Richiami:

- Matrice associata ad una f.ne lineare $\mathcal{L}$ in due basi date

$$\begin{tikzcd} V \arrow[r, "\mathcal{L}"] \arrow[d, "F_{\mathcal{B}_V}"'] & W \arrow[d, "F_{\mathcal{B}_W}"] \\ \mathbb{K}^n \arrow[r, "A"] & \mathbb{K}^m \end{tikzcd}$$

$A^i = F_{\mathcal{B}_W}(\mathcal{L}(v_i)) \quad (\mathcal{B}_V = \{v_1, \dots, v_n\})$

$A \in Mat_{m \times n}(\mathbb{K})$

- Matrice di cambiamento di base

$$\begin{tikzcd} V \arrow[r, "Id"] \arrow[d, "F_{\mathcal{B}_1}"'] & V \arrow[d, "F_{\mathcal{B}_2}"] \\ \mathbb{K}^n \arrow[r, "B"] & \mathbb{K}^n \end{tikzcd}$$

$B^i = F_{\mathcal{B}_2}(v_i) \quad (\mathcal{B}_1 = \{v_1, \dots, v_n\})$

$B \in Mat_{n \times n}(\mathbb{K})$

- Matrici invertibili: B inv. $\overset{def}{\Longleftrightarrow}$ S_B è invertibile

$$S_B^{-1} = S_{B^{-1}}.$$

- Matrice identità:

$$\begin{tikzcd} V \arrow[r, "Id_V"] \arrow[d, "F_{\mathcal{B}}"'] & V \arrow[d, "F_{\mathcal{B}}"] \\ \mathbb{K}^n \arrow[r, "\mathbb{1}_n"] & \mathbb{K}^n \end{tikzcd}$$

$\mathbb{1}_n = (e_1 | \dots | e_n) \in Mat_{n \times n}(\mathbb{K})$

$\cdot)$

$$\begin{tikzcd} V_1 \arrow[r, "\mathcal{L}_1"] \arrow[d, "F_1", "\simeq"'] & W_1 \arrow[d, "F_2"', "\simeq"] \\ V_2 \arrow[r, "\mathcal{L}_2"] & W_2 \end{tikzcd}$$

$\mathcal{L}_1 \sim \mathcal{L}_2$ sono simili.

$$\mathcal{L}_2 \circ F_1 = F_2 \circ \mathcal{L}_1$$

$$\mathcal{L}_2 = F_2 \circ \mathcal{L}_1 \circ F_1^{-1}$$

Teorema

$$\mathcal{L}_1 \sim \mathcal{L}_2 \iff rg(\mathcal{L}_1) = rg(\mathcal{L}_2), \quad \dim V_1 = \dim V_2, \quad \dim W_1 = \dim W_2.$$

$\cdot)$ Se $\mathcal{L} : V \to W$ ha rango r allora è simile alla matrice

$$\left(\begin{array}{c|c} \mathbb{1}_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$\begin{tikzcd} V \arrow[r, "\mathcal{L}"] \arrow[d, "\mathcal{B}_V"'] & W \arrow[d, "\mathcal{B}_W"] \\ \mathbb{K}^n \arrow[r] & \mathbb{K}^m \end{tikzcd}$$

$$(e_1 | \dots | e_r | 0 | \dots | 0)$$

- <u>Prodotto righe x colonne</u>

$$\begin{tikzcd} V_1 \arrow[r, "\mathcal{L}_2"] \arrow[d, "F_{\mathcal{B}_1}"'] & V_2 \arrow[r, "\mathcal{L}_1"] \arrow[d, "F_{\mathcal{B}_2}"'] & V_3 \arrow[d, "F_{\mathcal{B}_3}"] \\ \mathbb{K}^n \arrow[r, "S_B"] & \mathbb{K}^m \arrow[r, "S_A"] & \mathbb{K}^e \end{tikzcd} \Rightarrow \begin{tikzcd} V_1 \arrow[r, "\mathcal{L}_1 \circ \mathcal{L}_2"] \arrow[d, "F_{\mathcal{B}_1}"'] & V_3 \arrow[d, "F_{\mathcal{B}_3}"] \\ \mathbb{K}^n \arrow[r, "C", "S_{AB}"'] & \mathbb{K}^e \end{tikzcd}$$

$C = AB :=$ moltiplicazione righe x colonne di A e B

$$C^i_j = (AB)^i_j = A_i B^j = \sum_{k=1}^{m} a_{i\underline{k}}\, b_{\underline{k}j}$$

$$\begin{aligned} Mat_{e \times \underline{m}} \times Mat_{\underline{m} \times n} &\longrightarrow Mat_{e \times n} \\ (A\ ,\ B) &\longmapsto AB \end{aligned}$$

OSS: Il prodotto righe per colonne è associativo:

$$(AB)C \overset{?}{=} A(BC) \quad : \text{ sì! Infatti,}$$

$$S_{(AB)C} = S_{AB} \circ S_C = (S_A \circ S_B) \circ S_C$$

$$\overset{\uparrow}{=} S_A \circ (S_B \circ S_C) = S_A \circ S_{BC} = S_{A(BC)}$$

la composizione di f-ni è associativa

$\square$.

Esercizio: $\left[(AB)C\right]_i^j = \left[A(BC)\right]_i^j \quad \forall i,j.$

<u>OSS</u>:

$$S_{\mathbb{1}_n} : \mathbb{K}^n \longrightarrow \mathbb{K}^n$$
$$X \longmapsto \mathbb{1}_n X = X$$

$$S_{\mathbb{1}_n} = Id_{\mathbb{K}^n}.$$

Supponiamo che $B \in Mat_{n\times n}(\mathbb{K})$ invertibile.

$$S_B \circ S_B^{-1} = Id_{\mathbb{K}^n} = S_{\mathbb{1}_n}$$

$$\Rightarrow \quad S_B \circ S_{B^{-1}} = S_{\mathbb{1}_n}$$

$$\Rightarrow \quad S_{BB^{-1}} = S_{\mathbb{1}_n} \quad \Rightarrow \quad BB^{-1} = \mathbb{1}_n.$$

Similmente,

$$B^{-1}B = \mathbb{1}_n.$$

OSS: Se B è invertibile, allora B^{-1} è l'unica matrice t.c. $BB^{-1} = \mathbb{1}_n$.

cioè: Se B è invertibile e $BC = \mathbb{1}_n$ allora $C = B^{-1}$.

OSS: Se B è quadrata $n \times n$ ed esiste C t.c. $BC = \mathbb{1}_n$ allora B è invertibile e $C = B^{-1}$. (Esercizio).

Questo è falso se B non è quadrata:

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad C = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow BC = \frac{1}{2} + \frac{1}{2} = 1 = \mathbb{1}_1$$

Ma $CB = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \neq \mathbb{1}_2$

$S_B : \mathbb{R}^2 \to \mathbb{R} \qquad S_C : \mathbb{R} \to \mathbb{R}^2$

Es:

$$(a, b)\begin{pmatrix} x \\ y \end{pmatrix} = ax + by$$

$$(a, b)\begin{pmatrix} x & z \\ y & w \end{pmatrix} = (ax + by \mid az + bw)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} ax + by & az + bw \\ cx + dy & cz + dw \end{pmatrix}$$

Moltiplicazione a blocchi

$$T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \begin{array}{c} 2 \\ 2 \end{array}\left(\begin{array}{c|c} \overset{2}{A} & \overset{2}{0} \\ \hline 0 & B \end{array}\right) \quad \text{dove} \quad A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$T^2 = T\,T = \left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) = \left(\begin{array}{cc|cc} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

$$= \left(\begin{array}{c|c} X & 0 \\ \hline 0 & Y \end{array}\right) \qquad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^2 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = X$$

$$B^2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = Y$$

OSS:

$$T = \begin{array}{c|c} & \begin{array}{cccc} n_1 & n_2 & & n_n \end{array} \\ \begin{array}{c} n_1 \\ n_2 \\ \vdots \\ n_n \end{array} & \left(\begin{array}{c|c|c|c} A_1 & 0 & \cdots & 0 \\ \hline 0 & A_2 & \ddots & \vdots \\ \hline \vdots & 0 & \ddots & 0 \\ \hline 0 & \cdots & 0 & A_n \end{array}\right) \end{array}$$

matrice diagonale a blocchi

$$T^k = \begin{pmatrix} A_1^k & & & \\ & A_2^k & & \\ & & \ddots & \\ & & & A_n^k \end{pmatrix}$$

Es: Matrici diagonali : Se $D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{pmatrix} \in Mat_{n\times n}(\mathbb{K})$

allora $D^k = \begin{pmatrix} d_1^k & & & \\ & d_2^k & & \\ & & \ddots & \\ & & & d_n^k \end{pmatrix}$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$T = \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) \quad \text{a blocchi} \quad \text{allora} \quad T^2 = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

$$= \begin{pmatrix} A^2 & AB+BC \\ 0 & C^2 \end{pmatrix}$$

Per poter moltiplicare matrici a blocchi
i blocchi devono essere compatibili:

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{array}{c} \begin{array}{cc} 2 & 1 \end{array} \\ \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 0 & 0 & 7 \\ 0 & 0 & 8 \end{array}\right) \end{array} \begin{array}{c} 2 \\ \\ 1 \end{array}\left(\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ \hline 7 & -1 \end{array}\right) = \begin{pmatrix} X \\ Y \end{pmatrix} = \left(\begin{array}{c} AX+BY \\ \hline CY \end{array}\right) = \left(\begin{array}{cc} 28 & 7 \\ 61 & 22 \\ \hline 49 & -7 \\ 56 & -8 \end{array}\right)$$

$4 \times 3 \qquad 3 \times 2$

$$AX = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 19 & 28 \end{pmatrix} \qquad BY = \begin{pmatrix} 3 \\ 6 \end{pmatrix}\begin{pmatrix} 7 & -1 \end{pmatrix} = \begin{pmatrix} 21 & -3 \\ 42 & -6 \end{pmatrix}$$

$$CY = \begin{pmatrix} 7 \\ 8 \end{pmatrix}\begin{pmatrix} 7 & -1 \end{pmatrix} = \begin{pmatrix} 49 & -7 \\ 56 & -8 \end{pmatrix}$$

Es: (2 settimana 4):

$$V = \langle \{v_1, v_2, v_3\} \rangle \xrightarrow{L} \langle (w_1, w_2, w_3, w_4) \rangle = W$$

$$v_1 \longmapsto 2w_1 + 3w_2 - w_3 + w_4$$

$$v_2 \longmapsto w_1 - w_3 - w_4$$

$$v_3 \longmapsto 3w_1 + 6w_2 - w_3 + 3w_4$$

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 6 \\ -1 & -1 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$ rappresenta L in queste basi.

$$\mathcal{B}_V' = \mathcal{B}_{\ker L} \cup \{\dots\} = \{v_1, v_2, -2v_1 + v_2 + v_3\}$$

$$\ker L = \langle -2v_1 + v_2 + v_3 \rangle$$

$$\mathcal{B}_W' = (L(v_1), L(v_2), w_1, w_2).$$

Nelle basi $\mathcal{B}_V'$ e $\mathcal{B}_W'$

$$A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left(\begin{array}{c|c} \mathbb{1}_2 & 0 \\ \hline 0 & 0 \end{array}\right)$$

Es:

$$A = \begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix} \in Mat_{2\times 2}(\mathbb{C})$$

$$\operatorname{Ker} A = \left\langle \begin{pmatrix} i \\ -1 \end{pmatrix} \right\rangle \rightsquigarrow \mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ -1 \end{pmatrix} \right\}$$

$$\rightsquigarrow \mathcal{B}_2 = \left\{ A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

la matrice che rappresenta S_A in queste basi è

$$\operatorname{Can}(A) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \left(\begin{array}{c|c} \mathbb{1}_r & 0 \\ \hline 0 & 0 \end{array}\right) \qquad r = \operatorname{rg} A = 1.$$

Quindi

$$A = \begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} :$$

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "A"] \arrow[d, "F_{\mathcal{B}_1}"'] & \mathbb{C}^2 \arrow[d, "F_{\mathcal{B}_2}"] \\ \mathbb{C}^2 \arrow[r, "\operatorname{Can}(A)"'] & \mathbb{C}^2 \end{tikzcd}$$

Richiami:

B invertibile $= (B^1|\dots|B^n) \in Mat_{n\times n}(\mathbb{K})$.

allora

$$S_B^{-1} = F_{\mathcal{B}} \qquad \text{dove} \quad \mathcal{B} = \{B^1, B^2, \dots, B^n\}$$

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ -1 \end{pmatrix} \right\} \qquad \mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

$$F_{\mathcal{B}_1} = ? \qquad F_{\mathcal{B}_2} = ?$$

cioè ~~ch~~ chi sono B_1 e B_2 t.c. $F_{\mathcal{B}_1} = S_{B_1}$ e $F_{\mathcal{B}_2} = S_{B_2}$?

$$F_{\mathcal{B}_1} = S_B^{-1} \quad \text{dove} \quad B = \begin{pmatrix} 1 & i \\ 0 & -1 \end{pmatrix}$$

Quindi $B_1 = B^{-1}$, $B_2 = C^{-1}$ dove $C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

$$S_B : \begin{array}{l} e_1 \longmapsto B^1 \\ e_2 \longmapsto B^2 \end{array} \qquad S_B^{-1} : \begin{array}{l} B^1 \longmapsto e_1 \\ B^2 \longmapsto e_2 \end{array}$$

$$B^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_1 = B^1 \longmapsto e_1$$

$$B^2 = \begin{pmatrix} i \\ -1 \end{pmatrix} \qquad e_2 = i B^1 - B^2 \longmapsto i e_1 - e_2$$

$$B^{-1} = \begin{pmatrix} 1 & i \\ 0 & -1 \end{pmatrix} = B$$

verifichiamo

$$\begin{pmatrix} 1 & i \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & i \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2. \quad \checkmark$$

<u>Esercizio</u>: Calcolare C^{-1}, dove $C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

OSS (importante!):

$$\underset{m\times n}{A}\,\underset{n\times p}{B}\;,\;\underset{n\times p}{B}\,\underset{m\times n}{A} \quad \text{definite.} \iff p = m.$$

$$AB \overset{?}{=} BA \quad ??$$

Es:

$$\mathbb{R}^3 \xrightarrow{L_1} \mathbb{R}^2 \qquad\qquad \mathbb{R}^2 \xrightarrow{L_2} \mathbb{R}^2$$

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\2\end{pmatrix} \qquad\qquad \begin{pmatrix}1\\-1\end{pmatrix} \longmapsto \begin{pmatrix}3\\2\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\1\end{pmatrix} \longmapsto \begin{pmatrix}1\\3\end{pmatrix} \qquad\qquad \begin{pmatrix}1\\1\end{pmatrix} \longmapsto \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\0\\1\end{pmatrix} \longmapsto \begin{pmatrix}1\\4\end{pmatrix}$$

Trovare la matrice M che rappresenta $L_2 \circ L_1$ nelle basi standard di $\mathbb{R}^3$ e $\mathbb{R}^2$.

Sol.:

$$\mathcal{B}_1 = \left\{ v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, v_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, v_3 = \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$$

$$\mathcal{B}_2 = \left\{ w_1 = \begin{pmatrix}1\\-1\end{pmatrix}, w_2 = \begin{pmatrix}1\\1\end{pmatrix} \right\}$$

$$
\begin{tikzcd}
\mathbb{R}^3 \arrow[r, equal] \arrow[d, "\mathcal{E}_3"'] & \mathbb{R}^3 \arrow[r, "L_1"] \arrow[d, "\mathcal{B}_1"'] & \mathbb{R}^2 \arrow[r, equal] \arrow[d, "\mathcal{E}_2"] & \mathbb{R}^2 \arrow[r, "L_2"] \arrow[d, "\mathcal{B}_2"'] & \mathbb{R}^2 \arrow[d, "\mathcal{E}_2"] \\
\mathbb{R}^3 & \mathbb{R}^3 \arrow[l] \arrow[r] & \mathbb{R}^2 & \mathbb{R}^2 \arrow[l] \arrow[r] & \mathbb{R}^2
\end{tikzcd}
$$

$$
\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = B_1 \qquad
\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix} = A \qquad
\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = B_2 \qquad
\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = C
$$

La matrice cercata è :

$$C\, B_2^{-1}\, A\, B_1^{-1}$$

$$B_1 : \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad B_1^{-1} = ?$$

$$S_{B_1} : \quad e_1 \longmapsto B^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad S_{B_1}^{-1} : \quad B^1 \longmapsto e_1$$

$$e_2 \longmapsto B^2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad B^2 \longmapsto e_2$$

$$e_3 \longmapsto B^3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad B^3 \longmapsto e_3$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = ? \qquad B^1 + B^2 + B^3 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} - 2 B^2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2e_1$$

$$2e_1 = B^1 - B^2 + B^3 \Rightarrow e_1 = \frac{1}{2} B^1 - \frac{1}{2} B^2 + \frac{1}{2} B^3$$

$$S_{B_1}^{-1}(e_1) = \left(B_1^{-1}\right)^1 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \end{pmatrix} \qquad B_1^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$e_2 = ? \qquad e_2 = \frac{1}{2} B^1 + \frac{1}{2} B^2 - \frac{1}{2} B^3 \rightsquigarrow \left(B_1^{-1}\right)^2 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$e_3 = ? \qquad e_3 = -\frac{1}{2} B^1 + \frac{1}{2} B^2 + \frac{1}{2} B^3 \rightsquigarrow \left(B_1^{-1}\right) = \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad B_2^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

esercizio

La matrice cercata è:

$$\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$