Domande / Commenti / Suggerimenti ?

Richiami: <u>Disuguaglianza di Cauchy - Schwarz</u>

Sia (V, s) uno spazio euclideo, siano $v, w \in V$. Allora

$$|s(v,w)| \leq \|v\| \|w\|$$

In particolare, se $v, w \neq 0_V$,

$$-1 \leq \frac{s(v,w)}{\|v\|\|w\|} \leq 1$$

1) Abbiamo definito il coseno dell'angolo tra $v, w \neq 0_V$ come

$$\cos(\hat{vw}) := \frac{s(v,w)}{\|v\|\|w\|}.$$

Disuguaglianza triangolare

Siano $v, w \in V$, (V, s): spazio euclideo, allora

$$\|v+w\| \le \|v\| + \|w\|$$

dim:

$$\|v+w\|^2 = s(v+w, v+w) = s(v,v) + s(w,w) + 2s(v,w)$$

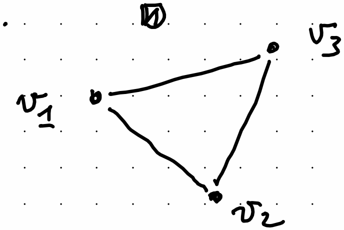
$$= \|v\|^2 + \|w\|^2 + 2s(v,w) \le \|v\|^2 + \|w\|^2 + 2|s(v,w)|$$

$$\underset{\text{C-S}}{\le} \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| = (\|v\| + \|w\|)^2$$

$$\Rightarrow \quad \|v+w\| \le \|v\| + \|w\|. \quad \blacksquare$$

Riformulazione:

$$\mathrm{dist}(v_1, v_2) \le \mathrm{dist}(v_1, v_3) + \mathrm{dist}(v_2, v_3)$$



dim:

$$\mathrm{dist}(v_1, v_2) \overset{\text{Def}}{=} \|v_1 - v_2\| = \|v_1 - v_3 + v_3 - v_2\| \underset{\text{dis. Triang.}}{\le} \|v_1 - v_3\| + \|v_3 - v_2\| =$$

$$= \mathrm{dist}(v_1, v_3) + \mathrm{dist}(v_2, v_3).$$

Geometria analitica del piano

Consideriamo lo spazio euclideo $(\mathbb{R}^2, \cdot)$:

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \Rightarrow X \cdot Y = X^t Y = x_1 y_1 + x_2 y_2$.

La norma di X è $\|X\| = \sqrt{X \cdot X} = \sqrt{x_1^2 + x_2^2}$

$\|X\| \geq 0 \quad \forall X \in \mathbb{R}^2$, $\|X\| = 0 \iff X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0_{\mathbb{R}^2}$.

$\|\lambda X\| = |\lambda| \, \|X\| \quad \forall \lambda \in \mathbb{R}, \forall X \in \mathbb{R}^2$

Es: $\left\| \begin{pmatrix} -3 \\ 6 \end{pmatrix} \right\| = \left\| -3 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\| = 3 \left\| \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\| = 3\sqrt{1^2 + 2^2} = 3\sqrt{5}$.

Un versore di $(\mathbb{R}^2, \cdot)$ è un vettore $X \in \mathbb{R}^2$ t.c. $\|X\| = 1$.

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ è un versore $\iff x_1^2 + x_2^2 = 1$.

$\iff \exists \ \theta \in \mathbb{R}$ t.c. $x_1 = \cos\theta$, $x_2 = \sin\theta$.

Versori di $(\mathbb{R}^2, \cdot) = \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \;\middle|\; \theta \in [0, 2\pi) \right\}$

Notazione: $P_\theta := \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$.

- $P_{\theta + 2k\pi} = P_\theta$, • $-P_\theta = P_{\theta+\pi}$, • $\cancel{P_\theta \cdot P_\mu \iff}$

$$P_\theta \cdot P_\mu = 0 \iff \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\mu \\ \sin\mu \end{pmatrix} = 0$$

$$\iff \cos\theta\cos\mu + \sin\theta\sin\mu = 0$$

$$\iff \cos(\theta-\mu) = 0$$

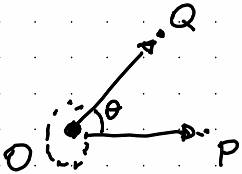
$$\iff \theta-\mu \in k\frac{\pi}{2} \quad \text{per qualche } k \in \mathbb{Z}.$$

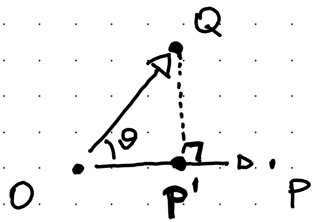
$$P_\theta \perp P_\mu \iff \mu = \theta \pm \frac{\pi}{2}$$

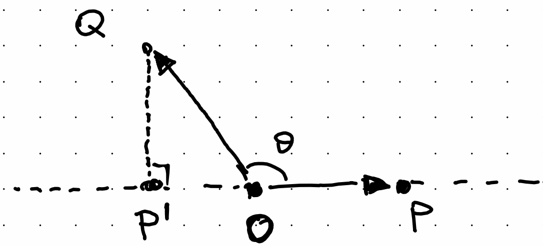
Rappresentazione grafica di $(\mathbb{R}^2, \cdot)$

Sia $V = V_0^2$. Il prodotto scalare standard di V_0^2 è

$$\vec{OP} \cdot \vec{OQ} = |\overline{OP}|\,|\overline{OQ}| \cos\vartheta$$



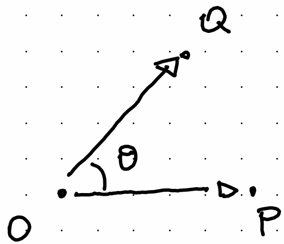




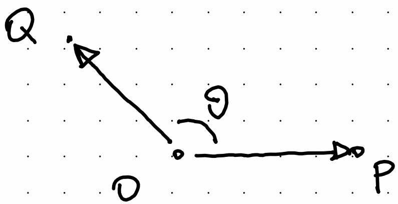
$$\vec{OP} \cdot \vec{OQ} = |\overline{OP}|\,|\overline{OP'}|$$

$$\vec{OP} \cdot \vec{OQ} = -|\overline{OP}|\,|\overline{OP'}|$$

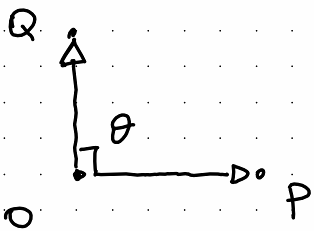
Oss: $\cdot$ è bilineare, simmetrico, definito positivo.



$$\overrightarrow{OP} \cdot \overrightarrow{OQ} > 0 \iff \theta \text{ è acuto.}$$

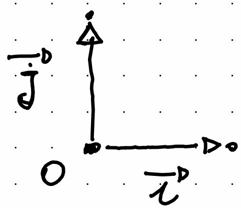


$$\overrightarrow{OP} \cdot \overrightarrow{OQ} < 0 \iff \theta \text{ è ottuso}$$



$$\overrightarrow{OP} \cdot \overrightarrow{OQ} = 0 \iff \theta = \frac{\pi}{2}.$$

Sia $\mathcal{B} = \{\vec{\imath}, \vec{\jmath}\}$ ~~una~~ base ortonormale $(\mathcal{V}_O^2, \cdot)$



$$F_{\mathcal{B}} : \mathcal{V}_O^2 \xrightarrow{\simeq} \mathbb{R}^2$$

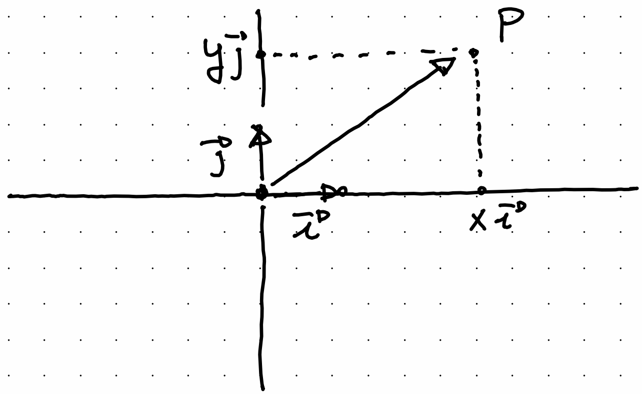
è un isomorfismo di spazi euclidei, i.e.

$$F_{\mathcal{B}}(\vec{OP}) \underset{\text{in } \mathbb{R}^2}{\cdot} F_{\mathcal{B}}(\vec{OQ}) = \vec{OP} \underset{\text{in } \mathcal{V}_O^2}{\cdot} \vec{OQ}$$

Infatti,

$$\begin{aligned} F_{\mathcal{B}}(\vec{OP}) \cdot F_{\mathcal{B}}(\vec{OQ}) &= \| F_{\mathcal{B}}(\vec{OP}) \| \, \| F_{\mathcal{B}}(\vec{OQ}) \| \cos\theta \\ &= |\overline{OP}| \, |\overline{OQ}| \cos\theta \\ &= \vec{OP} \cdot \vec{OQ} . \end{aligned}$$

Piano cartesiano



$\vec{OP} = x\,\vec{i} + y\,\vec{j}$

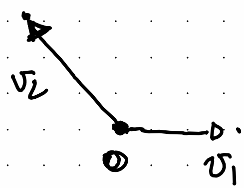
x = ascissa di $\vec{OP}$

y = ordinata di $\vec{OP}$

Da adesso in poi identifichiamo $\mathcal{V}_O^2 \cong \mathbb{R}^2$ tramite $\mathcal{B}$.

<u>OSS</u>:

$\mathcal{C} = \{v_1, v_2\}$



$|v_1| = 3$

$|v_2| = \sqrt{4+9} = \sqrt{13}$

$F_{\mathcal{C}}(v_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = 1 \neq \sqrt{13}$

Circonferenze: Sia $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^2$ e sia $r > 0$.

La circonferenza di centro c e raggio r è il luogo dei punti che hanno distanza r da c:

$$\mathcal{C}(c, r) = \{ X \in \mathbb{R}^2 \mid \|X - c\| = r \}$$

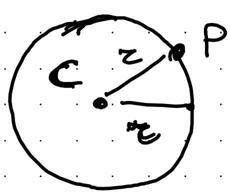
$$= \{ X \in \mathbb{R}^2 \mid \|X - c\|^2 = r^2 \}$$

$$= \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid (x_1 - c_1)^2 + (x_2 - c_2)^2 = r^2 \right\}.$$

$\mathcal{C}(c, r) : (x_1 - c_1)^2 + (x_2 - c_2)^2 = r^2$ Eq. cartesiana di $\mathcal{C}(c, r)$

$\mathcal{C}(c, r) : x_1^2 + x_2^2 - 2c_1 x_1 - 2c_2 x_2 + (c_1^2 + c_2^2 - r^2) = 0.$

Es: $\mathcal{C}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, 2\right) : x^2 + y^2 - 2x - 4y + 1 = 0$

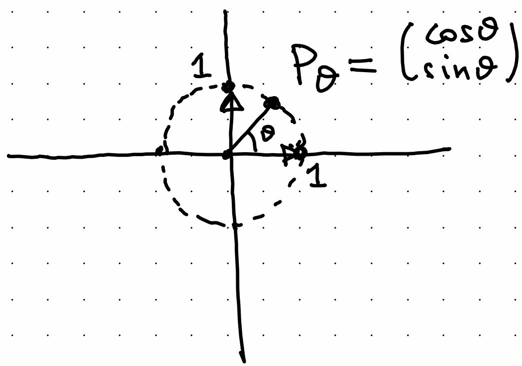
$\mathcal{C}(c, r) =$ 

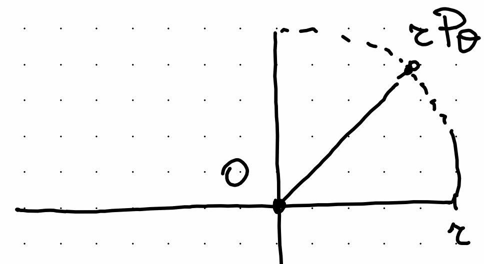
circonferenza unitaria di centro O:

$$\mathcal{C}(0,1) = \{ X \in \mathbb{R}^2 \mid \|X\| = 1 \} =$$

$$= \{ \text{versori di } (\mathbb{R}^2, \cdot) \}$$

$$= \left\{ P_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \;\middle|\; \theta \in [0, 2\pi) \right\}$$





$\dfrac{X}{\|X\|} = P_\theta$ è un versore

$$\mathcal{C}(0,r) = \{ X \in \mathbb{R}^2 \mid \|X\| = r \} = \{ X \mid X = r P_\theta \}$$

$$= \left\{ r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \;\middle|\; \theta \in [0, 2\pi) \right\}.$$

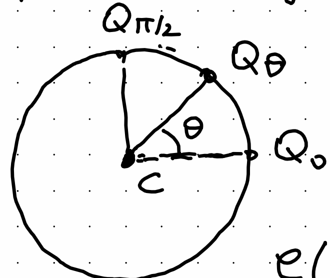
Eq. parametriche della circonferenza:

$$\mathcal{C}(C,r) = \{ X \in \mathbb{R}^2 \mid \|X - C\| = r \} \overset{X - C = r P_\theta}{=}$$

$$= \{ C + r P_\theta \mid \theta \in [0, 2\pi) \}$$

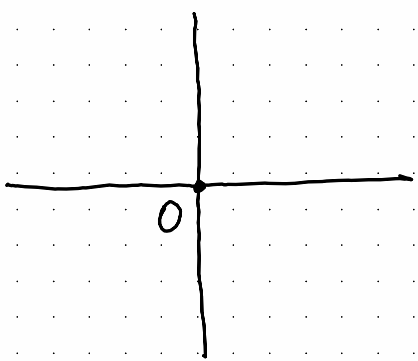
$$= \Big\{ \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}}_{\substack{\shortparallel \\ Q_\theta}} \;\Big|\; \theta \in [0, 2\pi) \Big\}$$

<u>Eq. parametriche di $\mathcal{C}(C,r)$.</u>



$\mathcal{C}(C,r)$

"senso anti-orario."



Es: Trovare un'equazione parametrica della circonferenza

$\mathcal{C}: \; x^2+y^2-3x+4y+4=0$

Sol.: Completiamo i quadrati:

$$\mathcal{C}: \left(x^2-2\cdot\frac{3}{2}x+\frac{9}{4}-\frac{9}{4}\right)+\left(y^2+2\cdot 2y+4\right)=0$$

$$\mathcal{C}: \left(x-\frac{3}{2}\right)^2-\frac{9}{4}+(y+2)^2=0$$

$$\Rightarrow \mathcal{C}: \left(x-\frac{3}{2}\right)^2+(y+2)^2=\left(\frac{3}{2}\right)^2$$

$\Rightarrow$ il centro di $\mathcal{C}$ è $C=\begin{pmatrix} 3/2 \\ -2 \end{pmatrix}$ e il raggio è $r=\frac{3}{2}$

$$\mathcal{C}=\left\{ C+rP_\theta \mid \theta\in[0,2\pi) \right\}=\left\{ \begin{pmatrix} 3/2 \\ -2 \end{pmatrix}+\frac{3}{2}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \;\middle|\; \theta\in[0,2\pi) \right\}.$$

Vettori direttori e normali di una retta

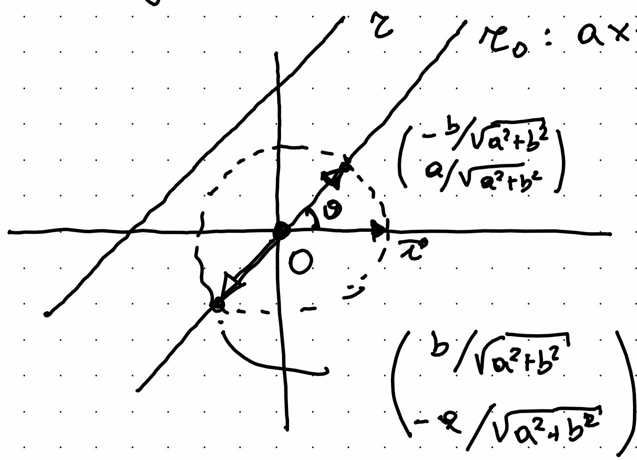
Sia $r: \; ax+by=c$ una retta $\big((a,b)\neq(0,0)\big)$.

Un vettore direttore di r è $v = \begin{pmatrix} -b \\ a \end{pmatrix}$ = base di $\ker(a,b)$

I versori direttori (o coseni direttori) di r sono

$$\pm \frac{v}{\|v\|} = \pm \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} -b \\ a \end{pmatrix}.$$

Es: $r: \; 2x+3y=1$ i versori direttori di r sono $\pm \frac{1}{\sqrt{13}} \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.



$(b<0)$

$$-\frac{b}{\sqrt{a^2+b^2}} = \cos\vartheta$$

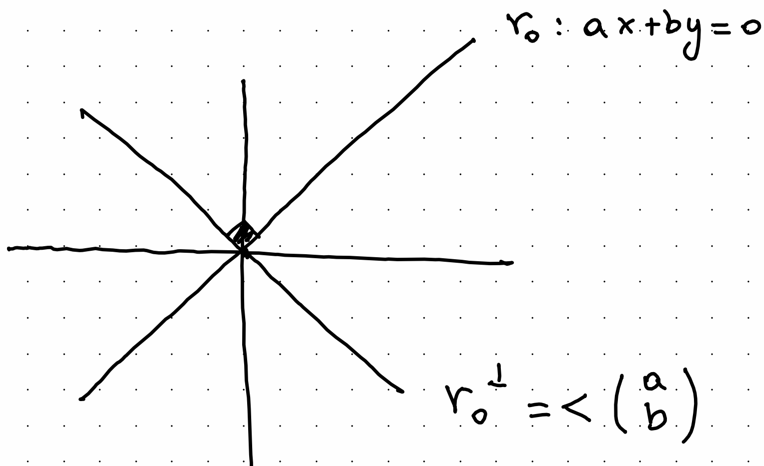
$$\frac{a}{\sqrt{a^2+b^2}} = \sin\vartheta = \cos\left(\frac{\pi}{2}-\vartheta\right).$$

I <u>versori normali</u> alla retta $r: ax+by=c$ sono i versori direttori di $r_0^{\perp}$.

$$r_0 = \left\langle \begin{pmatrix} -b \\ a \end{pmatrix} \right\rangle \Rightarrow r_0^{\perp} = \left\langle \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle$$

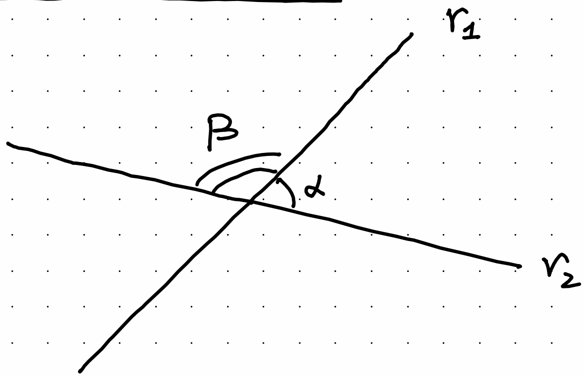
I <u>versori normali</u> sono

$$\pm \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \vec{n}$$



<u>Es</u>: $r: 2x+3y=3 \Rightarrow \pm\vec{n} = \pm\frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Angoli tra rette



$$\alpha + \beta = \pi$$

L'angolo tra r_1 e r_2 = angolo acuto = $\widehat{r_1 r_2} \in [0, \frac{\pi}{2}]$

$$\cos \widehat{r_1 r_2} = |\cos(v, w)| = \left| \frac{v \cdot w}{\|v\| \|w\|} \right|$$

dove $r_1 = P + \langle v \rangle$ e $r_2 = Q + \langle w \rangle$.

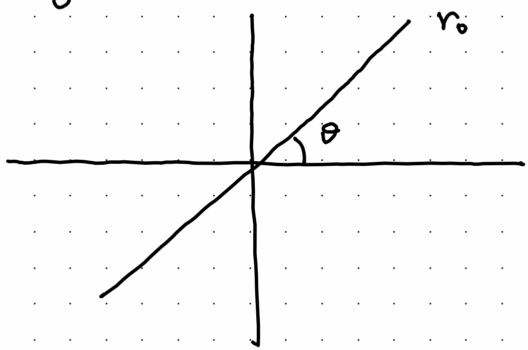
Pendenza o coefficiente angolare di una retta

Sia $r: ax+by=c$, non parallela a $x=0$ (asse delle ordinate). $\Rightarrow b \neq 0$.

Il numero $m = -\frac{a}{b}$ si chiama la pendenza o coefficiente angolare di r:

$r_0 : ax+by=0$, $r_0 = \left\langle \begin{pmatrix} -b \\ a \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\rangle$

θ = angolo con l'asse delle ascisse



$$m = -\frac{a}{b} = \frac{\sin\theta}{\cos\theta} = tg(\theta)$$

$$r: \quad y = \underbrace{tg(\theta)}_{m}\, x + c$$

Prop.: Se $r_1: y = m_1 x + q_1$ e $r_2: y = m_2 x + q_2$.
Allora l'angolo α tra r_1 e r_2 è t.c.

$$\operatorname{tg}(\alpha) = \frac{|m_1 - m_2|}{|1 + m_1 m_2|}$$

Es: Calcolare l'angolo tra

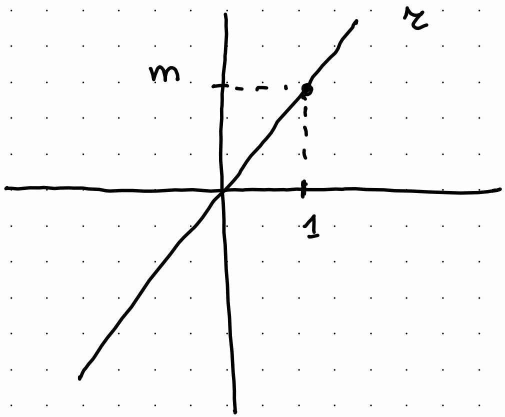
$r_1: 3x + y + 5 = 0$ e $r_2: 2x - y + 1 = 0$

Sol.:

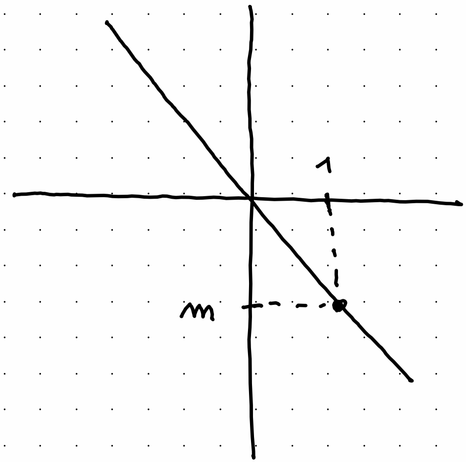
$r_1: y = -3x - 5$ $\qquad$ $r_2: y = 2x + 1$

$$\operatorname{tg}(\widehat{r_1 r_2}) = \frac{|-3-2|}{|1+(-3)2|} = \frac{5}{|-5|} = 1 \Rightarrow \widehat{r_1 r_2} = \frac{\pi}{4}$$

<u>oss</u>: $\quad r: \quad y = mx$



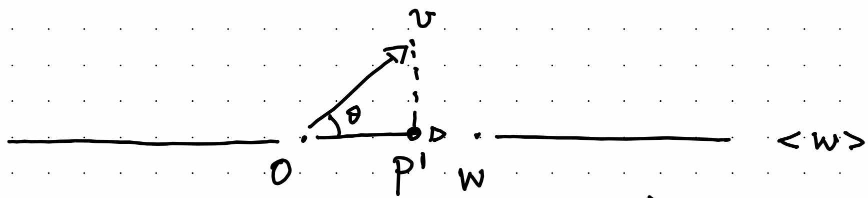
$m > 0$



$m < 0$

Distanza punto-retta

$v, w \in \mathbb{R}^2, \quad w = 0_{\mathbb{R}^2} \Rightarrow \quad pr_w(v) = \frac{v \cdot w}{w \cdot w}\, w$



$$\overrightarrow{OP'} = \frac{v \cdot w}{w \cdot w}\, w.$$

$r_0 = \langle w \rangle$

$$\mathrm{dist}(P, r_0) = \mathrm{dist}(P, pr_w(P)) = \| P - pr_w(P) \|$$

$$= \left\| P - \frac{P \cdot w}{w \cdot w}\, w \right\| = \text{distanza punto-retta per } O.$$

$r = Q + \langle w \rangle,$

$$\mathrm{dist}(P, r) = \mathrm{dist}(P - Q, \langle w \rangle) = \left\| (P-Q) - \frac{(P-Q) \cdot w}{w \cdot w}\, w \right\|$$

Es: $P = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\quad r = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle$

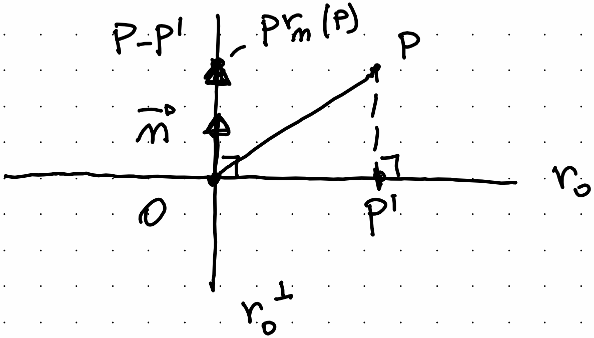
$$\mathrm{dist}(P, r) = \left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} 6/5 \\ -3/5 \end{pmatrix} \right\| = \frac{3}{5} \left\| \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\|$$

$$= \frac{3}{5} \sqrt{5} = \frac{3}{\sqrt{5}}.$$

Se $r: ax+by=c$ allora $r_0^{\perp} = \left\langle \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle$ $= n$



$P-P' = pr_n(P)$

$r = Q + \left\langle \begin{pmatrix} -b \\ a \end{pmatrix} \right\rangle.$

$P = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$\mathrm{dist}(P, r) = \| pr_n(P-Q) \|$$

$\|n\| = \sqrt{n \cdot n}$

$$= \left\| \frac{(P-Q)\cdot n}{n \cdot n} n \right\| = \frac{|(P-Q)\cdot n|}{n \cdot n} \|n\| =$$

$$= \frac{|(P-Q)\cdot n|}{\|n\|} = \frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}$$

__Es__: $P = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\quad r: 2x - y = 1$

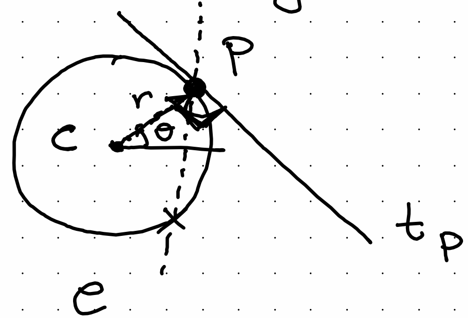
Allora

$$\text{dist}(P, r) = \frac{|2 \cdot 3 - 2 - 1|}{\sqrt{2^2 + (-1)^2}} = \frac{3}{\sqrt{5}}$$

Rette Tangenti ad una circonferenza

Sia $\mathcal{C} = \mathcal{C}(C, r)$, $P \in \mathcal{C}$

Sia t_P la retta tangente a $\mathcal{C}$ e passante per P



t_P è l'unica retta che passa per P e dista r da C:

$$P = C + r P_\theta \quad , \quad t_P = P + \langle v \rangle$$

$$v \perp P_\theta \iff v = P_{\theta + \frac{\pi}{2}} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

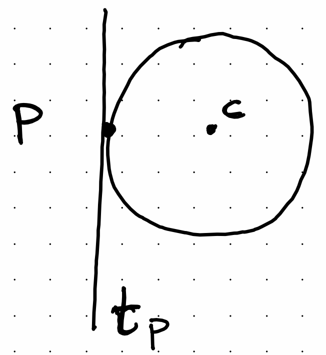
$$t_P = P + \left\langle \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \right\rangle \quad : \quad \cos\theta\, x + \sin\theta\, y = P \cdot P_\theta$$

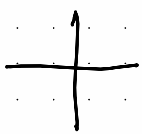
Es: $\mathcal{C}: x^2+y^2-3x+4y+4=0$, $P=\begin{pmatrix}0\\-2\end{pmatrix}\in\mathcal{C}$

Trovare t_P

Sol: $\mathcal{C}: \left(x-\frac{3}{2}\right)^2+\left(y+2\right)^2=\frac{9}{4}$ $\quad C=\begin{pmatrix}3/2\\-2\end{pmatrix}$ $\quad r=\frac{3}{2}$

$$P=\begin{pmatrix}3/2\\-2\end{pmatrix}+\frac{3}{2}\begin{pmatrix}-1\\0\end{pmatrix}=C+r\,P_{\pi}$$





$$t_P=P+\left\langle\begin{pmatrix}0\\1\end{pmatrix}\right\rangle$$

$: \; x=0$

$=$ asse delle ordinate.