Es 2:

1) $\mathcal{E} = \{1, x, x^2, \dots, x^5\}$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
\boxed{1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1}
\end{array}
$$

$$P_1(x) = (X + \lambda_1)^5 = X^5 + 5X^4\lambda_1 + 10X^3\lambda_1^2 + 10X^2\lambda_1^3 + 5X\lambda_1^4 + \lambda_1^5$$

$\vdots \qquad \vdots$

$$P_6(x) = (X + \lambda_6)^5 =$$

$$A = \left(F_{\mathcal{E}}(P_1) \middle| \dots \middle| F_{\mathcal{E}}(P_6) \right) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 5\lambda_1 & 5\lambda_2 & 5\lambda_3 & 5\lambda_4 & 5\lambda_5 & 5\lambda_6 \\ 10\lambda_1^2 & 10\lambda_2^2 & 10\lambda_3^2 & 10\lambda_4^2 & 10\lambda_5^2 & 10\lambda_6^2 \\ 10\lambda_1^3 & 10\lambda_2^3 & 10\lambda_3^3 & 10\lambda_4^3 & 10\lambda_5^3 & 10\lambda_6^3 \\ 5\lambda_1^4 & 5\lambda_2^4 & 5\lambda_3^4 & 5\lambda_4^4 & 5\lambda_5^4 & 5\lambda_6^4 \\ \lambda_1^5 & \lambda_2^5 & \lambda_3^5 & \lambda_4^5 & \lambda_5^5 & \lambda_6^5 \end{pmatrix}$$

$\{P_1, \dots, P_6\}$ è lin. Ind. $\iff$ $\det A \neq 0$.

$$\det A = 5 \cdot 10 \cdot 10 \cdot 5 \, \det\left(\mathrm{Van}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)^t\right)$$

$$= 5 \cdot 10 \cdot 10 \cdot 5 \prod_{i>j} (\lambda_i - \lambda_j) \neq 0 \iff \lambda_i \neq \lambda_j \quad \forall i \neq j.$$

<u>OSS</u>: Potevamo scegliere la base

$$\mathcal{B} = \{ x^5, 5x^4, 10x^3, 10x^2, 5x, 1 \}$$

$$A = \left(F_{\mathcal{B}}(p_1) \middle| \dots \middle| F_{\mathcal{B}}(p_6) \right) = \mathrm{Van}(\lambda_1, \dots, \lambda_6).$$

2. $S = \{\cos(x), \dots, \cos(nx)\}$ è lin. Ind. $\forall n$.

$$F : \langle S \rangle \longrightarrow \mathbb{R}^n$$

$$F(f) = \begin{pmatrix} f(0) \\ f''(0) \\ f^{iv}(0) \\ f^{2n}(0) \end{pmatrix}$$ è lineare.

$n = 3$:

$$\cos(kx)\Big|_{x=0} = 1 \qquad \forall k$$

$$-k\sin(kx)\Big|_{x=0} = 0$$

$$-k^2\cos(kx)\Big|_{x=0} = -k^2$$

$$+k^4\cos(kx)\Big|_{x=0} = k^4$$

$$A = \left(F(\cos x) \,\middle|\, F(\cos 2x) \,\middle|\, F(\cos 3x) \right)$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ -1 & -4 & -9 \\ 1 & 16 & 81 \end{pmatrix}$$

$$= Van(-1, -4, -9)^t$$

$\det A \neq 0 \iff rg\, A = 3 \iff$ le colonne di A sono lin. Ind.

$\Longrightarrow S$ è lin. Ind.

F lineare.

In generale

$$A = \mathrm{Van}(-1, -4, \dots, -n^2)$$

$\Rightarrow \det A \neq 0 \Rightarrow$ le colonne di A sono lin. Ind.

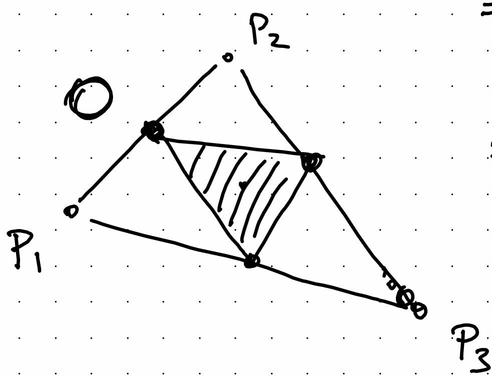
$\Rightarrow$ S è lin. Ind. $\forall n$.

F è lineare

Es 1 :

2. $\text{Area}\left(\frac{P_1+P_2}{2}, \frac{P_2+P_3}{2}, \frac{P_3+P_1}{2}\right) =$

$$= \text{Area}\left(O, \frac{P_3-P_1}{2}, \frac{P_3-P_2}{2}\right)$$

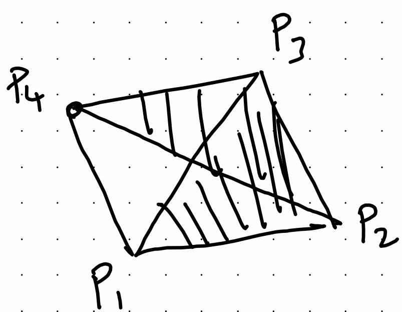


$$= \frac{1}{2}\left|\det\left(\frac{P_3-P_1}{2}, \frac{P_3-P_2}{2}\right)\right|$$

$$= \frac{1}{2}\left|\det\begin{pmatrix}\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\end{pmatrix}\left(P_3-P_1 \mid P_3-P_2\right)\right|$$

$$= \frac{1}{4}\,\text{Area}\left(P_1P_2P_3\right) = \frac{11}{4}$$

3.



Es3:

$$\det A(K) = K^2 (K-1)(K^2-1) = K^2 (K-1)^2 (K+1)$$

Se $K \neq 0$, $K \neq 1$ e $K \neq -1$, $rg(A_K) = 3$.

$rg A_K \geq 1$ perché $\exists$ 1-minore $\neq 0$

$rg A_K = 2 \iff$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \qquad A_{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

$rg A_0 = 1$ $\qquad$ $rg A_1 = 2$ $\qquad$ $rg A_{-1} = 2$

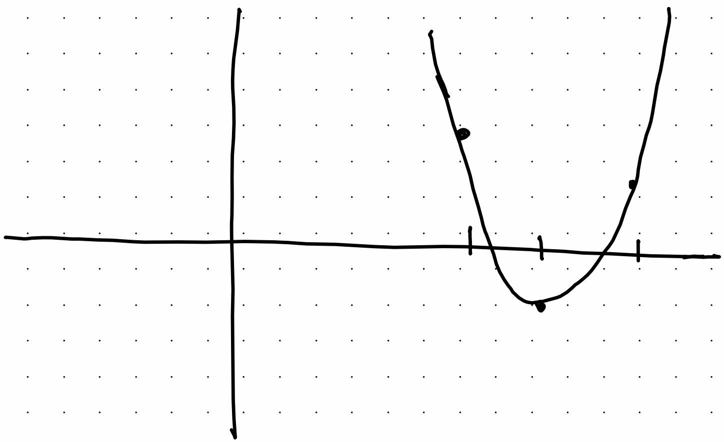
$rg A_K = 2 \iff K = 1$ oppure $K = -1$.

$rg A_K = 1 \iff K = 0$

Es4 :

$$\mathrm{Van}(8,9,10)^{-1}\begin{pmatrix}3\\-1\\1\end{pmatrix}=\begin{pmatrix}251\\-55\\3\end{pmatrix}$$

$$p(x)=251-55x+\underline{\underline{3}}x^2$$



Es 5:

5. Se $U_3 : Ax = b$ $\quad U_4 : A' x = b'$

$$\Rightarrow \quad U_3 \cap U_4 : \begin{cases} Ax = b \\ A'x = b' \end{cases}$$

$$U_3 : \quad -\frac{2}{3} x_1 - \frac{2}{3} x_2 + x_3 = 0$$

$$U_4 : \quad -x_1 + x_2 = 0$$

$$U_3 \cap U_4 : \begin{cases} -\frac{2}{3} x_1 - \frac{2}{3} x_2 + x_3 = 0 \\ -x_1 + x_2 = 0 \end{cases} \qquad \mathrm{rg} \begin{pmatrix} -2/3 & -2/3 & 1 \\ -1 & 1 & 0 \end{pmatrix} = 2.$$