base di $\mathbb{R}[x]_{\leq 2}$:

$\Rightarrow$ Una base per $\operatorname{Im} \mathcal{L}$ è

$$\{\mathcal{L}(1+x),\ \mathcal{L}(1+x+x^2)\} = \{1,\ 2+2x\}$$

oppure

$$\{1,\ 1+x\}$$

oppure

$$\{1,\ x\}$$

$\mathcal{L}$ è suriettiva.