Estendere ad una base

$\mathcal{I} = \{v_1, \dots, v_k\}$ lin. Ind. $\subset V$ f.g.

$\langle \mathcal{I} \rangle = V$ $\qquad\qquad$ $\langle \mathcal{I} \rangle \neq V$

$\mathcal{I}$ è una base $\qquad\qquad$ $\exists\, v \notin \langle \mathcal{I} \rangle \Rightarrow \mathcal{I} \cup \{v\}$ è lin. Ind.

$\langle \mathcal{I} \cup \{v\} \rangle = V$ $\qquad\qquad$ $\langle \mathcal{I} \cup \{v\} \rangle \neq V$

$\mathcal{I} \cup \{v\}$ è base

<u>Es</u>: $\mathcal{I} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^4$. Estendiamo $\mathcal{I}$ ad una base.

$$\mathcal{I} \cup \left\{ e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \rightsquigarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\rightsquigarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$ è una base che estende $\mathcal{I}$.

$$U_1 = \left\{ X \in \mathbb{C}^4 \;\middle|\; \begin{array}{l} x_1 + i x_2 = 0 \\ x_2 + i x_3 = 0 \end{array} \right\} \subset U_2 = \left\{ X \in \mathbb{C}^4 \;\middle|\; x_1 = -x_3 \right\}$$

$$x_1 = -i x_2 = -x_3$$

$$x_2 = -i x_3$$

$$p(x) = a_0 + a_1\,(x-1) + a_2\,(x-1)^2 + a_3\,(x-1)^3$$

$$a_0 = p(1)$$

$$p(x) - a_0 \;\; \begin{array}{|l} x-1 \\ \hline q(x) \end{array}$$

$$p(x) - a_0 = (x-1)\overbrace{\left[a_1 + a_2\,(x-1) + a_3(x-1)^2\right]}^{q(x)}$$

$$a_1 = q(1)$$

$U \ni v \neq 0$

$\langle v \rangle = U \qquad\qquad \langle v \rangle \neq U$

$\downarrow$

$\{v\}$ base $\qquad\qquad \exists\, v_2$ t.c. $\{v_1 v_2\}$ è lin. Ind.

$\langle v_1 v_2 \rangle = U \qquad\qquad \langle v_1 v_2 \rangle \neq U$

$\{v_1, v_2\}$ è base $\qquad\qquad \{v_1, v_2\}$ lin. Ind.

$\exists\, v_3 \ldots$

$$U = \{ X \in \mathbb{R}^4 \mid x_2 = 0 \} \ni \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = e_1$$

$$\langle e_1 \rangle \overset{?}{=} U, \text{ No} \rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}^{\cancel{\notin} \in U} \text{ è base ? Si.}$$

Se non lo fosse, $\dim U = 4 \Rightarrow U = \mathbb{R}^4$

Assurdo perché $e_2 \in \mathbb{R}^4$ ma $e_2 \notin U$.

$\Rightarrow \dim U = 3.$

$$U = \left\{ \begin{pmatrix} x_1 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} \middle| \, x_1, x_3, x_4 \in \mathbb{R} \right\} = \left\{ x_1 e_1 + x_3 e_3 + x_4 e_4 \,\middle|\, x_1, x_3, x_4 \in \mathbb{R} \right\}$$

$$= \langle e_1, e_3, e_4 \rangle$$

$$U = \{ x \mid x_2 = 0 \} \qquad W = \left\langle \overset{w_1}{\overset{\shortparallel}{\begin{pmatrix}1\\1\\1\\0\end{pmatrix}}}, \overset{w_2}{\overset{\shortparallel}{\begin{pmatrix}0\\1\\1\\1\end{pmatrix}}} \right\rangle \subset \mathbb{R}^4$$

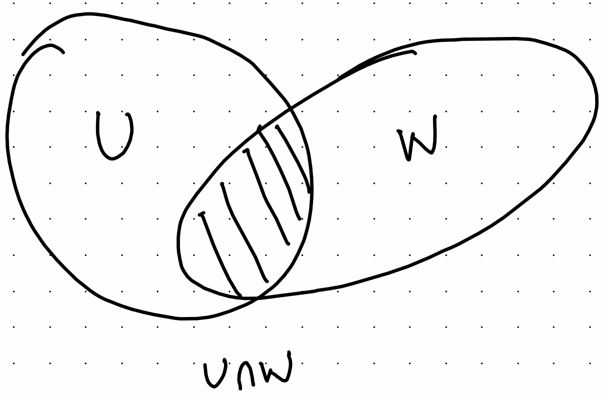
$\dim U = 3$ $\qquad$ $\dim W = 2$ $\qquad$ $w_1, w_2 \notin U.$

$$W \ni w_1 - w_2 = \begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \in U \qquad (\text{seconda componente } = 0)$$

dim U = cardinalità di una base di U.

$\dim U \cap W =$

$$U \cap W \subseteq U, \qquad U \cap W \subseteq W$$



$$\dim U \cap W \le \dim U = 3, \qquad \dim U \cap W \le \dim W = 2$$

$$\dim U \cap W \in \{0, 1, 2\}$$

$$\Rightarrow \begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \in U \cap W. \Rightarrow \dim U \cap W \in \{1, 2\}$$

Se $\dim U \cap W = 2 = \dim W$ allora $W \subseteq U$.

Ma $w_1 \notin U$ e quindi $W \not\subseteq U$.

Concludiamo che

$$\dim U \cap W = 1$$

e una base di $U \cap W$ è $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\} = \mathcal{B}_{U \cap W}$

$$\dim U+W \overset{\text{Grassmann}}{=} \dim U + \dim W - \dim U \cap W = 3+2-1 = 4$$

$$\Rightarrow \quad U+W = \mathbb{R}^4$$

Estendiamo $\mathcal{B}_{U \cap W}$ ad una base di $\mathbb{R}^4$:

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ è una base di $\mathbb{R}^4$ che estende $\mathcal{B}_{U \cap W}$.

$$U = \left\{ X \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x_1 + x_4 = 0 \\ x_2 + x_3 = 0 \end{array} \right\} \qquad W = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

Trovare una base di U, una di W, una di $U \cap W$, ed una di $U+W$ che estende le basi precedenti.

$$\begin{array}{ccccc} & & U+W & & \\ & \subseteq & & \supseteq & \\ U & & & & W \\ & \supseteq & & \subseteq & \\ & & U \cap W & & \end{array}$$

$$\begin{pmatrix} x \\ x \\ y \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\left. \begin{array}{l} \dim U = 2 \\ \dim W = 2 \end{array} \right\} =$$

$$U = \left\langle \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle = \left\{ x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; x_3, x_4 \right\}$$

$$\|$$

$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x_1 = -x_4 \\ x_2 = -x_3 \end{array} \right\} = \left\{ \begin{pmatrix} -x_4 \\ -x_3 \\ x_3 \\ x_4 \end{pmatrix} \;\middle|\; x_3, x_4 \in \mathbb{R} \right\}$$

$$U = \left\{ X \in \mathbb{R}^4 \;\middle|\; \begin{matrix} x_1 + x_4 = 0 \\ x_2 + x_3 = 0 \end{matrix} \right\} \qquad W = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$$\left.\begin{matrix} \dim U = 2 \\ \dim W = 2 \end{matrix}\right\} \Rightarrow \begin{matrix} \dim U \cap W = \{0, 1, 2\} \\ \dim U + W = \{4, 3, 2\} \end{matrix}$$

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \beta \\ \beta \end{pmatrix} \in U \iff \alpha = -\beta$$

$$U \cap W = \left\{ \begin{pmatrix} \alpha \\ \alpha \\ -\alpha \\ -\alpha \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\rangle$$

$$\dim U \cap W = 1 \Rightarrow \dim U + W = 2 + 2 - 1 = 3.$$

$$U = \left\{ X \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x_1 + x_4 = 0 \\ x_2 + x_3 = 0 \end{array} \right\} \qquad W = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$$U \cap W = \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\rangle \qquad \mathcal{B}_{U \cap W} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

Estendiamo $\mathcal{B}_{U \cap W}$ ad una base di U:

$$\mathcal{B}_U = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

Estendiamo $\mathcal{B}_{U \cap W}$ ad una base di W:

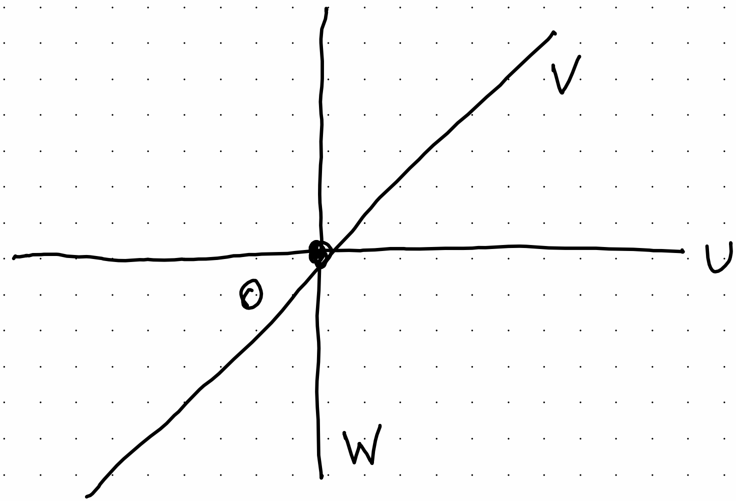
$$\mathcal{B}_W = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\mathcal{B}_U \cup \mathcal{B}_W = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$ è una base di $U + W$.

Somma diretta di sottospazi:

W, U : $\exists\ V$ t.c.

$$U \oplus V = W \oplus V = \mathbb{R}^4 \ ?$$



$$U = \left\{ X \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x_1 + x_4 = 0 \\ x_2 + x_3 = 0 \end{array} \right\} \qquad W = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

Stabilire se esiste un supplementare comune sia a U che a W, i.e. $V \subset \mathbb{R}^4$ t.c.

$$U \oplus V = W \oplus V = \mathbb{R}^4.$$

$$\mathcal{B}_U = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\} \qquad \mathcal{B}_W = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\Bigg\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}}_{U}, \underbrace{\overset{e_1}{\overset{\shortparallel}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}}, \overset{e_3}{\overset{\shortparallel}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}}}_{V} \Bigg\} \qquad \Bigg\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{W} \oplus \underbrace{\overset{e_1}{\overset{\shortparallel}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}}, \overset{e_3}{\overset{\shortparallel}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}}}_{V} \Bigg\}$$

~~Non è una base~~

$$V = \langle e_1, e_3 \rangle$$

<u>Coordinate</u>: Sia $\mathcal{B} = \{v_1, \dots, v_n\}$ una base di V.

Quindi 1) $\langle \mathcal{B} \rangle = V$, "$\mathcal{B}$ genera V".

2) $\mathcal{B}$ è lin. Ind.

1) vuol dire: $\forall v \in V \ \exists x_1, \dots, x_n \in \mathbb{K}$ t.c.

$$v = x_1 v_1 + x_2 v_2 + \dots + x_n v_n.$$

2) vuol dire: questi coefficienti sono <u>unici</u>.

$$v = x_1 v_1 + \dots + x_n v_n = y_1 v_1 + \dots + y_n v_n$$

$$\Rightarrow x_1 = y_1, \ x_2 = y_2, \dots, x_n = y_n.$$

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\qquad v_1 = 0 v_1 + \frac{1}{2} v_2 = -v_1 + v_2$

$v_1 = 1 v_1$

1) + 2) : $\forall v \in V \ \exists! \ x_1, \dots, x_n \in \mathbb{K}$ t.c.

$$v = x_1 v_1 + \dots + x_n v_n .$$

Quindi abbiamo una funzione

$$F_{\mathcal{B}} : V \longrightarrow \mathbb{K}^n$$

$$v = x_1 v_1 + \dots + x_n v_n \longmapsto \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$ = <u>vettore delle coordinate</u> di v nella base $\mathcal{B}$

Dipende da $\mathcal{B}$.

$x_1, \dots, x_n$: le <u>coordinate</u> di v nella base $\mathcal{B}$.

$$\mathcal{Y} = \left\{ \underset{v_1}{\underset{\shortparallel}{\begin{pmatrix}1\\1\end{pmatrix}}}, \underset{v_2}{\underset{\shortparallel}{\begin{pmatrix}2\\2\end{pmatrix}}} \right\} \subset \left\langle \begin{pmatrix}1\\1\end{pmatrix} \right\rangle = V = \langle v_1, v_2 \rangle$$

$$V \longrightarrow \mathbb{R}^2$$

$$v = x_1 v_1 + x_2 v_2 \longmapsto \begin{pmatrix}x_1\\x_2\end{pmatrix}$$

Questa mappa non è ben-definita!

$$v_1 \longmapsto \begin{pmatrix}-1\\1\end{pmatrix}$$

$$v_1 \longmapsto \begin{pmatrix}0\\1/2\end{pmatrix}$$

$\underline{Es}$: $\{1-x,\ 1+x\} = \mathcal{B} \subset \mathbb{R}[x]_{\leq 1}$

$$p(x) \overset{a_0 1 + a_1 x}{=} 2 - 3x = \boxed{a_0}(1-x) + \boxed{a_1}(1+x)$$

$$2 - 3x = a_0 + a_1 + (-a_0 + a_1)x$$

$$\Leftrightarrow \begin{cases} 2 = a_0 + a_1 \\ -3 = -a_0 + a_1 \end{cases} \Leftrightarrow \quad a_0 = 2 - a_1$$

$$-3 = -(2-a_1) + a_1 = -2 + 2a_1$$

$$\Rightarrow \begin{cases} a_0 = 2 - a_1 \\ a_1 = -\frac{1}{2} \end{cases} \qquad \begin{aligned} a_0 &= 5/2 \\ a_1 &= -1/2 \end{aligned}$$

$$F_{\mathcal{B}}(p(x)) = \begin{pmatrix} 5/2 \\ -1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

$\mathcal{E} = \{1, x\}$

$$F_{\mathcal{E}}(p(x)) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$\mathcal{B} = \{1, x+1, (x+1)^2, (x+1)^3\} \overset{\text{base}}{\subset} \mathbb{R}[x]_{\leq 3}$.

$|\mathcal{B}| = 4 = \dim \mathbb{R}[x]_{\leq 3}$. e gli elementi di $\mathcal{B}$ hanno gradi distinti e quindi $\mathcal{B}$ è lin. Ind.

$$p(x) = 3 + 2x + 4x^2 + 4x^3 \overset{(*)}{=} \underset{\checkmark}{a_0} + \underset{\checkmark}{a_1}(x+1) + a_2(x+1)^2 + a_3(x+1)^3$$

Trovare $F_{\mathcal{B}}(p(x)) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$.

$$a_0 = p(-1) = 3 - 2 + 4 - 4 = 1$$

$$p(x) - a_0 \overset{(*)}{=} (x+1)\underbrace{\left[a_1 + a_2(x+1) + a_3(x+1)^2\right]}_{= q_1(x)}$$

$$a_1 = q_1(-1)$$

$$p(x) = 3 + 2x + 4x^2 + 4x^3 \quad , \quad a_0 = 1$$

$$\begin{array}{rl|l}
p(x) - a_0 = & 4x^3 + 4x^2 + 2x + 2 & x+1 \\ \hline
- & 4x^3 + 4x^2 & 4x^2 + 2 \\ \hline
 & 2x + 2 & \\
 & 2x + 2 & \\ \hline
 & // &
\end{array}$$

Quindi

$$p(x) - a_0 = (x+1)(4x^2+2) \qquad =0 \qquad a_1 = (4x^2+2)\Big|_{x=-1} = 6$$

$p(x) = 3 + 2x + 4x^2 + 4x^3 \quad , \quad a_0 = 1 \quad , \quad a_1 = 6$

$$p(x) - a_0 - a_1 (x+1) = (x+1)^2 \underbrace{\left[a_2 + a_3 (x+1) \right]}_{q_2(x)}$$

$$a_2 = q_2(-1)$$

$$p(x) - a_0 - a_1 (x+1) = 4x^3 + 4x^2 + 2x + 2 - 6x - 6 =$$

$$\begin{array}{l|l} = 4x^3 + 4x^2 - 4x - 4 & x^2 + 2x + 1 \\ \hline -\ 4x^3 + 8x^2 + 4x & 4x - 4 = q_2(x) \\ \hline \quad -4x^2 - 8x - 4 & \\ \quad -4x^2 - 8x - 4 & \\ \hline \quad\quad // & \end{array}$$

$\Rightarrow \quad a_2 = -8$

$p(x) = 3 + 2x + 4x^2 + 4x^3 \quad , a_0 = 1 \;, a_1 = 6 \;, a_2 = -8$

$p(x) - a_0 - a_1(x+1) - a_2(x+1)^2 = a_3(x+1)^3$

$\parallel$

$4x^3 + 4x^2 + \underline{2x} + \underline{3} - \underline{1} - \underline{6x} - \underline{6} + 8(x^2+2x+1)$

$\parallel$

$4x^3 + \underline{4}x^2 - \underline{4}x - 4 + \underline{8x^2} + \underline{16x} + 8 \quad \Rightarrow \quad \boxed{F_{\mathcal{B}}(p(x)) = \begin{pmatrix} 1 \\ 6 \\ -8 \\ 4 \end{pmatrix}}$

$\parallel$

$4x^3 + 12x^2 + 12x + 4 \qquad\qquad a_3 = 4$

$\parallel \qquad\qquad \Longrightarrow$

$4(x^3 + 3x^2 + 3x + 1)$

$\parallel$

$(x+\underline{1})^3 = x^3 + 3x^2 + 3x + 1$

V_0^2