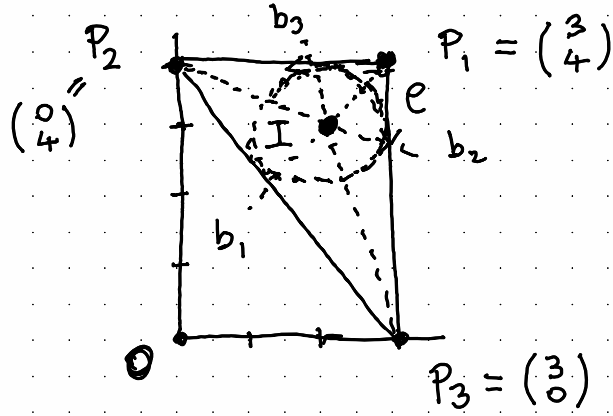
Es 1:



1) $\det\left(P_1 - P_2 \mid P_3 - P_2\right) = \det\begin{pmatrix}3 & 3\\0 & -4\end{pmatrix} = -12 \neq 0$

2) b_1: $\quad P_2 - P_1 = \begin{pmatrix}-3\\0\end{pmatrix}, \quad P_3 - P_1 = \begin{pmatrix}0\\-4\end{pmatrix}$

$$n_1 = \frac{P_2 - P_1}{\|P_2 - P_1\|} = \begin{pmatrix}-1\\0\end{pmatrix} \qquad n_2 = \frac{P_3 - P_1}{\|P_3 - P_1\|} = \begin{pmatrix}0\\-1\end{pmatrix}$$

$$b_1 = P_1 + \langle n_1 + n_2 \rangle =$$

$$= \begin{pmatrix}3\\4\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\end{pmatrix} \right\rangle \; : \; x - y = -1$$

b_2: $\quad -n_1, \quad P_3 - P_2 = \begin{pmatrix}3\\-4\end{pmatrix}, \quad n_3 = \frac{P_3 - P_2}{\|P_3 - P_2\|} = \frac{1}{\sqrt{25}}\begin{pmatrix}3\\-4\end{pmatrix}$

$$n_3 = \frac{P_3 - P_2}{\|P_3 - P_2\|} = \frac{1}{5}\begin{pmatrix}3\\-4\end{pmatrix} = \begin{pmatrix}3/5\\-4/5\end{pmatrix}$$

$$b_2 = P_2 + \langle -n_1 + n_3 \rangle = \begin{pmatrix}0\\4\end{pmatrix} + \left\langle \begin{pmatrix}8/5\\-4/5\end{pmatrix} \right\rangle$$

$$= \begin{pmatrix}0\\4\end{pmatrix} + \left\langle \begin{pmatrix}2\\-1\end{pmatrix} \right\rangle \; : \; x + 2y = 8$$

b_3: $\quad -n_3, \; -n_2$

$$b_3 = P_3 + \langle -n_3 - n_2 \rangle = P_3 + \left\langle \begin{pmatrix}3/5\\-9/5\end{pmatrix} \right\rangle = \begin{pmatrix}3\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\-3\end{pmatrix} \right\rangle$$

$$: \; 3x + y = 9$$

3) $I = b_1 \cap b_2 \cap b_3$

$$\begin{cases} x - y = -1 \\ x + 2y = 8 \\ 3x + y = 9 \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 2 & 8 \\ 3 & 1 & 9 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 3 & 9 \\ 0 & 4 & 12 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow I = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

4) Il centro di $\mathcal{C}$ è I.

$r = \text{dist}(I, \overline{P_1P_3}) = 1$

$\mathcal{C} = \left\{ I + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \;\middle|\; \theta \in [0, 2\pi) \right\}$

$: \quad (x-2)^2 + (y-3)^2 = 1$

Es2:

1) $r_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rangle$ $\quad r_1 : \begin{cases} y - z = 0 \\ x + y + z = 1 \end{cases}$

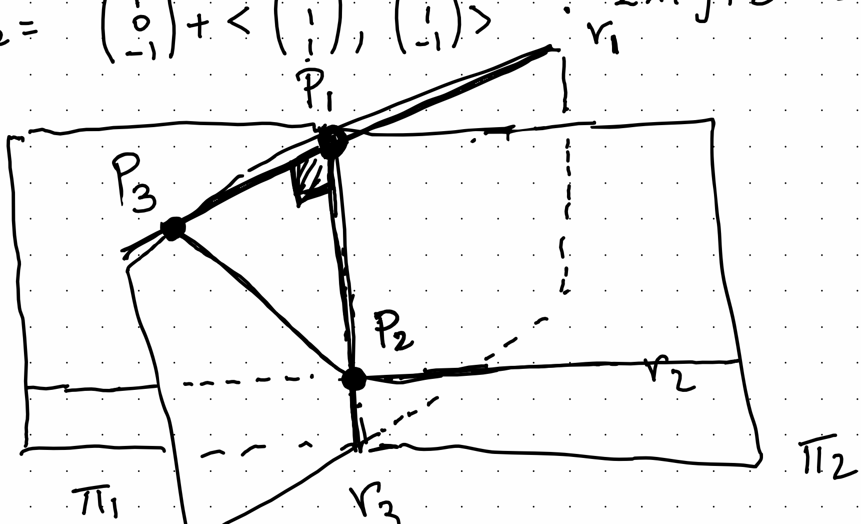
$r_2 : \begin{cases} x + y = 1 \\ x + z = 0 \end{cases}$ $\quad r_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

2) $\det \begin{pmatrix} 0 & -2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = -1 \neq 0 \Rightarrow r_1$ ed r_2 sono sghembe.

$$\operatorname{dist}(r_1, r_2) = \frac{\left| \det \begin{pmatrix} 0 & -2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right|}{\left\| \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\|} = \frac{1}{\left\| \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

3) $\pi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle : x + y + z = 1$

$\pi_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle : 2x + y + z = 1$



$r_3 : \begin{cases} x + y + z = 1 \\ 2x + y + z = 1 \end{cases} \Leftrightarrow r_3 : \begin{cases} y + z = 1 \\ x = 0 \end{cases} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$

4) $P_1 = r_1 \cap r_3 = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$

$P_2 = r_2 \cap r_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

5) $P_3 = e_1 \notin r_3 \;, \quad P_3 \in r_1$

$P_3 - P_1 \perp P_2 - P_1$

$\Rightarrow \quad \text{Area}\,(P_1 \widehat{P_2} P_3) = \frac{1}{2} \,\|P_3 - P_1\|\, \|P_2 - P_1\| = \frac{\sqrt{3}}{4}$

6) $P_4 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$\text{Area}\,(P_1 \widehat{P_3} P_4) = \frac{1}{2} \,\|(P_3 - P_1) \wedge (P_4 - P_1)\|$$

$$= \frac{1}{2} \left\| \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \right\| = \frac{1}{2} \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{4}$$

Es 3:

1) $P_A(x) = \det(x \mathbb{1}_4 - A)$

$$= \det \begin{pmatrix} x & 2 & 2 & -2 \\ 1 & x+1 & 2 & -2 \\ 1 & 2 & x+3 & -3 \\ 1 & 2 & 3 & x-3 \end{pmatrix}$$

$$= \det \begin{pmatrix} x & 2 & 2 & 0 \\ 1 & x+1 & 2 & 0 \\ 1 & 2 & x+3 & x \\ 1 & 2 & 3 & x \end{pmatrix}$$

$$= \det \begin{pmatrix} x & 2 & 2 & 0 \\ 1 & x+1 & 2 & 0 \\ 0 & 0 & x & 0 \\ 1 & 2 & 3 & x \end{pmatrix}$$

$$= x \det \begin{pmatrix} x & 2 & 2 \\ 1 & x+1 & 2 \\ 0 & 0 & x \end{pmatrix} = x^2 \det \begin{pmatrix} x & 2 \\ 1 & x+1 \end{pmatrix}$$

$$= x^2 (x^2 + x - 2) = x^2 (x-1)(x+2)$$

2) $Sp(A) = \{0, 1, -2\} \subset \mathbb{R}$

$ma_A(0) = 2$, $ma_A(1) = 1$, $ma_A(-2) = 1$

3) $mg_A(1) = 1$, $mg_A(-2) = 1$.

$$mg_A(0) = \dim \operatorname{Ker} A$$

$$\operatorname{Ker} A = \left\langle \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$mg_A(0) = 2 = ma_A(0)$$

$\Rightarrow$ A è diagonalizzabile.

4) $V_1(A) = \operatorname{Ker}(\mathbb{1}_4 - A) = \left\langle \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$

$V_{-2}(A) = \operatorname{Ker}(-2\mathbb{1}_4 - A) = \left\langle \begin{pmatrix} 2 \\ 2 \\ 3 \\ 3 \end{pmatrix} \right\rangle$

$$B = \begin{pmatrix} -1 & 1 & -2 & 2 \\ -1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -2 \end{pmatrix}$$

$$B^{-1}AB = D. \overset{?}{\Longleftrightarrow} AB = BD$$

$$\underline{AB^i = \lambda_i B^i} \overset{?}{\Longleftrightarrow}$$

$$A\left(B^1|\dots|B^n\right) = \left(B^1|\dots|B^n\right)\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$\left(AB^1|AB^2|\dots|AB^n\right) = \left(\lambda_1 B^1|\lambda_2 B^2|\dots|\lambda_n B^n\right)$$

$\underline{\text{Es 4}}$: $\quad p_1(x) = 1-x, \; p_2(x) = 2+x, \; p_3(x) = x+x^2$

$\mathcal{B} = \{p_1(x), p_2(x), p_3(x)\}$

1) $\det\left(F_e(p_1(x)) \,|\, F_e(p_2(x)) \,|\, F_e(p_3(x)) \right)$

$$= \det \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = 3 \neq 0$$

$\Rightarrow \mathcal{B}$ è una base.

2) $$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 3 \\ 1 & 0 & 2 \end{pmatrix}$$

3)
$$\begin{tikzcd} V \arrow[d, "F_e"'] & = & V \arrow[r, "T"] \arrow[d, "F_{\mathcal{B}}"] & V \arrow[d, "F_e"] \\ \mathbb{R}^3 & & \mathbb{R}^3 \arrow[ll, "B"'] \arrow[r, "A"] & \mathbb{R}^3 \end{tikzcd}$$

$$B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$C = AB^{-1}$: $\quad (A \,|\, B) \rightsquigarrow (\mathbb{1}_3 \,|\, C)$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 2 & 0 \\ 1 & -1 & 3 & -1 & 1 & 1 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & \frac{1}{3} & -2/3 & 8/3 \end{array}\right)$$

$$C = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 12 \\ 1 & -2 & 8 \end{pmatrix}$$

4) $\operatorname{Ker} C = \ldots = \left\langle \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} \right\rangle$

$\Leftrightarrow \operatorname{Ker} T = \langle 4x + x^2 \rangle$

5) $\mathrm{Im}\, C = \langle C^1, C^2 \rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -2/3 \end{pmatrix} \right\rangle$

$\mathrm{Im}\, T = \langle 1 + \frac{1}{3}x^2, -x - \frac{2}{3}x^2 \rangle$

$= \langle 3 + x^2, 3x + 2x^2 \rangle$

6) Si chiede se 1 è un autovalore di T.

$\det(\mathbb{1}_3 - C) = \det \begin{pmatrix} 0 & 0 & 0 \\ & * & \end{pmatrix} = 0$

$\Rightarrow$ 1 è un autovalore di C

e quindi anche di T.

$\Rightarrow \exists\ p(x) \neq 0$, t.c. $T(p(x)) = p(x)$.

$V_1(C) = \mathrm{Ker}(\mathbb{1}_3 - C) = \left\langle \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\rangle$

$p(x) = -1 + 2x + x^2$ ha la

proprietà richiesta.

Es5:

1) $\mathrm{rg}(A|b) = \mathrm{rg}\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{array}\right) =$

$= \mathrm{rg}\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -2 \end{array}\right) = \mathrm{rg}\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{array}\right) = 3$

$\mathrm{rg}(A) = 2 \Rightarrow$ Per il Teorema di Rouché-Capelli il sistema $AX=b$ non è risolubile, ovvero $b \notin \mathrm{Im}A$.

2) $P = P_{\mathrm{Im}A} = A\,(A^tA)^{-1}A^t =$

$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix} \left[\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}\right]^{-1}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$

$= \frac{1}{12}\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$= \frac{1}{12}\begin{pmatrix} 6 & 6 \\ 2 & -2 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$

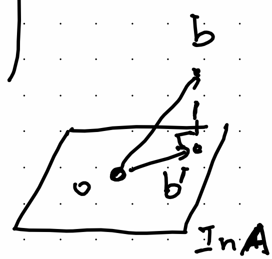
$$= \frac{1}{12} \begin{pmatrix} 6 & 6 \\ 2 & -2 \\ -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$= \frac{1}{12} \begin{pmatrix} 12 & 0 & 0 & 0 \\ 0 & 4 & -4 & 4 \\ 0 & -4 & 4 & -4 \\ 0 & 4 & -4 & 4 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

3) $$b' = Pb = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 6 \\ 3 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix} \in \mathrm{Im}\, A$$



4) $Ax = b'$:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow$$

L'unica soluzione è

$$x_0 = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

5) $\operatorname{dist}(b, \operatorname{Im} A) = \| b - pr_{\operatorname{Im} A}(b) \|$

$$= \| b - b' \| = \left\| \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\| = \sqrt{2}.$$