Annuncio: Controllate di aver consegnato Settimana 11.

Domande / Commenti / Suggerimenti ?

Annuncio: Il 22 facciamo lezione
in presenza o da remoto ?

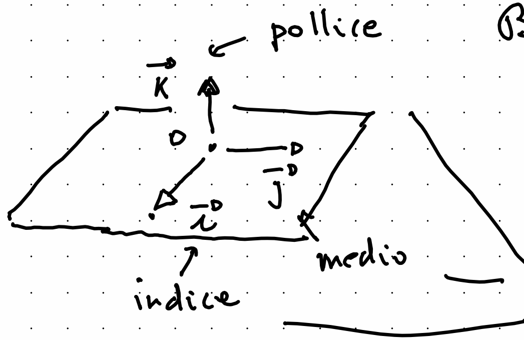
Geometria analitica dello spazio

In V_0^3 fissiamo la base standard:

$\mathcal{B} = \{\vec{i}, \vec{j}, \vec{k}\}$.

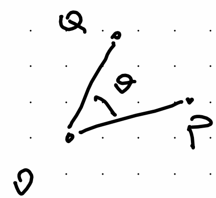
$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$

$\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k}$



regola della mano destra

$$\vec{OP} \cdot \vec{OQ} := |\vec{OP}|\,|\vec{OQ}| \cos\theta$$



$$F_{\mathcal{B}} : V_0^3 \xrightarrow{\simeq} \mathbb{R}^3 \qquad \mathcal{B} \longmapsto \{e_1, e_2, e_3\}$$

$$F_{\mathcal{B}}(\vec{OP}) \cdot F_{\mathcal{B}}(\vec{OQ}) = \vec{OP} \cdot \vec{OQ}$$

prod. scalare standard di $\mathbb{R}^3$

V_0^3

Una base $\mathcal{B} = \{v_1, v_2, v_3\}$ di $\mathbb{R}^3$ si dice

<u>equiversa</u> se $\det(v_1 | v_2 | v_3) > 0$ $\leadsto$ mano destra

<u>contraversa</u> se $\det(v_1 | v_2 | v_3) < 0$ $\longrightarrow$ mano sinistra

Il <u>prodotto vettoriale</u> o <u>prodotto vettore</u>

<u>Def</u>: Il prodotto vettoriale è la funzione

$$\wedge : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

definito come

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \wedge \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \det \begin{pmatrix} x_2 & y_2 \\ x_3 & y_3 \end{pmatrix} \\ -\det \begin{pmatrix} x_1 & y_1 \\ x_3 & y_3 \end{pmatrix} \\ \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \end{pmatrix}$$

"X vettor Y"

<u>Es</u>:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}$$

$$(X \wedge Y) \cdot Z = \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \wedge \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right] \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

Prodotto misto di X, Y, Z.

$$= \det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

sviluppando la 3ª colonna

<u>OSS</u>:

$$(X \wedge Y) \cdot e_1 = \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \end{pmatrix} = \det \begin{pmatrix} x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}$$

$$(X \wedge Y) \cdot e_2 = -\det \begin{pmatrix} x_1 & y_1 \\ x_3 & y_3 \end{pmatrix}$$

$$(X \wedge Y) \cdot e_3 = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$$

$$\Rightarrow \quad X \wedge Y = \begin{pmatrix} (X \wedge Y) \cdot e_1 \\ (X \wedge Y) \cdot e_2 \\ (X \wedge Y) \cdot e_3 \end{pmatrix} = (X \wedge Y) \cdot e_1 \; e_1 + (X \wedge Y) \cdot e_2 \; e_2 + (X \wedge Y) \cdot e_3 \; e_3$$

$$= \sum_{i=1}^{3} (X \wedge Y) \cdot e_i \; e_i = \sum_{i=1}^{3} \det(X | Y | e_i) \, e_i$$

Proprietà di $\wedge$:

1) $v \wedge w$ è ortogonale sia a v che a w

Infatti, $v \wedge w \cdot v = \det(v|w|v) = 0$

$v \wedge w \cdot w = \det(v|w|w) = 0$.

2) $v \wedge w = - w \wedge v$

Infatti, " $\det\begin{pmatrix} x_i & y_i \\ x_j & y_j \end{pmatrix} = -\det\begin{pmatrix} y_i & x_i \\ y_j & x_j \end{pmatrix}$ "

3) È bilineare :

$$(\alpha v_1 + \beta v_2) \wedge w = \alpha \, v_1 \wedge w + \beta \, v_2 \wedge w$$

(È un conto che trovate sulle dispense).

4) $v \wedge w = 0_{\mathbb{R}^3} \iff \mathrm{rg}\,(v|w) \leq 1$

$\iff \{v, w\}$ è lin. Dip.

Infatti, questo segue dal Teorema degli orlati.

5) Se $v \wedge w \neq 0_{\mathbb{R}^3}$ allora la base

$\mathcal{B} = (v, w, v \wedge w)$ è equiversa

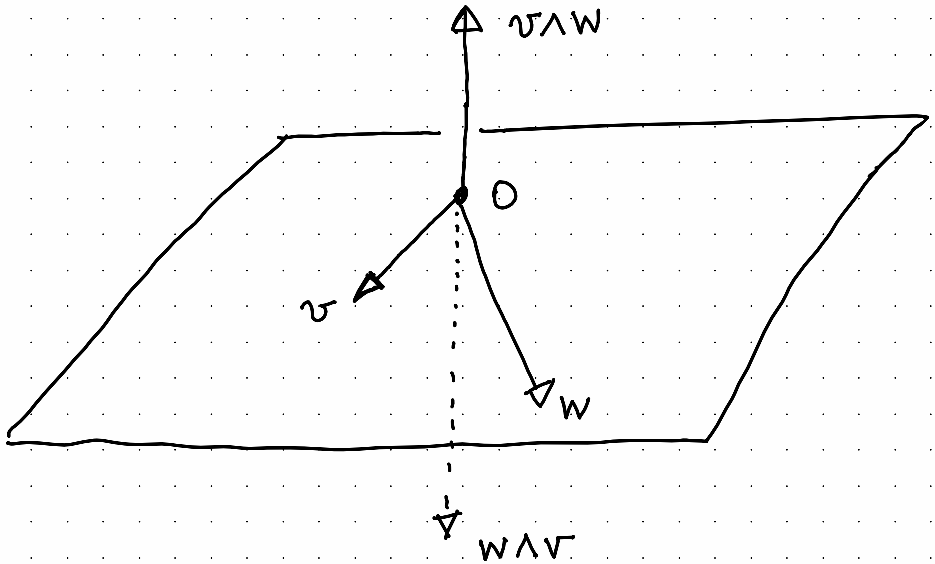
<u>Prop.</u>: $\det(v|w|v \wedge w) = \|v \wedge w\|^2 > 0$

<u>dim</u>:

$$\det(v|w|v\wedge w) = \det\left(v|w| \sum_{i=1}^{3} \det(v|w|e_i)\, e_i\right)$$

$$= \sum_{i=1}^{3} \det(v|w|e_i) \det(v|w|e_i)$$

$$= \sum \det(v|w|e_i)^2 = \|v \wedge w\|^2 \qquad \blacksquare$$


v∧w
O
v
w
w∧v

Es: $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad w = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \Rightarrow v \wedge w = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$

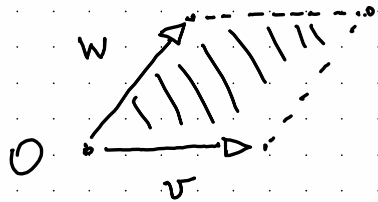
$$\det (v \,|\, w \,|\, v \wedge w) = \det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & -3 \end{pmatrix}$$

$$= 3 \det \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & -1 \end{pmatrix}$$

$$= 3 \det \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 4 & 5 & 0 \end{pmatrix} = 3 \cdot 9 = 27$$

$$\| v \wedge w \|^2 = \left\| 3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\|^2 = 9 \left\| \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\|^2 = 9 \cdot 3 = 27$$

Dati $v, w \in \mathbb{R}^3$ sia $\mathcal{P}(v,w)$ il parallelogramma che ha v e w come lati



$= \mathcal{P}(v,w)$

<u>Teorema</u>: $\quad \text{Area}\,(\mathcal{P}(v,w)) = \|v \wedge w\|$

<u>dim</u>: È un conto:

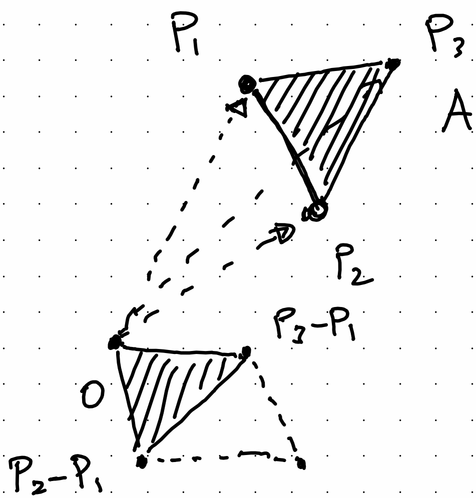
$$\text{Area}\ \mathcal{P}(v,w) = \|v\|\,\|w\| \sin \widehat{vw}.$$

<u>Es</u>: Siano $v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ e $w = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$

$$\text{Area } \mathcal{P}(v,w) = \| v \wedge w \| =$$
$$= \left\| \begin{pmatrix} -1 \\ -4 \\ -3 \end{pmatrix} \right\| = \sqrt{1^2+4^2+3^2} = \sqrt{26}$$

<u>Es</u>: Calcolare l'area del Triangolo di vertici $P_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $P_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$.

<u>Sol.</u>:

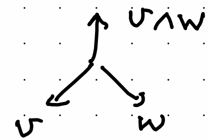


$$\text{Area } \widehat{P_1 P_2 P_3} = \frac{1}{2} \text{Area } \mathcal{P}(P_3-P_1, P_2-P_1)$$
$$= \frac{1}{2} \| (P_3-P_1) \wedge (P_2-P_1) \|$$
$$= \frac{1}{2} \left\| \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\| = \frac{1}{2} \left\| \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} \right\|$$
$$= \frac{3}{2} \left\| \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\| = \frac{3}{2} \sqrt{5}.$$

Ricapitolando: $v \wedge w$ ha

1) Direzione: ortogonale al piano $\langle v, w \rangle$

2) Verso: come la mano destra



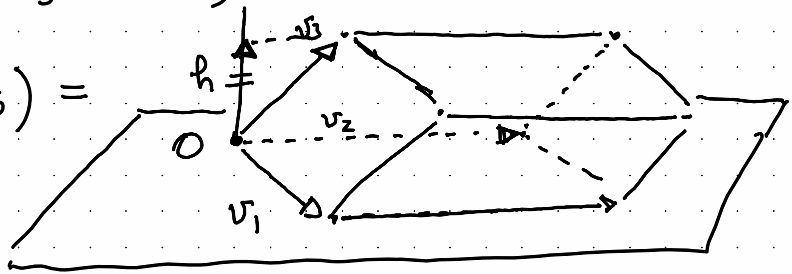
3) Norma: Area $P(v,w)$.

OSS: Il prodotto vettoriale $\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ non è un'operazione associativa su $\mathbb{R}^3$:

Es:
$$(e_1 \wedge e_2) \wedge (e_2 + e_3) = e_3 \wedge (e_2 + e_3) = e_3 \wedge e_2 + e_3 \wedge e_3 = -e_1$$
$$\neq$$
$$e_1 \wedge (e_2 \wedge (e_2 + e_3)) = e_1 \wedge (e_1) = 0$$

Il determinante 3×3 è un volume "orientato".

Siano $v_1, v_2, v_3 \in \mathbb{R}^3$, sia

$P(v_1, v_2, v_3) =$ 

Teorema: $\mathrm{Vol}\, P(v_1, v_2, v_3) = |\det(v_1|v_2|v_3)|$

dim:

$$\mathrm{Vol}\, P(v_1, v_2, v_3) = \mathrm{Area}\, P(v_1, v_2) \times \text{altezza}$$

$$= \|v_1 \wedge v_2\| \, \|pr_{v_1\wedge v_2}(v_3)\|$$

$$= \|v_1 \wedge v_2\| \frac{|(v_1\wedge v_2)\cdot v_3|}{\|v_1\wedge v_2\|} = |(v_1\wedge v_2)\cdot v_3| = |\det(v_1|v_2|v_3)|.$$

$\blacksquare$

Es: Determinare $K \in \mathbb{R}$ t.c. il volume del parallelepipedo di spigoli

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ K \\ 2 \end{pmatrix}$$

sia uguale a 1

Sol.:

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & K \\ 1 & 0 & 2 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & K-2 \\ 1 & 0 & 0 \end{pmatrix} = \det \begin{pmatrix} 1 & -1 \\ 1 & K-2 \end{pmatrix} = K-1$$

$$\mathrm{Vol}\, \mathcal{P}(v_1, v_2, v_3) = |K-1|$$

$$|K-1| = 1 \iff K = 2 \text{ oppure } K = 0.$$

$\blacksquare$

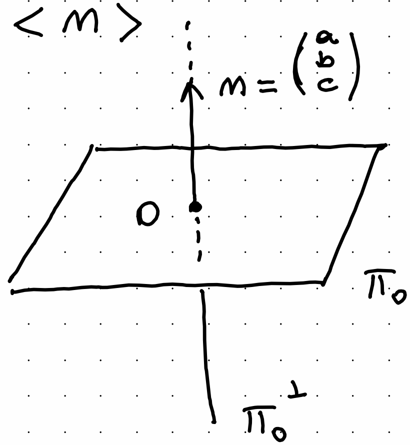
Versori normal ad un piano

Sia $\quad \pi : ax+by+cz=d \quad$ un pieno di $\mathbb{R}^3$.

$\quad \pi_0 : ax+by+cz=0$

$\Rightarrow \pi_0 : n \cdot X = 0 \quad$ dove $\; n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\pi_0^{\perp} = \langle n \rangle$



I versori normali a π sono $\quad \pm \frac{1}{\|n\|} n = \pm \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Se $X_0 \in \pi$ allora

$$\pi : \; n \cdot X = n \cdot X_0$$

Se $\pi = X_0 + \langle v_1, v_2 \rangle$, Allora

$$\pi_0 = \langle v_1, v_2 \rangle \quad \text{e} \quad \pi_0^{\perp} = \langle v_1 \wedge v_2 \rangle$$

Le eq. cartesiane di π sono

$$(v_1 \wedge v_2) \cdot X = (v_1 \wedge v_2) \cdot X_0 .$$

<u>Es</u>: Trovare delle eq cartesiane del piano

$$\pi = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \right\rangle .$$

<u>Sol.</u>: $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \overset{\text{prodotto misto.}}{=} \det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix}$

$= -2$

$$\pi : \; -5x + 2y + z = -2$$

Versori normali e direttori di una retta di $\mathbb{R}^3$

Sia $r : \begin{cases} ax+by+cz=d \\ a'x+b'y+c'z=d' \end{cases}$ una retta di $\mathbb{R}^3$

$\left(\Leftrightarrow \operatorname{rg}\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix} = 2 \Leftrightarrow \operatorname{rg}\begin{pmatrix} a & a' \\ b & b' \\ c & c' \end{pmatrix} = 2 \Leftrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right).$

Lo spazio di giacitura di r è

$$r_0 : \begin{cases} ax+by+cz=0 \\ a'x+b'y+c'z=0 \end{cases}$$

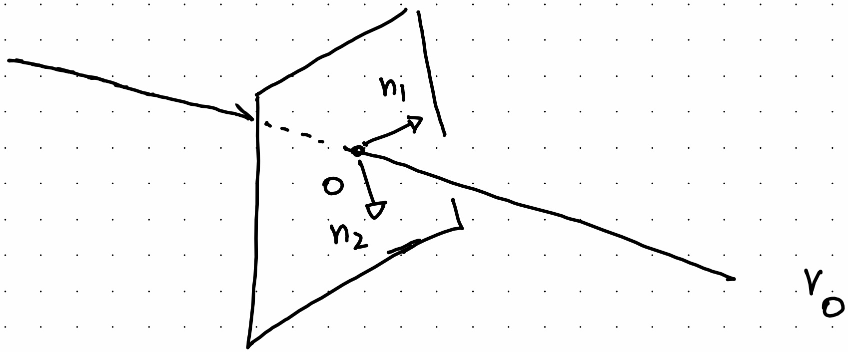
$$r_0 = \left\{ X \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot X = 0 \;;\; \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \cdot X = 0 \right\}$$

$$= \left\{ X \in \mathbb{R}^3 \;\middle|\; X \text{ è ortogonale a } \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \right\rangle \right\}.$$

$$= \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \right\rangle \Rightarrow v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \text{ è}$$

un vettore direttore di $r \Rightarrow \boxed{r = X_0 + \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \right\rangle}$

$n_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $n_2 = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}$ sono vettori <u>normali</u> alla retta
(ortogonali)

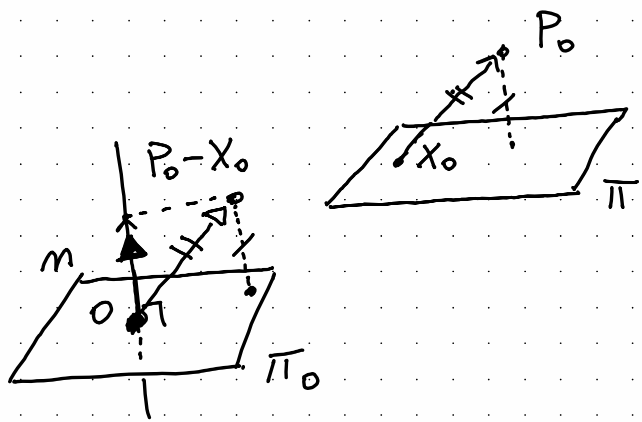


I versori normali :

$$\frac{1}{\sqrt{a^2+b^2c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \frac{1}{\sqrt{a'^2+b'^2+c'^2}} \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}$$

Distanza punto-piano

Sia $P_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \in \mathbb{R}^3$ e sia $\pi \subset \mathbb{R}^3$ un piano.
Calcoliamo $\mathrm{dist}(P_0, \pi)$. Supponiamo $\pi = X_0 + \pi_0$
Sia n un vettore normale a π_0.



$$\mathrm{dist}(P_0, \pi) = \mathrm{dist}(P_0 - X_0, \pi_0) = \| \mathrm{pr}_n (P_0 - X_0) \|$$

$$= \frac{|(P_0 - X_0) \cdot n|}{\|n\|}$$

Se $\pi_0 = \langle v_1, v_2 \rangle$ allora possiamo scegliere

$$n = v_1 \wedge v_2$$

e quindi

$$\mathrm{dist}(P_0, \pi) = \frac{|(P_0 - X_0) \cdot n|}{\|n\|} = \frac{|(P_0 - X_0) \cdot v_1 \wedge v_2|}{\|v_1 \wedge v_2\|}$$

$$= \frac{|\det(v_1 | v_2 | P_0 - X_0)|}{\|v_1 \wedge v_2\|}$$

Se $\pi : ax + by + cz = d$, allore possiemo scegliere $n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $n \cdot X_0 = d$. Quindi

$$\mathrm{dist}(P_0, \pi) = \frac{|(P_0 - X_0) \cdot n|}{\|n\|} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Es: $P_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\quad \pi: 2x + 3y - z = 2$

$$\operatorname{dist}(P_0, \pi) = \frac{|2\cdot 1 + 3\cdot 2 - 3 - 2|}{\sqrt{2^2+3^2+(-1)^2}} = \frac{3}{\sqrt{14}}$$

Es: $P_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\pi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle$

$$\operatorname{dist}(P_0, \pi) = \frac{\left| \det\left(\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \middle| P_0 - X_0 \right) \right|}{\left\| \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\|}$$

$$= \frac{\left| \det \begin{pmatrix} 3 & 1 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 3 \end{pmatrix} \right|}{\left\| \begin{pmatrix} -4 \\ -6 \\ 2 \end{pmatrix} \right\|} = \frac{\left| \det \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 2 & 3 \end{pmatrix} \right|}{\sqrt{4^2+6^2+2^2}} =$$

$$= \frac{\left| -2 \det \begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix} \right|}{\sqrt{56}} = \frac{\left| -6 \det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \right|}{\sqrt{56}} = \frac{6}{\sqrt{56}} = \frac{3}{\sqrt{14}}.$$

Distanza retta-retta

Sia $r_1 = X_1 + \langle v_1 \rangle$ e $r_2 = X_2 + \langle v_2 \rangle$ due rette di $\mathbb{R}^3$.

$$\mathrm{dist}(r_1, r_2) = \min_{s,t \in \mathbb{R}} \mathrm{dist}(X_1 + s v_1, X_2 + t v_2)$$

dist è invariante per traslazioni

$$= \min_{s,t \in \mathbb{R}} \mathrm{dist}(X_1 - X_2, t v_2 - s v_1)$$

$$= \mathrm{dist}(X_1 - X_2, \langle v_1, v_2 \rangle)$$

distanza "punto-piano"

$v_1 \wedge v_2 \neq 0_{\mathbb{R}^3}$

$$= \| pr_{v_1 \wedge v_2}(X_1 - X_2) \|$$

$$= \frac{|\det(v_1 | v_2 | X_1 - X_2)|}{\| v_1 \wedge v_2 \|}$$

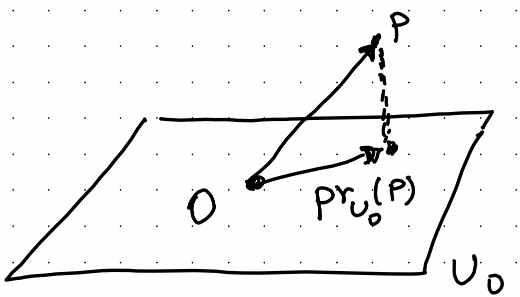
Se r_1 ed r_2 sono parallele con giacitura $r_0 = \langle v \rangle$

$$\mathrm{dist}(r_1, r_2) = \mathrm{dist}(X_1 - X_2, r_0) = \text{distanza } \underline{\text{punto-retta}}$$

$$= \| (X_1 - X_2) - pr_v(X_1 - X_2) \|$$

<u>Richiami</u>: Se $U_0 \subset V$ è un sp. vett., allora

$$\mathrm{dist}(P, U_0) = \| P - pr_{U_0}(P) \|$$



Es:

$$r_1 : \begin{cases} x+y-z=2 \\ 2x-y+2z=1 \end{cases} \qquad r_2 : \begin{cases} x-2y-2z=1 \\ 2x+y-z=2 \end{cases}$$

Sol.:

$$r_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \right\rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \right\rangle$$

$$r_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} \right\rangle$$

$$\operatorname{dist}(r_1, r_2) = \frac{\left| \det \begin{pmatrix} 1 & 4 & 0 \\ -4 & -3 & 1 \\ -3 & 5 & 0 \end{pmatrix} \right|}{\left\| \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \wedge \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} \right\|} = \frac{17}{\sqrt{1299}}$$

Es: $P_0 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad r = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\rangle$

$$\text{dist}(P_0, r) = \left\| \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} - pr_{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} - \frac{-1}{9} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} 2/9 \\ -8/9 \\ 2/9 \end{pmatrix} \right\| = \frac{2}{9} \left\| \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} \right\|$$

$$= \frac{2}{9} \sqrt{1 + 4^2 + 1} = \frac{2\sqrt{18}}{9}$$