Retta per due punti distinti di $\mathbb{R}^2$

Dati $P_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ e $P_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in \mathbb{R}^2$, $P_0 \neq P_1$,
l'unica retta passante per P_0 e P_1 è

$$\boxed{-(y_1 - y_0)(x - x_0) + (x_1 - x_0)(y - y_0) = 0}$$

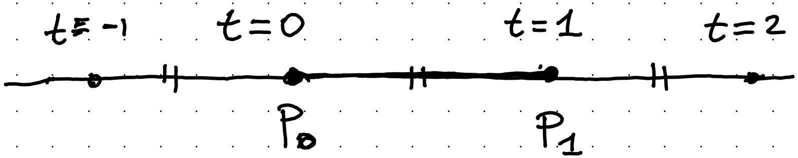
Se $y_1 \neq y_0$ e $x_1 \neq x_0$ allora possiamo scrivere questa equazione come

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}$$

Eq. parametrica di tale retta:

$r = P_0 + \langle v \rangle \ni P_1 \quad \exists\, t \in \mathbb{R}$ t.c. $P_1 = P_0 + tv$
$\Rightarrow \quad P_1 - P_0 = tv$. Poiché $t \neq 0$, $P_1 - P_0$
può essere preso come vettore direttore. Quindi

$$\boxed{r = P_0 + \langle P_1 - P_0 \rangle}$$



$$r = P_0 + \langle \underbrace{P_1 - P_0}_{\text{vett. direttore.}} \rangle \ni P_t$$

$$P_t = P_0 + t(P_1 - P_0) = (1-t)P_0 + t\,P_1$$

(combinazione affine di P_0 e P_1)

$$= s\,P_0 + t\,P_1 \qquad s + t = 1$$

$$\left.\begin{array}{l} s, t \geq 0 \\ s + t = 1 \end{array}\right\} \text{combinazione convessa.}$$

Es: Calcolare eq. par. e cartesiane della retta passante per $P_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ e $P_1 = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

Sol.: $P_0 \neq P_1$. Inoltre $x_0 \neq y_0$ e $x_1 \neq y_1$.

$$r: \quad \frac{x-2}{-4-2} = \frac{y-3}{5-3}$$

ovvero

$$-\frac{1}{6}x + \frac{1}{3} = \frac{1}{2}y - \frac{3}{2}$$

ovvero

$$-\frac{1}{6}x + \frac{1}{2}y = \frac{11}{6} \quad \text{ovvero} \quad x + 3y = 11 :$$

Le eq. par. sono

$$r = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \langle \underset{\substack{\uparrow \\ P_1 - P_0}}{\begin{pmatrix} -6 \\ 2 \end{pmatrix}} \rangle = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \langle \underset{\substack{\uparrow \\ \neq P_1 - P_0}}{\begin{pmatrix} -3 \\ 1 \end{pmatrix}} \rangle .$$

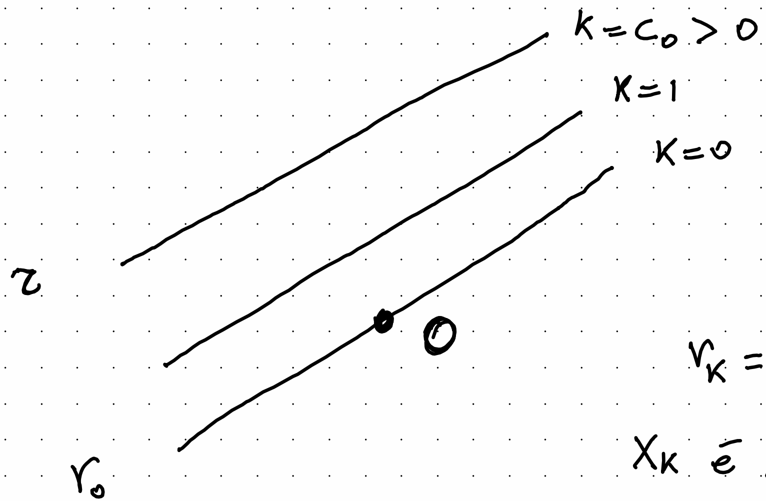
Fascio improprio di rette

Sia $r: a_0 x + b_0 y = c_0$. Come sono fatte le rette parallele a r?

Sono tutte e solo le rette di equazione

$$r_K: \; a_0 x + b_0 y = K$$

al variare di $K \in \mathbb{R}$



Eq. param. di r_K

$$r_K = X_K + \left\langle \begin{pmatrix} -b_0 \\ a_0 \end{pmatrix} \right\rangle$$

X_K è soluzione di
$a_0 x + b_0 y = K$.

Es: Trovare eq. par. e cartesiane della retta passante per $P_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ e parallela alla retta

$$r: \ 2x+3y=-2$$

Sol.: La retta cercata è del tipo

$$r_K: \ 2x+3y = K.$$

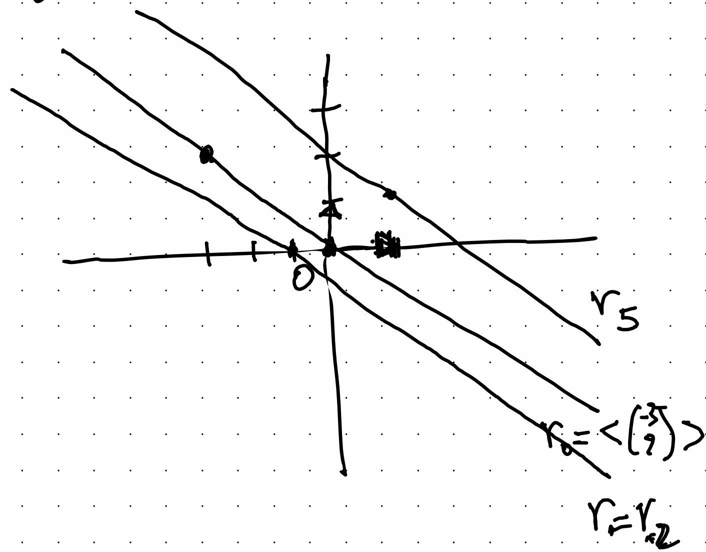
Imponiamo il passaggio per P_0:

$$2+3=K$$

Quindi la retta cercata è

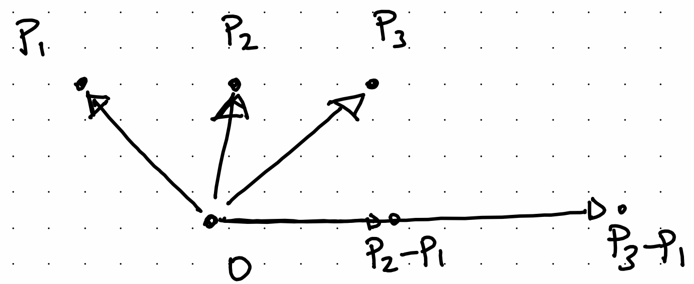
$$r_5: \ 2x+3y=5.$$

$$r_5 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\rangle$$



<u>3 punti allineati:</u>

Siano $P_1, P_2, P_3 \in \mathbb{R}^2$. Condizioni di allineamento:
i.e. $\exists$ retta r t.c. $P_1, P_2, P_3 \in r$.

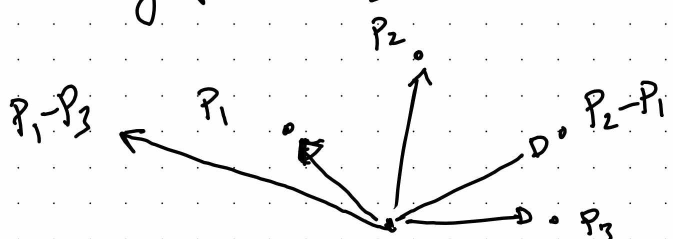


P_1, P_2, P_3 sono allineati $\iff P_3 \in P_1 + \langle P_2 - P_1 \rangle$

$$\iff P_3 - P_1 \in \langle P_2 - P_1 \rangle$$

$\iff \operatorname{rg}(P_2 - P_1 \mid P_3 - P_1) \leq 1 \iff \det(P_2 - P_1 \mid P_3 - P_1) = 0$

$\iff \text{Area}(P_1 \widehat{P_2} P_3) = 0$



3 rette concorrenti

3 rette r_1, r_2, r_3 si dicono concorrenti se $\exists! P_0 \in \mathbb{R}^2$ t.c.
$P_0 \in r_1 \cap r_2 \cap r_3$.

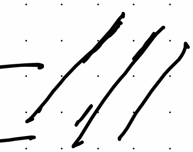
$r_1 : (A_1 | b_1)$ $\quad r_2 : (A_2 | b_2)$ $\quad r_3 : (A_3 | b_3)$

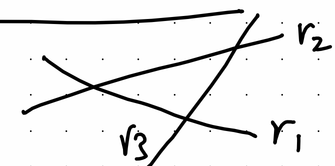
$r_1 : a_1 x + b_1 y = c_1$ $\quad r_2 : a_2 x + b_2 y = c_2$ $\quad r_3 : a_3 x + b_3 y = c_3$

$$r_1 \cap r_2 \cap r_3 : \begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \\ a_3 x + b_3 y = c_3 \end{cases}$$

$$\begin{aligned} 2x + 3y &= 1 \\ 4x + 6y &= 2 \\ 6x + 9y &= 3 \end{aligned}$$

	$rg\begin{pmatrix} A_1 \\ \hline A_2 \\ \hline A_3 \end{pmatrix}$	$rg\left(\begin{array}{c\|c} A_1 & c_1 \\ \hline A_2 & c_2 \\ \hline A_3 & c_3 \end{array}\right)$
$r_1 \equiv r_2 \equiv r_3$	1	1
parallele non concorrenti	1	2
concorrenti	2	2
pos. generica	2	3





Domanda:

$$X_0 + \left\langle \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle . \quad \rightsquigarrow \quad \ker(a, b) = \left\langle \begin{pmatrix} -b \\ a \end{pmatrix} \right\rangle$$

$$-bx + ay = c \quad \text{dove } c = (-b, a) X_0 .$$

Geometria affine di $\mathbb{R}^3$

I sottospazi affini di $\mathbb{R}^3$ sono punti, rette e piani:

Rette:

$r = X_0 + \langle v \rangle$, $v \in \mathbb{R}^3 \setminus \{0_{\mathbb{R}^3}\}$, $X_0 \in \mathbb{R}^3$. Eq. param.

$$r: \begin{cases} ax+by+cz=d \\ a'x+b'y+c'z=d' \end{cases} \quad \text{con} \quad \operatorname{rg}\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix} = 2$$

NB: $\begin{cases} x+y+z=1 \\ 2x+2y+2z=2 \end{cases}$ è equivalente a $x+y+z=1$

le cui soluzioni formano un piano.

Piani:

$$\pi = X_0 + \langle v_1, v_2 \rangle \qquad \text{con} \quad rg\,(v_1 | v_2) = 2$$

le eq. cartesiane sono

$$\pi : \ ax + by + cz = d \qquad \text{con} \quad (a, b, c) \neq (0,0,0)$$

$$\Leftrightarrow \ rg\,(a, b, c) = 1$$

Es: $\pi: \ x + y + z = 1$ è un piano di

eq. parametriche

$$\pi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

Condizioni di parallelismo di due rette

•) $r = X_0 + \langle v_1 \rangle$, $S = Y_0 + \langle v_2 \rangle$

sono parallele se e solo se $rg(v_1 | v_2) = 1$

•) $r : \begin{cases} ax + by + cz = d \\ a'x + b'y + c'z = d' \end{cases}$ $S = Y_0 + \langle v_2 \rangle$

sono parallele se e solo se $\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

•) $r : \begin{cases} ax + by + cz = d \\ a'x + b'y + c'z = d' \end{cases}$ $S : \begin{cases} \alpha x + \beta y + \gamma z = \delta \\ \alpha' x + \beta' y + \gamma' z = \delta' \end{cases}$

sono parallele se e solo se

$Ker \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix} = Ker \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix}$

se solo se $rref \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix} = rref \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix}$

Es:

$$r:\begin{cases} x+y=3 \\ 2x+z=3 \end{cases} \qquad s:\begin{cases} x-y+z=\cancel{-}1 \\ 2x+y+\frac{1}{2}z=\frac{11}{2} \end{cases}$$

$$A_r=\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{pmatrix} = rref(A_r)$$

$$A_s=\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1/2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -3/2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{pmatrix} = rref(A_s)$$

$rref(A_r)=rref(A_s) \Rightarrow \operatorname{Ker} A_r=\operatorname{Ker} A_s \Rightarrow$ sono parallele.

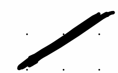
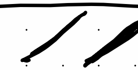
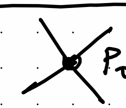
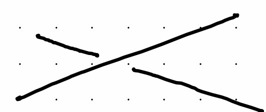
Condizioni di incidenza di due rette

i) $r = X_0 + \langle v_1 \rangle \qquad s = Y_0 + \langle v_2 \rangle$

Sono incidenti se e solo se $X_0 - Y_0 \in \langle v_1, v_2 \rangle$

$\iff \exists$ soluzione di $(v_1 | v_2) X = X_0 - Y_0$

$\iff \operatorname{rg}(v_1 | v_2) = \operatorname{rg}(v_1 | v_2 | X_0 - Y_0)$.

	$\operatorname{rg}(v_1 \| v_2)$	$\operatorname{rg}(v_1 \| v_2 \| X_0 - Y_0)$	
$r \equiv s$ coincidenti	1	1	
$r // s$, $r \not\equiv s$ parallele non coinc.	1	2	
$r \cap s = \{P_0\}$ incidenti	2	2	
SGHEMBE	2	3	

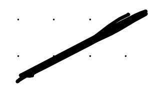
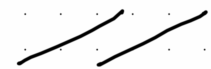
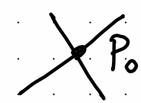
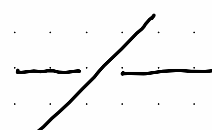
Due rette sono sghembe se non giacciono sullo stesso piano.

Posizione reciproca di 2 rette

$r: \ A_r X = b_r \qquad\qquad s = X_0 + \langle v \rangle$

$r \cap s \neq \emptyset \iff \exists\, t \in \mathbb{R}$ t.c. $A_r(X_0 + tv) = b_r$

$\iff \exists\, t : \ t(A_r v) = b_r - A_r X_0$

	rg $A_r v$	rg $(A_r v \mid b_r - A_r X_0)$	
$r \equiv s$	0	0	
$r /\!/ s, r \not\equiv s$	0	1	
$r \cap s = \{P_0\}$	1	1	
SGHEMBE	1	2	

Esercizio :

Fare il caso cartesiano / cartesiano.

$r: A_r v = b_r \qquad s: A_s v = b_s.$

Es: Stabilire la posizione reciproca delle due rette

$$r: \begin{cases} 2x+3y-z=5 \\ x+2y+z=6 \end{cases} \qquad s = \begin{pmatrix}1\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle.$$

Sol.:

$$A_r = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix} \quad b_r = \begin{pmatrix}5\\6\end{pmatrix}, \quad X_0 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \; v = \begin{pmatrix}1\\0\\1\end{pmatrix}.$$

$$A_r v = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \Rightarrow \text{Non sono parallele.}$$

$$b - A_r X_0 = \begin{pmatrix}5\\6\end{pmatrix} - \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}5\\6\end{pmatrix} - \begin{pmatrix}4\\4\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

$$\Rightarrow \quad rg(A_r v) = 1 = rg(A_r v \mid b - A_r X_0)$$

$\Rightarrow$ r ed s sono incidenti in un unico punto P_0.

Dobbiamo trovare P_0.

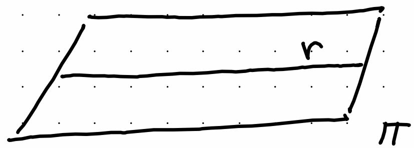
$$t(A_r v) = b - A_r X_0 \iff t\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \iff t = 1$$

$$\Rightarrow \quad P_0 = X_0 + tv = X_0 + v = \begin{pmatrix}2\\1\\2\end{pmatrix}.$$

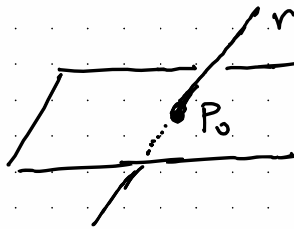
Posizione retta / piano

r: retta , π: piano.

$r \subset \pi$

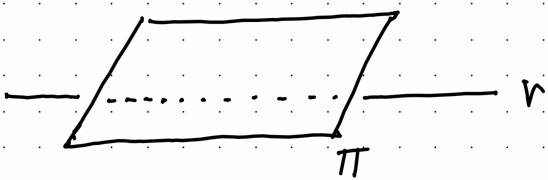


$r \cap \pi = \{P_0\}$



$r \parallel \pi$:

$r \cap \pi = \emptyset$.



·) $r = X_0 + \langle v_1 \rangle$ $\qquad \pi = Y_0 + \langle v_2, v_3 \rangle$.

	$rg(v_1 \mid v_2 \mid v_3)$	$rg(v_1 \mid v_2 \mid v_3 \mid X_0 - Y_0)$
$r \subset \pi$	2	2
$r \parallel \pi$, $r \cap \pi = \emptyset$	2	3
$r \cap \pi = \{P_0\}$.	3	3

Es:

$$r = \begin{pmatrix}1\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}1\\2\\3\end{pmatrix} \right\rangle \qquad \pi = \begin{pmatrix}1\\-1\\1\end{pmatrix} + \left\langle \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix} \right\rangle$$

$$\det \begin{pmatrix}1 & 1 & 2\\2 & 2 & 1\\3 & 1 & 3\end{pmatrix} = \det \begin{pmatrix}1 & 1 & 2\\0 & 0 & -3\\3 & 1 & 3\end{pmatrix} = 3 \det \begin{pmatrix}1 & 1\\3 & 1\end{pmatrix} = -6 \neq 0.$$

$\Rightarrow$ $r \cap \pi = \{P_0\}$. Dobbiamo trovare P_0:

$$P_0 = \begin{pmatrix}1\\1\\1\end{pmatrix} + t_0 \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix} + t_1 \begin{pmatrix}1\\2\\1\end{pmatrix} + t_2 \begin{pmatrix}2\\1\\3\end{pmatrix}$$

$$t_1 \begin{pmatrix}1\\2\\1\end{pmatrix} + t_2 \begin{pmatrix}2\\1\\3\end{pmatrix} - t_0 \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1 & 2 & -1 & 0\\2 & 1 & -2 & 2\\1 & 3 & -3 & 0\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1 & 2 & -1 & 0\\0 & -3 & 0 & 2\\0 & 1 & -2 & 0\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 2 & -1 & 0\\0 & 1 & -2 & 0\\0 & -3 & 0 & 2\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c}1 & 2 & -1 & 0\\0 & 1 & -2 & 0\\0 & 0 & -6 & 2\end{array}\right) \Rightarrow t_0 = -\frac{2}{6} = -\frac{1}{3}$$

$$\Rightarrow P_0 = \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2/3\\1/3\\0\end{pmatrix}$$

Esercizio: Fare tutti gli altri casi.