Es 1: $\mathcal{B}_1 = \{1-x, 1+x, x+x^2, 2x-x^2+2x^3\}$

$\mathcal{B}_2 = \{1, x-1, (x-1)^2, (x-1)^3\}$

$$p(x) = 2-3x+2x^2-x^3$$

$$p(x) = a_0(1-x) + a_1(1+x) + a_2(x+x^2) + a_3(2x-x^2+2x^3)$$

$$= (a_0+a_1) + (-a_0+a_1+a_2+2a_3)x + (a_2-a_3)x^2 + 2a_3x^3$$

$$a_0 = \frac{11}{4}, \quad a_1 = -\frac{3}{4}, \quad a_2 = \frac{3}{2}, \quad a_3 = -\frac{1}{2}$$

$$p(x) = b_0 + b_1(x-1) + b_2(x-1)^2 + b_3(x-1)^3$$

$$b_0 = 0, \quad b_1 = -2, \quad b_2 = -1, \quad b_3 = -1$$

Es2:

$$U = \left\{ X \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x_1 - 2x_2 + x_4 = 0, \\ x_3 + x_4 = 0 \end{array} \right\}, \quad W = \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$U = \left\{ X \;\middle|\; \begin{array}{l} x_1 = 2x_2 - x_4 \\ x_3 = -x_4 \end{array} \right\} = \left\{ \begin{pmatrix} 2x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} \;\middle|\; x_2, x_4 \in \mathbb{R} \right\}$$

$$= \left\{ x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_4 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

soluzioni - base

quindi è un sottospazio vettoriale.

Inoltre, dato che

$$u_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ e } u_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

sono lin. Ind. e generano U, formano una base.

$\dim U = 2$, $\dim W = 2 \Rightarrow \dim U \cap W \in \{0, 1, 2\}$

$\dim U+W = \dim U + \dim W - \dim U \cap W \in \{4, 3, 2\}$

Dato che $w_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \in U \cap W$ e $w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \notin U$

concludiamo che

$$U \cap W = \langle w_1 \rangle, \quad \dim U \cap W = 1$$

Dalla Formula di Grassmann otteniamo

$$\dim U+W = 4 - 1 = 3$$

$\mathcal{B}_U = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ è una base di U che estende la base di $U \cap W$.

$\mathcal{B}_W = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ è una base di W che estende la base di $U \cap W$

Una base di $U+W$ è

$$\mathcal{B}_U \cup \mathcal{B}_W = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\square$

Es3:

$$U_1 = \left\{ X \in \mathbb{C}^3 \;\middle|\; \begin{array}{l} x_1 + i x_2 = 0 \\ x_2 + i x_3 = 0 \end{array} \right\} \qquad U_2 = \{ X \in \mathbb{C}^3 \mid x_1 = -x_3 \}$$

$$U_1 = \left\{ X \;\middle|\; \begin{array}{l} x_1 = -i x_2 \\ x_2 = -i x_3 \end{array} \right\} = \left\{ X \;\middle|\; \begin{array}{l} x_1 = -x_3 \\ x_2 = -i x_3 \end{array} \right\}$$

$$= \left\langle \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix} \right\rangle \subseteq U_2$$

$$U_2 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$ è un s. ~~pp~~. vett. di $\mathbb{C}^3$.

$\dim U_2 = 2 > 1 = \dim U_1$. Quindi $U_1 \subsetneq U_2$.

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix} \right\} \subset \mathcal{B}_2 = \left\{ \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subset \mathcal{B}_3 = \left\{ \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$F_{\mathcal{B}_1} : \quad U_1 \longrightarrow \mathbb{C}$$
$$x \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix} \longmapsto x$$

$$F_{\mathcal{B}_2} : \quad U_2 \longrightarrow \mathbb{C}^2$$
$$x \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

$$F_{\mathcal{B}_3} : \quad U_3 \longrightarrow \mathbb{C}^3$$
$$x \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\underline{ES4}$: Vedi la lezione sostituendo a t, $1-t$.

1) NO

2) $\forall t \in \mathbb{R}$

3) $U_0 \cap W_0 \rightsquigarrow$ base $\left\{ \begin{pmatrix} -1 \\ -1 \\ -3 \\ 1 \end{pmatrix} \right\}$

$\rightsquigarrow \left\{ \begin{pmatrix} -1 \\ -1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

Es5:

$$U = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \qquad V = \left\langle \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$$W = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$$W_t = \left\langle \begin{pmatrix} 1+t \\ 0 \\ t \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \qquad \forall t \quad U \oplus W_t = V \oplus W_t = \mathbb{R}^4.$$

$\Rightarrow$ non è unico.