Richiami: Il determinante è una funzione

$$\det = \det^{(n)} : \mathrm{Mat}_{n\times n}(\mathbb{K}) \longrightarrow \mathbb{K}$$

tale che

$$\left.\begin{aligned} &\det(P_{ij}A) = -\det(A) \\ &\det(D_i(\lambda)A) = \lambda\det(A) \\ &\det(F_{ij}(c)A) = \det(A) \end{aligned}\right] \text{ multilineare + alternante sulle righe}$$

$$\det(\mathbb{1}_n) = 1$$

Prop.: $\det(A) \neq 0 \iff \mathrm{rref}(A) = \mathbb{1}_n \iff A$ è invertibile.

$$\det\begin{pmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} = a_{11}a_{22}\cdots a_{nn}$$

$$\det^{(n)}(A) = \sum_{j=1}^{n} a_{ij} \underbrace{\det^{(n-1)}(A_{i,j})(-1)^{i+j}}_{C_{i,j}}$$ : Sviluppo di Laplace della riga $i$.

operazioni elementari sulle colonne:

$$A \overset{C^i \leftrightarrow C^j}{\rightsquigarrow} A P_{ij}$$

$$A \overset{C^i \mapsto \lambda C^i}{\rightsquigarrow} A D_i(\lambda)$$

$$A \overset{C^i \mapsto C^i + \mu C^j}{\rightsquigarrow} A F_{ji}(\mu)$$

$\Leftarrow (AB)^t = B^t A^t$

3) $\det(A) = \sum_{i=1}^{n} a_{ij} \det(A_{i,j}) (-1)^{i+j}$ : Sviluppo di Laplace della colonna $j$.

Teorema: 1) $rg(A^t) = rg(A)$

2) $\det(A^t) = \det(A)$.

Per calcolare il determinante:

1) operare sulle colonne o sulle righe per creare una riga o una colonna con tanti zeri.

2) Sviluppare quella colonna o riga.

Es:

$$A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & -1 & 3/2 \\ 5 & 7 & -3/2 \end{pmatrix}$$

$$\det A = \det \begin{pmatrix} 0 & 4 & 1 \\ 0 & -1 & 3/2 \\ 8 & 7 & -3/2 \end{pmatrix} = 8 \det \begin{pmatrix} 4 & 1 \\ -1 & 3/2 \end{pmatrix} =$$

$$= 8 \det \begin{pmatrix} 0 & 7 \\ -1 & 3/2 \end{pmatrix} = -8 \det \begin{pmatrix} -1 & 3/2 \\ 0 & 7 \end{pmatrix}$$

$$= 56$$

$\mathrm{Ker}\, A = ?$ $\qquad A \underset{\text{sulle righe}}{\rightsquigarrow} \mathrm{rref}(A) = R$

$\Rightarrow \quad \mathrm{Ker}\, A = \mathrm{Ker}\, R.$

$$\begin{tikzcd}
\cdot \arrow[r, "A"] \arrow[d, "C"', "\simeq"] & \cdot \arrow[d, "\simeq"', "B"] \\
\cdot \arrow[r, "R"] & \cdot
\end{tikzcd}$$

$$\begin{tikzcd}
\cdot \arrow[r, "A"] \arrow[d, equal] & \cdot \arrow[d, "C"', "\simeq"] \\
\cdot \arrow[r, "CA"] & \cdot
\end{tikzcd}$$

operazioni sulle righe

$$\begin{tikzcd}
\cdot \arrow[r, "A"] \arrow[d, "\simeq"', "C"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "AC^{-1}"] & \cdot
\end{tikzcd}$$

operazioni sulle colonne

# Utilizzo del determinante per il calcolo dell'inversa

Per $n = 2$: $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ è invertibile $\Leftrightarrow$ $\det(A) = ad - bc \neq 0$.

In questo caso

$$A^{-1} = \frac{1}{\det(A)} \underbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{\text{"Matrice Aggiunta di } A\text{"}}$$

**Def**: Dato $n \geq 1$ ed una matrice quadrata $A \in Mat_{n \times n}(\mathbb{K})$ e dati un indice di riga $i$ ed un indice di colonna $j$, definiamo il <u>cofattore $(i,j)$ di $A$</u> come il numero

$$C_{i,j} = C_{i,j}(A) = \det(A_{i,j})(-1)^{i+j}$$

**OSS**: 
$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{i,j} = \sum_{i=1}^{n} a_{ij} C_{ij} .$$

La matrice <u>aggiunta</u> (o dei cofattori) di $A$ è la matrice che ha come componenti i cofattori e la denotiamo con $Agg(A)$:

$$Agg(A) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix} \in Mat_{n \times n}(\mathbb{K})$$

Es: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$C_{11}(A) = +d$ $\qquad$ $C_{12}(A) = -c$

$C_{21}(A) = -b$ $\qquad$ $C_{22}(A) = +a$

$$Agg(A) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det A} Agg(A)^t$$

**Teorema**: Sia $A \in Mat_{n \times n}(\mathbb{K})$. Allora

$$\underset{n\times \cancel{n}}{A}\ \underset{\cancel{n}\times n}{Agg(A)^t} = \underset{n \times n}{\det(A)\, \mathbb{1}_n} = \begin{pmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \det(A) \end{pmatrix}$$

**Es**: $(n=2)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \det(A) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

✓

**dim**: Dobbiamo dimostrare

$$\left[A\, Agg(A)^t\right]_i^j = \begin{cases} \det(A) & \text{se } i = j \\ 0 & \text{se } i \neq j \end{cases}$$

$i = j$:

$$[A \operatorname{Agg}(A)^t]_i^i = A_i \, [\operatorname{Agg}(A)^t]^i$$

$$= A_i \begin{pmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{in} \end{pmatrix} = a_{i1} c_{i1} + a_{i2} c_{i2} + \dots + a_{in} c_{in}$$

$$= \sum_{j=1}^{n} a_{ij} c_{ij} = \det(A)$$

$i \neq j$ :

$$\left[ A \, Agg(A)^t \right]_i^j = A_i \left( Agg(A)^t \right)^j$$

$$= A_i \left( Agg(A)_j \right)^t$$

$$= \sum_{k=1}^{n} a_{ik} \, c_{jk}$$

Sia $B$ la matrice che ha le stesse righe di $A$ a parte la $j$-esima che poniamo uguale ad $A_i$

$$B = \begin{pmatrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_i \\ \vdots \\ A_n \end{pmatrix} \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} \begin{pmatrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_j \\ \vdots \\ A_n \end{pmatrix} = A$$

Sviluppiamo la riga $j$

$$0 = \det(B) \overset{\checkmark}{=} \sum_{k=1}^{n} b_{jk}\, C_{jk}(B)$$

$$= \sum_{k=1}^{n} a_{ik}\, C_{jk}(B)$$

$$= \sum_{k=1}^{n} a_{ik}\, C_{jk}(A)$$

**COR**: Se $A$ è invertibile, $\boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{Agg}(A)^t}$ Formula di Cramer per l'inversa.

**Es**: Calcolare l'inversa di

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \\ 1 & -1 & 3/2 \end{pmatrix}$$

con la formula di Cramer.

**Sol.**: $\det A = \det \begin{pmatrix} 0 & 0 & -1/2 \\ 2 & 2 & 1 \\ 1 & -1 & 3/2 \end{pmatrix} = 2 \neq 0$

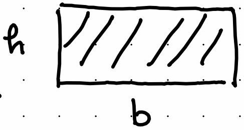
$C_{11} = + \det \begin{pmatrix} 2 & 1 \\ -1 & 3/2 \end{pmatrix} = 4 \quad C_{21} = - \det \begin{pmatrix} -1 & 1 \\ -1 & 3/2 \end{pmatrix} = 1/2 \quad C_{31} = + \det \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} = -3$

$C_{12} = - \det \begin{pmatrix} 2 & 1 \\ 1 & 3/2 \end{pmatrix} = -2 \quad C_{22} = + \det \begin{pmatrix} 1 & 1 \\ 1 & 3/2 \end{pmatrix} = 1/2 \quad C_{32} = - \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = 1$

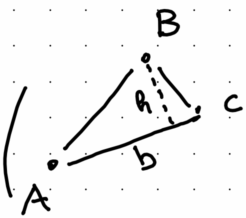
$C_{13} = + \det \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} = -4 \quad C_{23} = - \det \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = 0 \quad C_{33} = + \det \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} = 4$

$$\Rightarrow \quad A^{-1} = \frac{1}{2} \begin{pmatrix} 4 & 1/2 & -3 \\ -2 & 1/2 & 1 \\ -4 & 0 & 4 \end{pmatrix}$$

<u>Applicazioni del determinante</u>: Il determinante è un'area (se $n=2$) "orientata".

<u>Aree</u>: Area( 

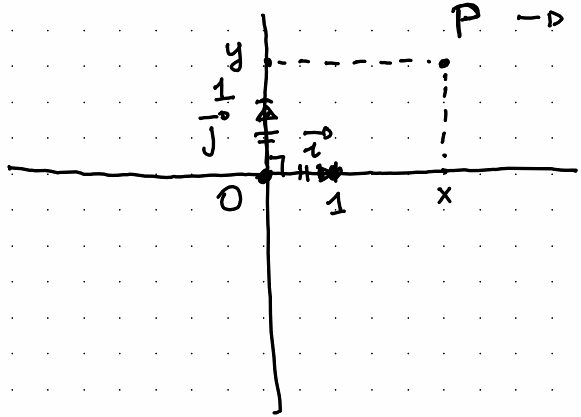
) $= bh$

triangolo: Area( 

) $= \frac{1}{2}\, b \cdot h$

$$\text{Area}\left(\widehat{ABC}\right) = \frac{bh}{2}.$$

Identifichiamo $\mathbb{R}^2$ con il piano cartesiano:



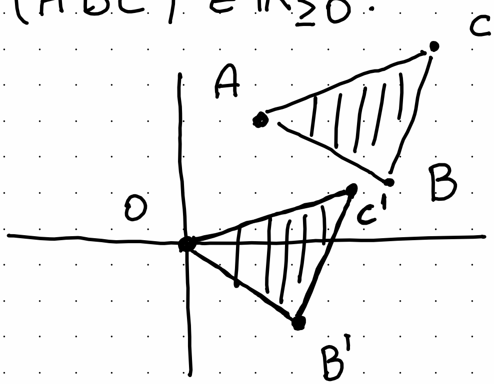

$P \to \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

<u>oss</u>:

$F_e : V_0^2 \xrightarrow{\simeq} \mathbb{R}^2$

$e = \{ \vec{i}, \vec{j} \}$

$\text{Area}(\overset{\triangle}{ABC}) \in \mathbb{R}_{\geq 0}$.




$\text{Area}(ABC) = \text{Area}(OB'C')$

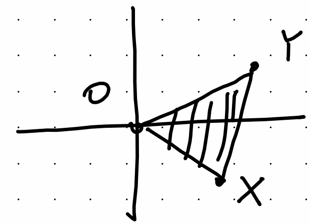
dove

$B' = B - A$, $C' = C - A$

$$\text{Area} : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}_{\geq 0}$$

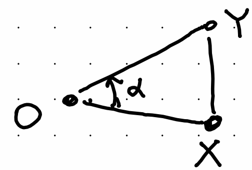
$$(X, Y) \longmapsto \text{Area}\,(O\hat{X}Y) :$$

$$\text{Area}\,(O\hat{X}Y) = \text{Area}\,(X,Y)$$

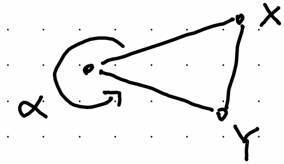



Definiamo la f-ne $A : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ come

$$A\,(X,Y) = \begin{cases} \text{Area}\,(X,Y) & \text{se} \\ -\,\text{Area}\,(X,Y) & \text{se} \end{cases}$$




$0 \leq \alpha \leq 180^\circ$



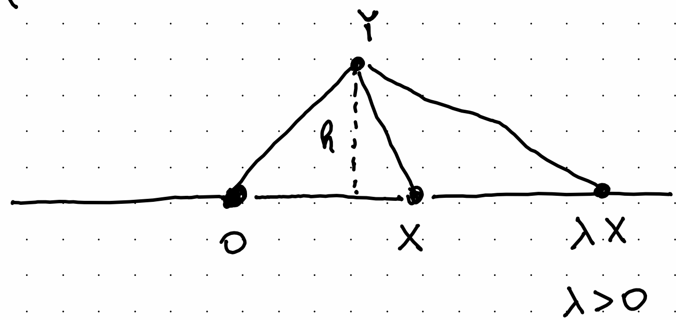

$180^\circ \leq \alpha \leq 360^\circ$

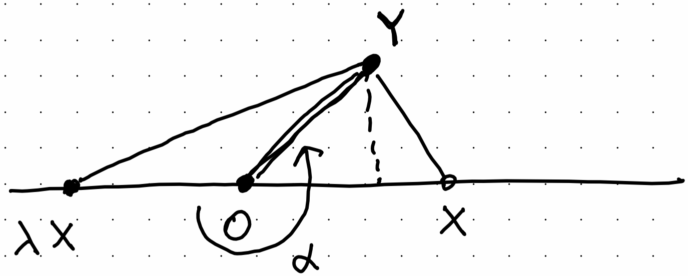
Proprietà:

$$A(Y, X) = -\,A\,(X,Y) \qquad [\text{ alternante }]$$

$A(\lambda X, Y) = ?$ $\quad$ $\text{Area}(\lambda X, Y) = \lambda\, A(X,Y)$ se $\lambda > 0$

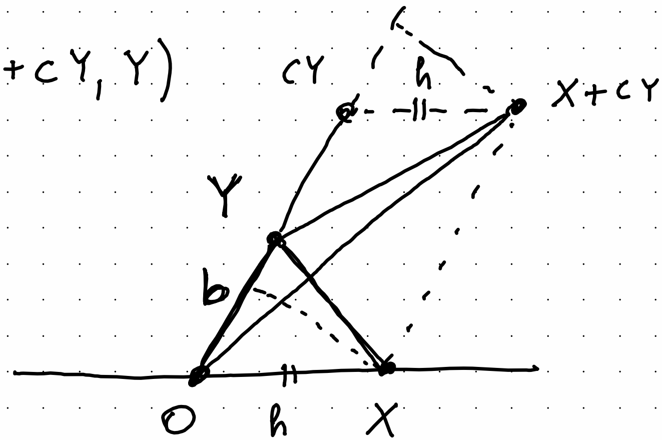
$= -|\lambda|\, A(X,Y)$ se $\lambda < 0$



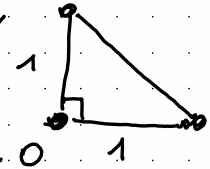




$$A(\lambda X, Y) = \lambda\, A(X,Y).$$

$A(X+cY, Y)$




$$A(X+cY, Y) = A(X, Y)$$

$A(e_1, e_2) = \text{Area}\,($ 

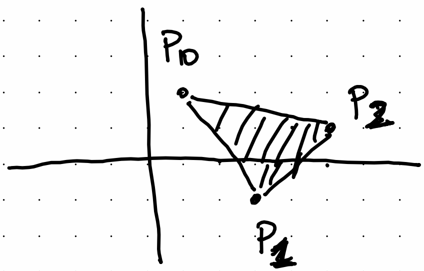
$) = \frac{1}{2}$

$$\Rightarrow \quad 2\,A(X, Y) = \det(X, Y) \quad \Rightarrow \quad \boxed{A(X,Y) = \frac{1}{2}\det(X,Y)}$$

$$\Rightarrow \quad \text{Area}(X, Y) = \left| \frac{1}{2} \det(X, Y) \right|$$

**Es:** Calcolare l'area del triangolo di vertici

$$P_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad P_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

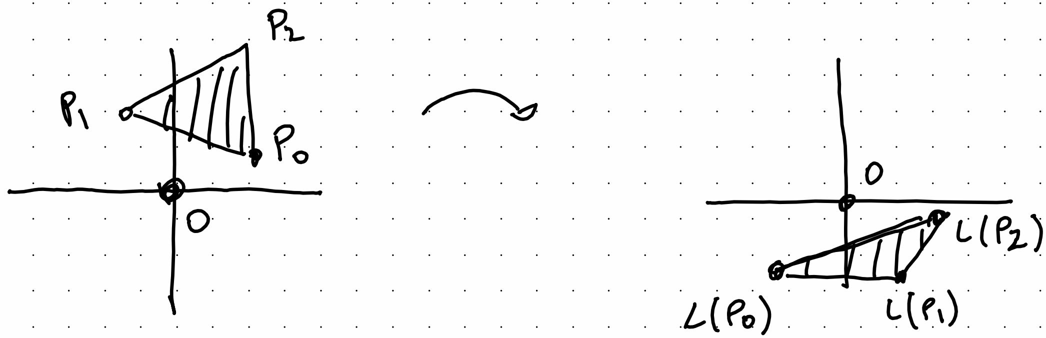



Sol.

$$\text{Area}\,(\widehat{P_0 P_1 P_2}) = \text{Area}\,\Big(0,\ \underset{\begin{pmatrix} 2 \\ -3 \end{pmatrix}}{P_1 - P_0},\ \underset{\begin{pmatrix} 4 \\ -1 \end{pmatrix}}{P_2 - P_0}\Big) =$$

$$= \frac{1}{2}\left|\det\begin{pmatrix} 2 & 4 \\ -3 & -1 \end{pmatrix}\right| = \frac{1}{2}\,|-2+12| = 5.$$

$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ lineare, $L = S_C$ $C \in Mat_{2\times 2}(\mathbb{R})$




$$A\big(L(X), L(Y)\big) = \frac{1}{2} \det\big(L(X) \,|\, L(Y)\big)$$

$$= \frac{1}{2} \det\big(CX \,|\, CY\big)$$

$$= \frac{1}{2} \det\big(C\,(X|Y)\big) \overset{\text{Binet}}{=} \frac{1}{2} \det(C) \det(X|Y)$$

$$\text{Area}\big(L(X), L(Y)\big) = |\det(L)|\ \text{Area}(X|Y).$$

# Matrice di Vandermonde

<u>Def</u>: Dati $x_1, \dots, x_n \in \mathbb{K}$ definiamo

$$Van(x_1, \dots, x_n) = V(x_1, \dots, x_n) = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & & x_n^{n-1} \end{pmatrix} \in Mat_{n \times n}(\mathbb{K})$$

Essa si chiama <u>matrice di Vandermonde</u> associata ai "p.ti" $x_1, \dots, x_n$.

$$V(0,1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad V(-1,-2) = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$$

$$V(1,-1,2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}, \quad V(-1,-1,2) = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

L'abbiamo già vista:

$$V = \mathbb{K}[x]_{\leq n-1}$$

$$\begin{tikzcd} V \arrow[r, equal] \arrow[d, "F"'] & V \arrow[d, "F_e"] \\ \mathbb{K}^n & \mathbb{K}^n \arrow[l, "\mathrm{Van}(x_1, \dots, x_n)"'] \end{tikzcd}$$

$$F(p(x)) = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_n) \end{pmatrix}$$

$$\mathrm{Van}(x_1, \dots, x_n) = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix}$$

$$\begin{array}{ccccc} F(1) & F(x) & F(x^2) & \dots & F(x^{n-1}) \end{array}$$

$$F(p(x)) = \begin{pmatrix} p(x_1) \\ \vdots \\ p(x_n) \end{pmatrix} \quad \text{è lineare}$$

$p(x) \in \operatorname{Ker} F \Rightarrow p(x_1)=0,\ p(x_2)=0, \dots, p(x_n)=0$

$\Rightarrow p(x) = q(x)\,(x-x_1)(x-x_2)\cdots(x-x_n)$

Se $x_i \neq x_j \ \forall i \neq j$ allora $q(x)=0 \Rightarrow p(x)=0$

$\Rightarrow \operatorname{Ker} F = \{0_V\} \Rightarrow F$ è un isomorfismo.

Se $x_i = x_j$ per un certo $i \neq j$. allora

$\operatorname{Ker} F \neq 0$.

Es: $F(p(x)) = \begin{pmatrix} p(1) \\ p(1) \\ p(2) \end{pmatrix} \quad 0 \neq (x-1)(x-2) \in \operatorname{Ker} F$

Quindi,

$$\mathrm{Van}(x_1,\dots,x_n) \text{ è invertibile} \iff F \text{ è invertibile}$$
$$\iff x_i \neq x_j \quad \forall\, i \neq j$$

In questo caso $\mathrm{Van}(x_1,\dots,x_n)$ è una matrice di cambiamento di base

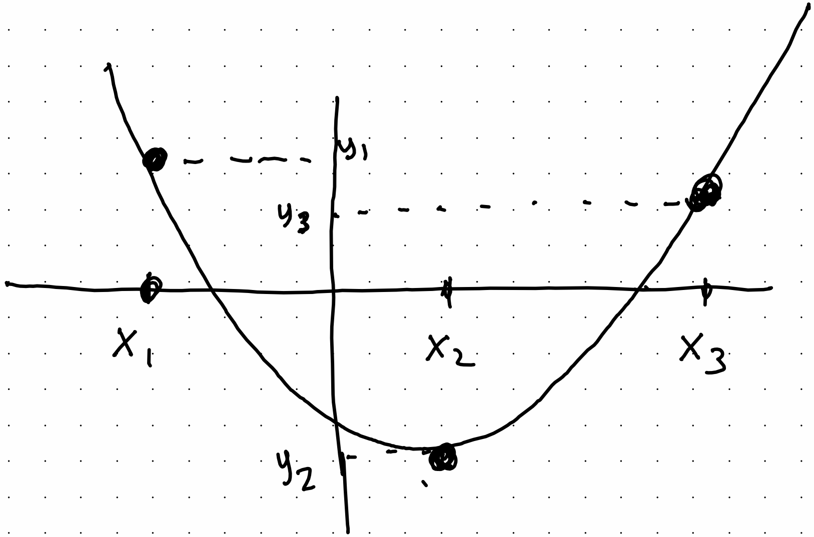
$$\begin{tikzcd}
V \arrow[r, equal] \arrow[d, "F_{\mathcal{B}} = F"'] & V \arrow[d, "F_e"] \\
\mathbb{K}^n & \mathbb{K}^n \arrow[l, "\mathrm{Van}(x_1,\dots,x_n)"]
\end{tikzcd}$$

$\mathcal{B} = \{b_1, \dots, b_n\}$

$b_1,\dots,b_n$ polinomi di Lagrange.

$$F(b_i) = e_i$$

$m=3$




$p(x)$ : polinomio interpolatore dei tre punti.

"Definizione"

$$F^{-1}\left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = p(x) \quad \text{t.c.} \quad \begin{array}{l} p(x_1)=y_1 \\ p(x_2)=y_2 \\ p(x_3)=y_3 \end{array}$$

ha grado $\leq 2$