1. Comunicazioni
2. Domande?

$B \in \mathrm{Mat}_{n\times n}(\mathbb{K})$ <u>quadrata</u> è invertibile

$$\Updownarrow$$

$\{B^1, \dots, B^n\}$ formano una base di $\mathbb{K}^{\cancel{m}\, n}$

Se $B \in \mathrm{Mat}_{m\times n}(\mathbb{K})$ e $\{B^1, \dots, B^n\}$ ~~fa~~ è una <u>base</u> di $\mathbb{K}^m$ allora $m = n$ e quindi B è quadrata e invertibile.

Se $\{B^1, \dots, B^n\}$ sono lin. indipendenti in $\mathbb{K}^m$ allora S_B è <u>iniettiva</u> e quindi ammette inversa destra: $\exists\, C \in \mathrm{Mat}_{n\times m}$ t.c.

$$BC = \mathbb{1}_m.$$

Tecniche di calcolo di

1) una base del nucleo
2) una base dell' immagine
3) dell' inversa (se esiste)
4) ...

di ~~un' applicazione lineare~~ una matrice.

L' algoritmo di eliminazione di Gauss

Per illustrare l' algoritmo consideriamo un esempio:

Es:

$$A = \begin{pmatrix} 0 & \boxed{1} & 2 & 3 & \boxed{1} & 2 & \boxed{2} & 1 & 5 \\ 0 & \boxed{1} & 2 & 3 & \boxed{2} & 3 & \boxed{3} & 2 & 8 \\ 0 & \boxed{1} & 2 & 3 & \boxed{-1} & 0 & \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 2 & 3 & \boxed{2} & 3 & \boxed{3} & 2 & 8 \end{pmatrix} \in Mat_{4\times 9}(\mathbb{R})$$

$$S_A : \mathbb{R}^9 \longrightarrow \mathbb{R}^4$$

Vogliamo calcolare una base di <u>$\operatorname{Ker} A$</u>:

$$\mathcal{B}_{col(A)} = \{A^2, A^5, A^7\}$$

Estendiamola ad una base di $\mathbb{R}^4$

$$\mathcal{B} = \{A^2, A^5, A^7, e_1\}$$

$$\begin{tikzcd} \mathbb{R}^9 \arrow[r, "A"] \arrow[d, equal, "F_e"'] & \mathbb{R}^4 \arrow[d, "F_{\mathcal{B}}"] \\ \mathbb{R}^9 \arrow[r] & \mathbb{R}^4 \end{tikzcd}$$

$$R = \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 2 & 2 & 1 & 5 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \\ 0 & 1 & 2 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \end{pmatrix}$$

$$R = \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{Ker} A = \operatorname{Ker} R =$$

$$= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_9 \end{pmatrix} \in \mathbb{R}^9 \;\middle|\; \begin{array}{l} \underline{x_2} + 2x_3 + 3x_4 + x_6 - x_8 + x_9 = 0 \\ \underline{x_5} + x_6 + 0x_8 + 2x_9 = 0 \\ \underline{x_7} + x_8 + x_9 = 0 \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ -2x_3 - 3x_4 - x_6 + x_8 - x_9 \\ x_3 \\ x_4 \\ -x_6 - 2x_9 \\ x_6 \\ -x_8 - x_9 \\ x_8 \\ x_9 \end{pmatrix} \;\middle|\; \underbrace{x_1, x_3, x_4, x_6, x_8, x_9}_{\text{variabili libere o indipendenti}} \in \mathbb{R} \right\} =$$

$$
= \left\{ \begin{pmatrix} x_1 \\ -2x_3 - 3x_4 - x_6 + x_8 - x_9 \\ x_3 \\ x_4 \\ -x_6 - 2x_9 \\ x_6 \\ -x_8 - x_9 \\ x_8 \\ x_9 \end{pmatrix} \;\middle|\; \underbrace{x_1, x_3, x_4, x_6, x_8, x_9}_{\text{variabili libere o indipendenti}} \in \mathbb{R} \right\} =
$$

$$
= \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_8 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_9 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; x_1, x_3, x_4, x_6, x_8, x_9 \in \mathbb{R} \right\}
$$

$$
= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle
$$

L' insieme

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

"$x_1=1$" "$x_3=1$" "$x_4=1$" "$x_6=1$" "$x_8=1$" "$x_9=1$"

Soluzioni-base di $RX=0$.

è una base di $\operatorname{Ker} R = \operatorname{Ker} A$.

La matrice R si chiama la forma a scala ridotta o forma di Hermite di A.

In inglese:

Row Reduced Echelon Form

e quindi in MATLAB:

$R = \text{rref}(A)$

Matrici simboliche in MATLAB :

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: 1) syms a b c d

$A = sym([a, b; c, d])$

2)

righe colonne

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$: $A = sym('a', [m, n])$

L'idea per trovare una base di $\operatorname{Ker} A$ è

trovare una matrice R t.c.

1) $\operatorname{Ker} A = \operatorname{Ker} R$

2) $\operatorname{Ker} R$ ha una base facile da calcolare.

$$
\begin{array}{ccc}
\mathbb{K}^n & \xrightarrow{\quad A \quad} & \mathbb{K}^m \\
\| & & \Big\downarrow F_{\mathcal{B}} = S_C \\
\mathbb{K}^n & \xrightarrow{\quad R \quad} & \mathbb{K}^m
\end{array}
$$

$$\mathcal{B} = \{ A^{j_1}, \dots, A^{j_r}, e_{i_{r+1}}, \dots, e_{i_m} \}$$

Trovare R è equivalente a trovare C:

$$C = \left(A^{j_1} \,|\, A^{j_2} \,|\, \dots \,|\, A^{j_r} \,|\, e_{i_{r+1}} \,|\, \dots \,|\, e_{i_m} \right)^{-1}$$

Operazioni elementari sulle righe di una matrice

I) Scambio di due righe

$$R_i \longleftrightarrow R_j$$

II) Moltiplicare una riga per uno scalare $\lambda \neq 0$

$$R_i \longmapsto \lambda R_i \qquad (\lambda \neq 0)$$

III) Aggiungere ad una riga un multiplo di un'altra

$$R_i \longmapsto R_i + c R_j \qquad (c \in \mathbb{K})$$

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 2 & 2 & 1 & 5 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \\ 0 & 1 & 2 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \end{pmatrix}$$

$$\begin{array}{c} R = \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccccccccc} & e_1 & & & e_2 & & e_3 & & \end{array} \end{array}$$

$$A^{j_1} \longmapsto e_1$$
$$A^{j_2} \longmapsto e_2$$
$$\vdots$$
$$A^{j_r} \longmapsto e_r \, .$$

Utilizzare I) II) III) in maniera che

$$A^2 \longmapsto e_1$$
$$A^5 \longmapsto e_2$$
$$A^7 \longmapsto e_3$$

$$A=\begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 2 & 2 & 1 & 5 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \\ 0 & 1 & 2 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \end{pmatrix} \rightsquigarrow \begin{array}{c} \begin{array}{ccccccccc} & e_1 & & & & & & & \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 2 & 2 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & -2 & -2 & -1 & -1 & -5 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 \end{pmatrix} \end{array}$$

$R_2 \mapsto R_2 - R_1$

$R_3 \mapsto R_3 - R_1$

$R_4 \mapsto R_4 - R_1$

$$\rightsquigarrow \begin{array}{c} \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccccccccc} & e_1 & & & e_2 & & & & \end{array} \end{array}$$

$R_1 \mapsto R_1 - R_2$

$R_3 \mapsto R_3 + 2R_2$

$R_4 \mapsto R_4 - R_2$

$$\rightsquigarrow \begin{array}{c} \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccccccccc} & e_1 & & & e_2 & & e_3 & & \end{array} \end{array} = R$$

$R_1 \mapsto R_1 - R_3$

$R_2 \mapsto R_2 - R_3$

Es:

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 3 & 1 & 4 \\ 0 & 2 & 0 & 2 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{pmatrix} 0 & 3 & 1 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

$$\xrightarrow[R_1 \mapsto \frac{1}{3} R_1]{} \begin{pmatrix} 0 & 1 & 1/3 & 4/3 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

$$\xrightarrow[R_3 \mapsto R_3 - 2R_1]{} \begin{pmatrix} 0 & \boxed{1} & 1/3 & 4/3 \\ 0 & \boxed{0} & 1 & 1 \\ 0 & \boxed{0} & -2/3 & -2/3 \end{pmatrix} \xrightarrow[R_1 \mapsto R_1 + \frac{1}{2} R_3]{} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2/3 & -2/3 \end{pmatrix}$$

$$\xrightarrow[R_3 \mapsto \frac{3}{2} R_3]{} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \xrightarrow[R_3 \mapsto R_3 + R_2]{} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \mapsto R_1 - \frac{1}{3} R_2 \\ R_3 \mapsto R_3 + \frac{2}{3} R_2}]{} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Es:

$$A = \begin{pmatrix} 0 & 2 & 4 & 1 & 3 \\ 0 & 3 & 6 & 3 & 6 \\ 0 & 4 & 8 & 2 & 6 \end{pmatrix} \underset{R_1 \mapsto \frac{1}{2} R_1}{\rightsquigarrow} \begin{pmatrix} 0 & 1 & 2 & 1/2 & 3/2 \\ 0 & 3 & 6 & 3 & 6 \\ 0 & 4 & 8 & 2 & 6 \end{pmatrix}$$

$$\underset{\substack{R_2 \mapsto R_2 - 3R_1 \\ R_3 \mapsto R_3 - 4R_1}}{\rightsquigarrow} \begin{pmatrix} 0 & \boxed{1} & 2 & 1/2 & 3/2 \\ 0 & \boxed{0} & 0 & 3/2 & 3/2 \\ 0 & \boxed{0} & 0 & 0 & 0 \end{pmatrix} \qquad 6 - \frac{12}{2} = 0$$

$$\underset{R_2 \mapsto \frac{1}{\frac{3}{2}} R_2}{\rightsquigarrow} \begin{pmatrix} 0 & 1 & 2 & 1/2 & 3/2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underset{R_1 \mapsto R_1 - \frac{1}{2} R_2}{\rightsquigarrow} \begin{pmatrix} 0 & \boxed{1} & 2 & \boxed{0} & 1 \\ 0 & \boxed{0} & 0 & \boxed{1} & 1 \\ 0 & \boxed{0} & 0 & \boxed{0} & 0 \end{pmatrix} = \mathrm{rref}(A)$$

Es:

$$A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow[\substack{R_2 \mapsto R_2 - R_1 \\ R_3 \mapsto R_3 - R_1}]{} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$