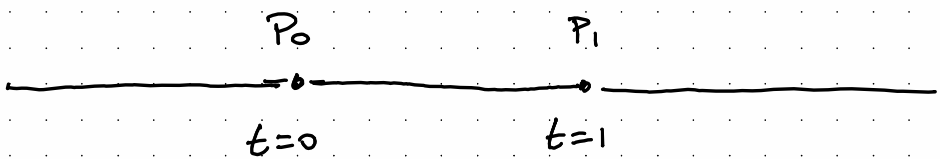
Combinazioni affini e convesse

Retta r per 2 punti $P_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq P_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in \mathbb{R}^2$:

$$r: \quad -(y_1 - y_0)(x - x_0) + (x_1 - x_0)(y - y_0) = 0$$

ed eq. parametriche

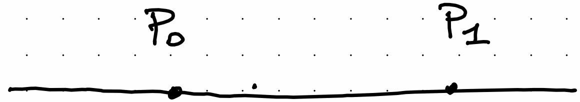
$$r = P_0 + \langle P_1 - P_0 \rangle$$



$$r = \{ P_0 + t(P_1 - P_0) = P_t \mid t \in \mathbb{R} \}$$

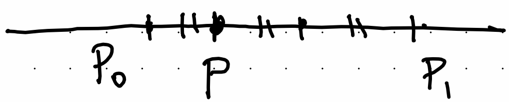
$$P_t = (1-t)P_0 + tP_1 = t_0 P_0 + t_1 P_1 \qquad (\underline{\underline{t_0 + t_1 = 1}})$$

combinazione affine.

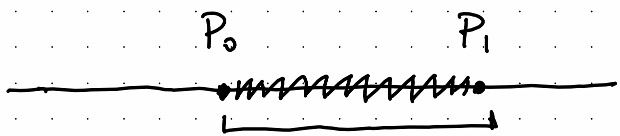


$$P = t_0 P_0 + t_1 P_1 \qquad t_0 + t_1 = 1$$

$$t_0 = \frac{1}{3} \qquad t_1 = \frac{2}{3}$$

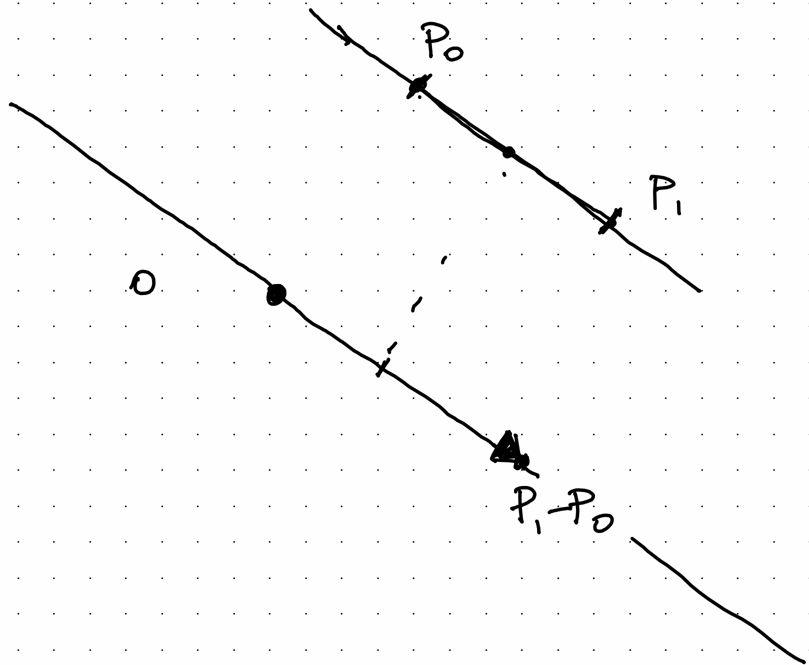


$$P = \frac{1}{3} P_0 + \frac{2}{3} P_1$$



$t_0 P_0 + t_1 P_1 \qquad 0 \leq t_0, t_1 \leq 1 \qquad \overline{P_0 P_1} = \{$

$$\overline{P_0 P_1} = \{ t_0 P_0 + t_1 P_1 \mid t_0 + t_1 = 1 ,\ 0 \leq t_0, t_1 \leq 1 \}.$$



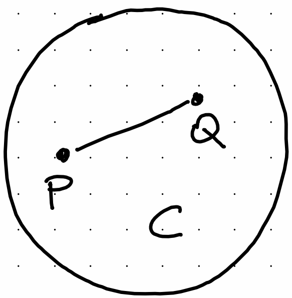
$$\overline{P_0 P_1} = \{ t_0 P_0 + t_1 P_1 \mid t_0 + t_1 = 1 ,\ t_0, t_1 \geq 0 \}.$$

<u>combinazione convessa</u>

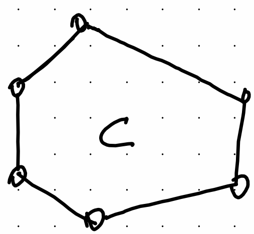
Un sottoinsieme $C \subset \mathbb{R}^2$ si dice <u>convesso</u> se $\forall\, P, Q \in C$ il segmento

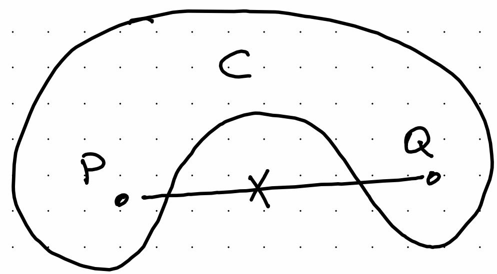
$$\overline{PQ} = \{ t_0 P + t_1 Q \mid t_0 + t_1 = 1,\ t_0, t_1 \geq 0 \}$$

è contenuto in C.

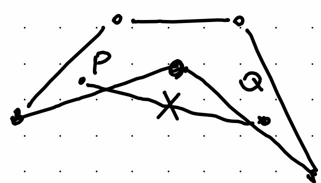


convesso





non convesso



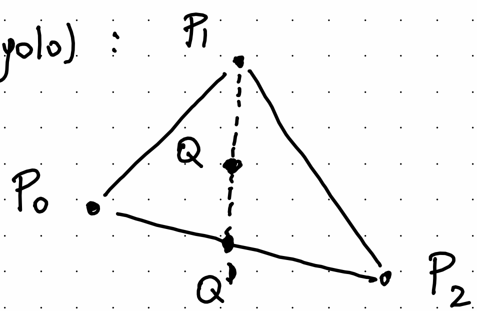
Def: Dati $P_0, P_1, \dots, P_n \in \mathbb{R}^2$, una loro **combinazione convessa** è un vettore di $\mathbb{R}^2$ della forma

$$t_0 P_0 + t_1 P_1 + \dots + t_n P_n$$

tali che

1) $t_0 + t_1 + \dots + t_n = 1$
2) $t_0, t_1, \dots, t_n \geq 0$.

Es (Triangolo):



$Q = s_1 P_1 + s' Q'$ con $s_1 + s' = 1$, $s_1, s' \geq 0$.

$Q' = t_0 P_0 + t_2 P_2$ con $t_0 + t_2 = 1$ e $t_0, t_2 \geq 0$.

$$\Rightarrow Q = s_1 P_1 + s'(t_0 P_0 + t_2 P_2) = s' t_0 P_0 + s_1 P_1 + s' t_2 P_2.$$

$$s' t_0 + s_1 + s' t_2 = s'(t_0 + t_2) + s_1 = s' + s_1 = 1$$

$$s' t_0,\ s_1,\ s' t_2 \geq 0$$

$$\mathrm{Conv}(P_0, \dots, P_n) = \{ t_0 P_0 + \dots + t_n P_n \mid t_0 + \dots + t_n = 1,\ t_0, \dots, t_n \geq 0 \}.$$

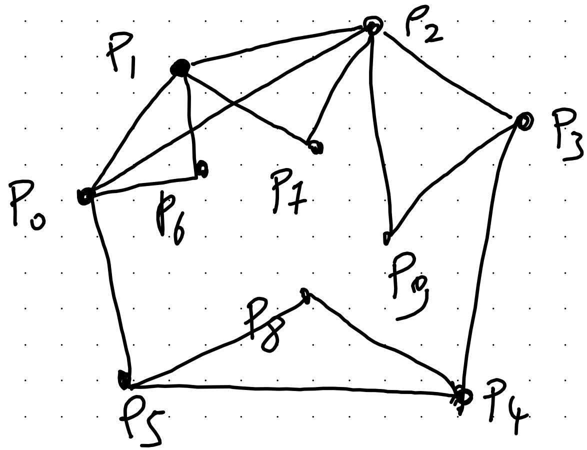
si chiama l'<u>inviluppo convesso</u> di $P_0, P_1, \dots, P_n$.

<u>Proprietà</u>:

1) $\mathrm{Conv}(P_0, \dots, P_n)$ è <u>convesso</u>.

2) $\mathrm{Conv}(P_0, \dots, P_n)$ è il piccolo convesso che contiene $P_0, \dots, P_n$.

<u>Es</u>:



$\mathrm{Conv}(P_0, \dots, P_9)$

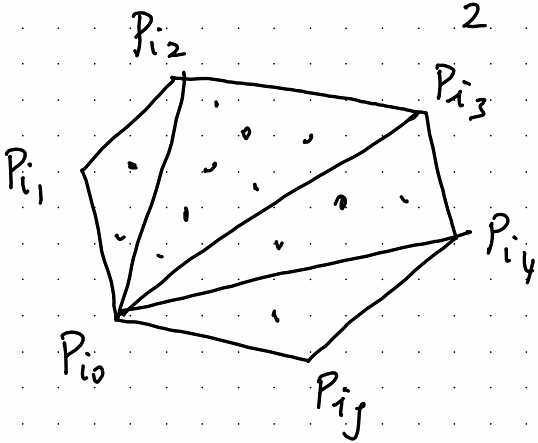
$= \mathrm{Conv}(P_0, \dots, P_5)$

$$\mathrm{Conv}\left(0, \tfrac{1}{2}P, P\right) = \mathrm{Conv}(0, P). \quad P \neq 0.$$

$\mathrm{Conv}(P_0, \dots, P_n) =$ Poligono di vertici $P_{i_0}, \dots, P_{i_k}$.

$$\mathrm{Area}\left(\mathrm{Conv}(P_0, \dots, P_m)\right) = \sum \mathrm{Aree}\left(\overset{\triangle}{P_{i_0} P_{i_j} P_{i_{j+1}}}\right)$$

$$= \frac{1}{2} \sum_j \left| \det\left(P_{i_j} - P_{i_0} \mid P_{i_{j+1}} - P_{i_0} \right) \right|$$



Def: Una **combinazione affine** di $P_0, \dots, P_n \in \mathbb{R}^2$ è un vettore della forma

$$t_0 P_0 + t_1 P_1 + \dots + t_n P_n \quad \in \mathbb{R}^2$$

con $t_0 + t_1 + \dots + t_n = 1$, $t_0, \dots, t_n \in \mathbb{R}$.

$$\mathrm{Aff}(P_0, \dots, P_n) = \{ t_0 P_0 + \dots + t_n P_n \mid t_0 + \dots + t_n = 1,\ t_0, \dots, t_n \in \mathbb{R} \}$$

Prop.: $\mathrm{Aff}(P_0, \dots, P_n) = P_0 + \langle P_1 - P_0, P_2 - P_0, \dots, P_n - P_0 \rangle$ è un sottospazio affine.

dim:

$$t_0 P_0 + \dots + t_n P_n = (t_0 + 1 - 1) P_0 + t_1 P_1 + \dots + t_n P_n$$

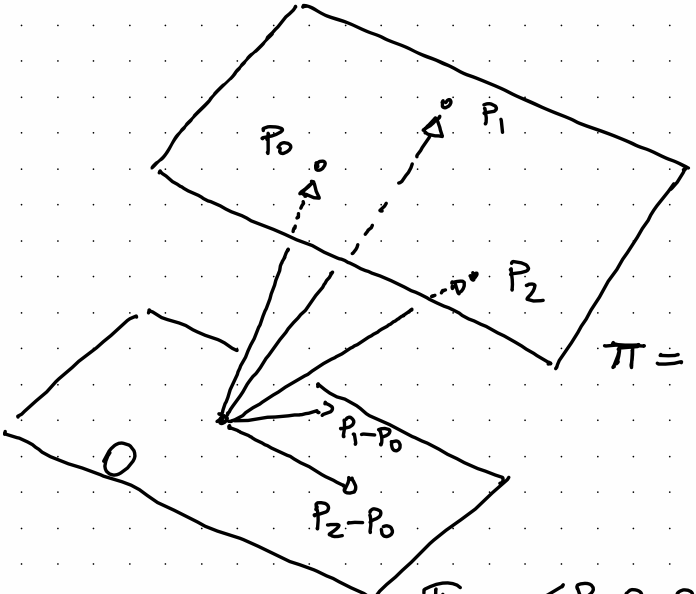
$$= P_0 + (t_0 - 1) P_0 + t_1 P_1 + \dots + t_n P_n =$$

$$= P_0 + (-t_1 - t_2 - \dots - t_n) P_0 + t_1 P_1 + \dots + t_n P_n$$

$$= P_0 + t_1 (P_1 - P_0) + t_2 (P_2 - P_0) + \dots + t_n (P_n - P_0). \qquad \blacksquare$$

Piano per 3 punti di $\mathbb{R}^3$

Siano $P_0, P_1, P_2 \in \mathbb{R}^3$ distinti.



$$\pi = \mathrm{Aff}(P_0, P_1, P_2) = P_0 + \langle P_1 - P_0, P_2 - P_0 \rangle$$

$$\pi_0 = \langle P_1 - P_0, P_2 - P_0 \rangle$$

Es: Stabilire se i 3 punti

$$P_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad P_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

sono allineati e nel caso non lo siano trovare eq. par. e cart. del piano che li contiene.

Sol.:

$$rg\left(P_1 - P_0 \mid P_2 - P_0\right) = rg\begin{pmatrix} 0 & 0 \\ 1 & -2 \\ 0 & 1 \end{pmatrix} = 2$$

$\Rightarrow$ P_0, P_1, P_2 non sono allineati.

$$\pi = P_0 + \langle P_1 - P_0, P_2 - P_0 \rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\rangle$$

$$\mathrm{Ker}\begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \Rightarrow \pi_0 : x_1 = 0$$

$$\Rightarrow \pi : x_1 = 1$$

Sistemi con parametro

Studiare (= "stabilire se è risolubile e nel caso lo sia trovare tutte le soluzioni") il seguente sistema lineare in 6 variabili $x_1, \dots, x_6 \in \mathbb{R}$ dipendente da un parametro $K \in \mathbb{R}$:

$$\begin{cases} x_1 + (K-1)x_2 - (K-1)x_4 - (K-1)x_6 = K-3 \\ 3x_1 + 3(K-1)x_2 + x_3 - (K-1)x_4 - (K+1)x_6 = K^2+K-7 \\ x_1 + (K-1)x_2 + x_3 + (K-1)x_4 + (K-2)(K-1)x_5 + (4K-6)x_6 = K^2-3 \end{cases}$$

Sol.:

$$\left(\begin{array}{cccccc|c} 1 & K-1 & 0 & -(K-1) & 0 & -(K-1) & K-3 \\ 3 & 3(K-1) & 1 & -(K-1) & 0 & -(K+1) & K^2+K-7 \\ 1 & K-1 & 1 & K-1 & (K-2)(K-1) & 4K-6 & K^2-3 \end{array}\right)$$

$$\left(\begin{array}{cccccc|c} 1 & K-1 & 0 & -(K-1) & 0 & -(K-1) & K-3 \\ 3 & 3(K-1) & 1 & -(K-1) & 0 & -(K+1) & K^2+K-7 \\ 1 & K-1 & 1 & K-1 & (K-2)(K-1) & 4K-6 & K^2-3 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc|c} 1 & K-1 & 0 & -(K-1) & 0 & -(K-1) & K-3 \\ 0 & 0 & 1 & 2(K-1) & 0 & 2K-4 & K^2-2K+2 \\ 0 & 0 & 1 & 2(K-1) & (K-2)(K-1) & 5K-7 & K^2-K \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc|c} 1 & K-1 & 0 & -(K-1) & 0 & -(K-1) & K-3 \\ 0 & 0 & 1 & 2(K-1) & 0 & 2K-4 & K^2-2K+2 \\ 0 & 0 & 0 & 0 & (K-2)(K-1) & 3K-3 & K-2 \end{array}\right)$$

$$\left(\begin{array}{cccccc|c} 1 & K-1 & 0 & -(K-1) & 0 & -(K-1) & K-3 \\ 0 & 0 & 1 & 2(K-1) & 0 & 2K-4 & K^2-2K+2 \\ 0 & 0 & 0 & 0 & (K-2)(K-1) & 3K-3 & K-2 \end{array}\right)$$

Se $K=1$ l'ultima riga è

$$\left(\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

quindi il sistema non è risolubile.

Se $K=2$ otteniamo

$$\left(\begin{array}{cccccc|c} 1 & 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc|c} 1 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$ a scala ridotta.

$$
\begin{array}{c}
\begin{array}{cccccc} \bullet & \checkmark & \bullet & \checkmark & \checkmark & \bullet \end{array} \\
\left(\begin{array}{cccccc|c}
1 & 1 & 0 & -1 & 0 & 0 & -1 \\
0 & 0 & 1 & 2 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right)
\end{array}
\qquad
\begin{cases}
x_1 + x_2 - x_4 = 0 \\
x_3 + 2x_4 = 0 \\
x_6 = 0
\end{cases}
$$

Se $K=2$ le soluzioni sono quindi

$$
\begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \left\langle \underset{\text{"}x_2\text{"}}{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}, \underset{\text{"}x_4\text{"}}{\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{\text{"}x_5\text{"}}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}} \right\rangle
$$

$$\left(\begin{array}{cccccc|c} 1 & K-1 & 0 & -(K-1) & 0 & -(K-1) & K-3 \\ 0 & 0 & 1 & 2(K-1) & 0 & 2K-4 & K^2-2K+2 \\ 0 & 0 & 0 & 0 & (K-2)(K-1) & 3K-3 & K-2 \end{array}\right)$$

Se $(K-2)(K-1) \neq 0$

$$\sim \left(\begin{array}{cccccc|c} 1 & K-1 & 0 & -(K-1) & 0 & -(K-1) & K-3 \\ 0 & 0 & 1 & 2(K-1) & 0 & 2K-4 & K^2-2K+2 \\ 0 & 0 & 0 & 0 & 1 & \frac{3}{K-2} & \frac{1}{K-1} \end{array}\right)$$

Le soluzioni sono

$$\begin{pmatrix} K-3 \\ 0 \\ K^2-2K+2 \\ 0 \\ 1/K-1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -(K-1) \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} K-1 \\ 0 \\ -2(K-1) \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} K-1 \\ 0 \\ -(2K-4) \\ 0 \\ -3/K-2 \\ 1 \end{pmatrix} \right\rangle .$$

Es: Studiare la posizione reciproca delle seguenti due rette di $\mathbb{R}^3$:

$$r: \begin{cases} 2x+3y-z=5 \\ x+2y+z=6 \end{cases} \qquad s = \begin{pmatrix}1\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$$

Sol.:

$$r: \; AX=b \qquad s = X_0 + \langle v \rangle$$

$$r \cap s \neq \emptyset \iff A(X_0 + t v) = b \iff t\, Av = b - AX_0$$

	$rg(Av)$	$rg(Av \mid b-AX_0)$
$r \equiv s$	0	0
$r /\!/ s,\ r\cap s=\emptyset$	0	1
$r \cap s = \{P_0\}$	1	1
sghembe	1	2

$\rightsquigarrow P_0 = X_0 + t_0 v$, $\; t_0 AX = b - AX_0$

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix} \quad b = \begin{pmatrix}5\\6\end{pmatrix}$$

$$X_0 = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad v = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$Av = \begin{pmatrix}1\\2\end{pmatrix}$$

$$b - AX_0 = \begin{pmatrix}5\\6\end{pmatrix} - \begin{pmatrix}4\\4\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

$\Rightarrow r \cap s = \{P_0\}$ con

$$P_0 \overset{t=1}{=} X_0 + v = \begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\2\end{pmatrix}$$

__Es__: Stabilire la posizione reciproca di

$$r: \begin{cases} x+2y-3z=2 \\ 2x+3y+z=1 \end{cases} \qquad \pi = \begin{pmatrix}1\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} \right\rangle.$$

__Sol.__:

$r: \ AX=b \qquad \pi = X_0 + \langle v_1, v_2 \rangle.$

$$A(X_0+t_1v_1+t_2v_2) = b \iff (Av_1 | Av_2)\begin{pmatrix}t_1\\t_2\end{pmatrix} = b - AX_0$$

	$\mathrm{rg}(Av_1 \mid Av_2)$	$\mathrm{rg}(Av_1 \mid Av_2 \mid b-AX_0)$
	$\cancel{\emptyset}$	
$r \subset \pi$	1	1
$r \parallel \pi$, $r\cap\pi=\emptyset$	1	2
$r\cap\pi=\{P_0\}$.	2	2

$P_0 = X_0 + t_1v_1 + t_2v_2$ dove

$$\begin{pmatrix}t_1\\t_2\end{pmatrix} = (Av_1|Av_2)^{-1}(b-AX_0)$$

$$A = \begin{pmatrix}1 & 2 & -3\\ 2 & 3 & 1\end{pmatrix}$$

$$v_1 = \begin{pmatrix}1\\-1\\1\end{pmatrix}, \quad v_2 = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\mathrm{rg}(Av_1|Av_2) = \mathrm{rg}\begin{pmatrix}-4 & -4\\ 0 & 11\end{pmatrix} = 2$$

$b - AX_0 =$

$$\begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}1 & 2 & -3\\ 2 & 3 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}0\\6\end{pmatrix} = \begin{pmatrix}2\\-5\end{pmatrix}$$

$$\begin{pmatrix}t_1\\t_2\end{pmatrix} = \begin{pmatrix}-4 & -4\\ 0 & 11\end{pmatrix}^{-1}\begin{pmatrix}2\\-5\end{pmatrix}$$

$$= -\frac{1}{44}\begin{pmatrix}11 & 4\\ 0 & -4\end{pmatrix}\begin{pmatrix}2\\-5\end{pmatrix}$$

$$= \begin{pmatrix}-1/22\\ -5/11\end{pmatrix}$$

$$\Rightarrow \; P_0 = X_0 - \frac{1}{22}\begin{pmatrix}1\\-1\\1\end{pmatrix} - \frac{5}{11}\begin{pmatrix}1\\2\\3\end{pmatrix} =$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{22}\begin{pmatrix}1\\-1\\1\end{pmatrix} - \frac{5}{11}\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{1}{22}\begin{pmatrix}11\\3\\-9\end{pmatrix}$$