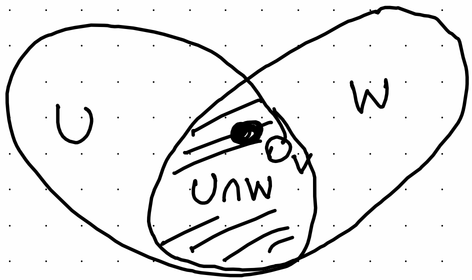
Richiami: Formula di Grassmann

$U, W \subset V$ s.sp. vett., V f.g. allora

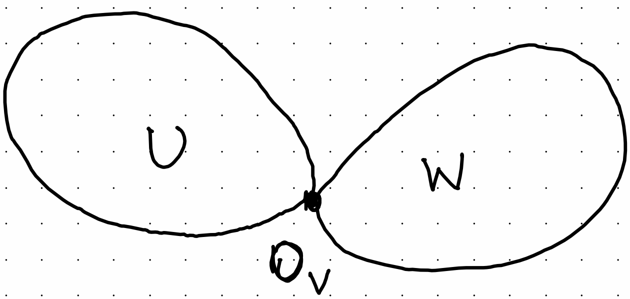
$$\dim(U+W) + \dim(U \cap W) = \dim U + \dim W$$



Def: U e W formano **somma diretta** se $U \cap W = \{0_V\}$

Scriviamo

$$U + W =: U \oplus W$$



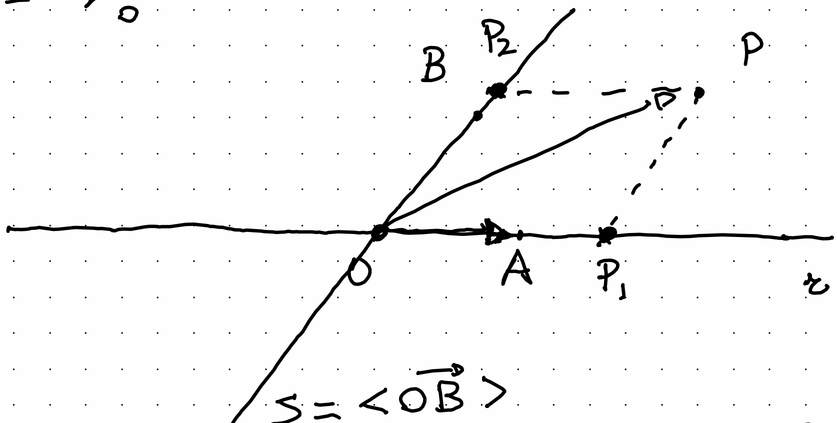
Può succedere che

$$V = U \oplus W$$

Es: $V = \mathbb{R}^2 \supset U = \langle e_1 \rangle, W = \langle e_2 \rangle$

Allora $\left.\begin{array}{l} U + W = \mathbb{R}^2 \\ U \cap W = \{0_{\mathbb{R}^2}\} \end{array}\right\} \iff \mathbb{R}^2 = \langle e_1 \rangle \oplus \langle e_2 \rangle$

Es: $V = \mathcal{V}_0^2$



$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{OP_2}$

$r = \langle \overrightarrow{OA} \rangle$

$s = \langle \overrightarrow{OB} \rangle$

$$\mathcal{V}_0^2 = \langle \overrightarrow{OA} \rangle \oplus \langle \overrightarrow{OB} \rangle. \quad := \begin{cases} \mathcal{V}_0^2 = \langle \overrightarrow{OA} \rangle + \langle \overrightarrow{OB} \rangle \\ \langle \overrightarrow{OA} \rangle \cap \langle \overrightarrow{OB} \rangle = \{ \overrightarrow{OO} \} \end{cases}$$

Domanda: Sia V f.g.. Sia $U \subset V$ un s.sp. vettoriale.

Esiste $W \subset V$ s.sp. tale che $V = U \oplus W$? si

Infatti, sia $\mathcal{B}_U = \{v_1, \dots, v_k\}$ una base di U.

Possiamo estendere $\mathcal{B}_U$ ad una base

$$\mathcal{B}_V = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$$

di V. Quindi basta prendere $W = \langle v_{k+1}, \dots, v_n \rangle$.

Infatti, ① $U \cap W = \{0_V\}$ perché: $x_1 v_1 + \dots + x_k v_k = y_{k+1} v_{k+1} + \dots + y_n v_n$

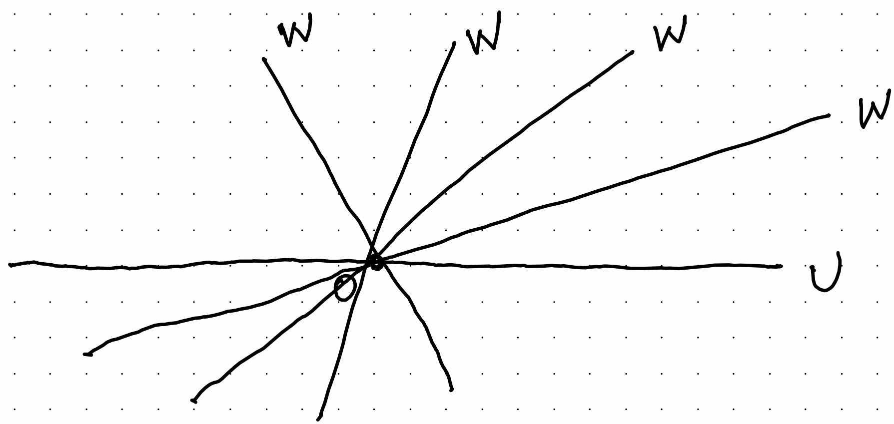
$\Rightarrow x_1 v_1 + \dots + x_k v_k - y_{k+1} v_{k+1} - \dots - y_n v_n = 0_V \xRightarrow[\text{Lin Ind.}]{\mathcal{B}_V \text{ è}} x_1 = \dots = x_k = y_{k+1} = \dots = y_n = 0.$

② $U + W = V$. $\dim(U+W) = \dim U + \dim W - \underbrace{\dim(U \cap W)}_{= 0} = n$

Def: Un sottospazio complementare o supplementare di un sottospazio vettoriale U di V, è un sottospazio vettoriale $W \subset V$ tale che

$$V = U \oplus W.$$

Domanda: Un complementare è unico? NO

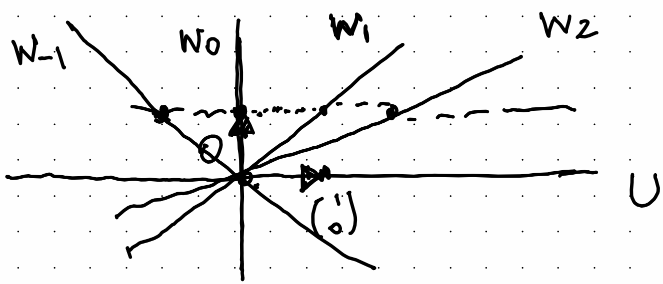


Es: Trovare tutti i complementari in $\mathbb{R}^2$ di $U = \langle e_1 \rangle = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\}$

Sol.: Se W è un complementare di U, allora

- $\dim W = 1$
- $W \cap U = \{ 0_{\mathbb{R}^2} \}$.

$$W_y = \left\langle \begin{pmatrix} y \\ 1 \end{pmatrix} \right\rangle \qquad y \in \mathbb{R}$$



Es: $V = \mathbb{R}[x]_{\leq 5}$

$$U_1 = \{ p(x) \in V \mid p(0) = 0 \}$$

$$U_2 = \{ p(x) \in V \mid x^2+1 \text{ divide } p(x) \}$$

$$U_3 = \{ p(x) \in V \mid p(-x) = p(x) \ \forall x \in \mathbb{R} \}$$

1) Dimostrare che U_1, U_2, U_3 sono sottosp. vettoriali di V ed esibire una loro base

2) Trovare una base di $U_1 \cap U_2$, $U_1 \cap U_3$, $U_2 \cap U_3$

3) Trovare dei complementari a U_3 e a $U_1 \cap U_2$: ovvero W_1 e W_2 t.c.

$$U_3 \oplus W_1 = V = (U_1 \cap U_2) \oplus W_2 .$$

$$V = \mathbb{R}[x]_{\leq 5} = \langle 1, x, x^2, x^3, x^4, x^5 \rangle$$

$$U_1 = \{ p(x) \in V \mid p(0) = 0 \}$$

$$U_2 = \{ p(x) \in V \mid x^2+1 \text{ divide } p(x) \}$$

$$U_3 = \{ p(x) \in V \mid p(-x) = p(x) \ \forall x \in \mathbb{R} \}$$

1) $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

$p(x) \in U_1 \iff a_0 = 0 \iff p(x) \in \langle x, x^2, \dots, x^5 \rangle$.

$U_1 = \langle x, x^2, x^3, x^4, x^5 \rangle$ e quindi è un s.sp. vettoriale.

$p(x) \in U_2 \iff \exists\, q(x)$ t.c. $p(x) = (x^2+1)\, q(x)$.

$q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$

$0 = (x^2+1)\, 0 \in U_2$

$\alpha (x^2+1)\, q_1(x) + \beta (x^2+1)\, q_2(x) = (x^2+1)(\alpha q_1 + \beta q_2)$

$p(x) = b_0 (x^2+1) + b_1 (x^2+1) x + b_2 (x^2+1) x^2 + b_3 (x^2+1) x^3$.

$\in \langle x^2+1, (x^2+1)x, (x^2+1)x^2, (x^2+1)x^3 \rangle$

$$V = \mathbb{R}[x]_{\leq 5} \simeq \langle 1, x, x^2, x^3, x^4, x^5 \rangle$$

$$U_1 = \{ p(x) \in V \mid p(0) = 0 \}$$

$$U_2 = \{ p(x) \in V \mid x^2+1 \text{ divide } p(x) \}$$

$$U_3 = \{ p(x) \in V \mid p(-x) = p(x) \quad \forall x \in \mathbb{R} \}$$

$$U_2 \subseteq \langle x^2+1, (x^2+1)x, (x^2+1)x^2, (x^2+1)x^3 \rangle =: \overline{U}_2$$

$$\dim \overline{U}_2 = 4$$

Osserviamo che $1, x \notin U_2$.

$$\Rightarrow \quad \langle 1, x \rangle \cap U_2 = \{ 0_V \}$$

$$\Rightarrow \quad \dim(\langle 1, x \rangle + U_2) = \dim \langle 1, x \rangle + \dim U_2 = 2 + \dim U_2$$

$$\Rightarrow \quad \dim U_2 = \dim(\langle 1, x \rangle + U_2) - 2 \leq 6 - 2 = 4$$

$$V = \mathbb{R}[x]_{\leq 5} \simeq \langle 1, x, x^2, x^3, x^4, x^5 \rangle$$

$$U_1 = \{ p(x) \in V \mid p(0) = 0 \}$$

$$U_2 = \{ p(x) \in V \mid x^2+1 \text{ divide } p(x) \}$$

$$U_3 = \{ p(x) \in V \mid p(-x) = p(x) \quad \forall x \in \mathbb{R} \}$$

$$U_2 \subseteq \langle x^2+1, (x^2+1)x, (x^2+1)x^2, (x^2+1)x^3 \rangle =: \overline{U_2}$$

$$\dim \overline{U_2} = 4$$

Dato che i generatori di $\overline{U_2}$ stanno in U_2, e U_2 è un s.sp. vettoriale,

$$U_2 = \overline{U_2}.$$

$$\dim U_2 = 4,$$

$$V = \mathbb{R}[x]_{\leq 5} \simeq < 1, x, x^2, x^3, x^4, x^5 >$$

$$U_1 = \{ p(x) \in V \mid p(0) = 0 \}$$

$$U_2 = \{ p(x) \in V \mid x^2+1 \text{ divide } p(x) \}$$

$$U_3 = \{ p(x) \in V \mid p(-x) = p(x) \ \forall x \in \mathbb{R} \}$$

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

$$\parallel$$

$$p(-x) = a_0 - a_1 x + a_2 x^2 - a_3 x^3 + a_4 x^4 - a_5 x^5$$

$$p(x) \in U_3 \iff p(x) = p(-x) \iff a_1 = a_3 = a_5 = 0.$$

$$U_3 = < 1, x^2, x^4 >$$

$U_1 = \langle x, x^2, x^3, x^4, x^5 \rangle$ $\quad$ $\dim U_1 = 5$

$U_2 = \langle x^2+1, (x^2+1)x, (x^2+1)x^2, (x^2+1)x^3 \rangle$ $\quad$ $\dim U_2 = 4$

$U_3 = \langle 1, x^2, x^4 \rangle$ $\quad$ $\dim U_3 = 3$

$U_1 \cap U_2 = \langle (x^2+1)x, (x^2+1)x^2, (x^2+1)x^3 \rangle$

$U_1 \cap U_3 = \langle x^2, x^4 \rangle$

$U_2 \cap U_3 = \langle x^2+1, (x^2+1)x^2 \rangle$

$W_2 + U_1 \cap U_2 = V$

$\Uparrow$

$W_1 = \langle x, x^3, x^5 \rangle$

$$\left.\begin{array}{l} W_2 = \langle 1, x, x^2 \rangle \cap (U_1 \cap U_2) = \{0\} \\ \dim W_2 = 3, \ \dim U_1 \cap U_2 = 3 \end{array}\right] \Rightarrow \dim\big(W_2 + (U_1 \cap U_2)\big) = 6 = \dim V$$

Es: Siano $m, n \geq 1$ due numeri naturali.

Sia $V = Mat_{m \times n}(\mathbb{K}) = \{\text{matrici } m \times n\}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in V$$

Trovare una base di V e calcolare la sua dimensione.

$$\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Sol.: $\mathbb{R}^2 = Mat_{2 \times 1}(\mathbb{R}) = \langle \underset{e_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{e_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \rangle$

$$Mat_{2 \times 2}(\mathbb{R}) \ni A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} =$$

$$\mathrm{Mat}_{2\times 2}(\mathbb{R}) \ni$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{E_{11}} + a_{12} \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{E_{12}} + a_{21} \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{E_{21}} + a_{22} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{E_{22}}$$

Definiamo $E_{ij} \in \mathrm{Mat}_{m\times n}(\mathbb{K})$ come

$$(E_{ij})_{k,l} = \begin{cases} 1 & \text{se } k = i \text{ e } l = j \\ 0 & \text{altrimenti.} \end{cases}$$

3×2:

$$E_{32} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{Mat}_{m\times n}(\mathbb{K}) = \langle E_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n \rangle$$

$$A = \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} a_{ij} E_{ij}$$

$$\dim \mathrm{Mat}_{m\times n}(\mathbb{K}) = m\, n$$

Def: Sia $A \in Mat_{m \times n}(\mathbb{K})$. Una componente a_{ij} di A si dice <u>diagonale</u> se $i = j$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

A si dice <u>quadrata</u> se $m = n$.

Una matrice <u>quadrata</u> A si dice <u>simmetrica</u> se

$$a_{ij} = a_{ji} \quad \forall i,j = 1, \dots, n$$

<u>Es</u>:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & 4 \\ -3 & -5 \end{pmatrix}$$

simmetrica — non-simmetrica — non è simmetrica

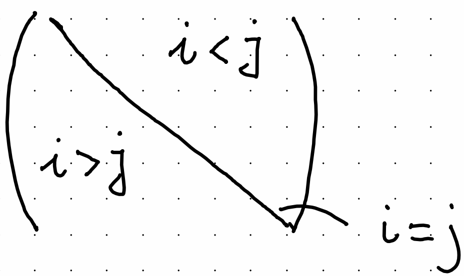
$$\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{3} & a_{32} \end{matrix}$$

$A \in Mat_{n \times n}(\mathbb{K})$

$$A = \sum_{1 \le i,j \le n} a_{ij} E_{ij}$$

è simmetrica se $a_{ij} = a_{ji}$

$$A = \sum_{1 \le i \le j \le n} a_{ij} E_{ij} + \sum_{1 \le j < i \le n} a_{ij} E_{ij}$$



$$\begin{matrix} a_{11} & \boxed{a_{12} \quad a_{13}} & \\ & \quad a_{23} & \\ \boxed{a_{21}} & a_{22} & \\ \boxed{a_{31} \quad a_{32}} & & a_{33} \end{matrix}$$

$i<j$

$i=j$

$i>j$

$$a_{ij} = a_{ji}$$

$$A = \sum_{1 \le i < j \le n} a_{ij} E_{ij} + \sum_{1 \le i \le n} a_{ii} E_{ii} + \sum_{1 \le i < j \le n} a_{ij} E_{ji}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = a_{11} E_{11} + a_{12} \underline{E_{12}} +$$

$$+ a_{12} \underline{E_{21}} + a_{22} E_{22}$$

$$= a_{11} E_{11} + a_{22} E_{22} + a_{12} (E_{12} + E_{21})$$

$$Sym_n(\mathbb{K}) = \{ A \in Mat_{n \times n}(\mathbb{K}) \mid A \text{ simmetrica} \}$$

è un sottospazio vettoriale di $Mat_{n \times n}(\mathbb{K})$

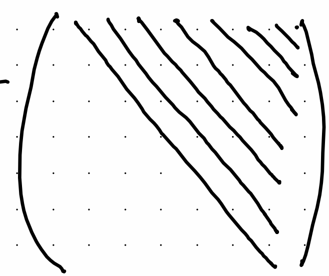
A simmetrica

$$A = \sum_{1\le i<j\le n} a_{ij}\,(E_{ij}+E_{ji}) + \sum_{1\le i\le n} a_{ii}E_{ii}$$

Una base di $\mathrm{Sym}_n(\mathbb{K})$ è

$$\{E_{ij}+E_{ji},\ E_{kk} \mid 1\le i<j\le n,\ 1\le k\le n\}$$

$$\dim \mathrm{Sym}_n(\mathbb{K}) = n+(n-1)+(n-2)+\dots+1$$



$$=$$

$$
\begin{array}{cccccccc}
 & 1 & 2 & 3 & 4 & 5 & \dots & n-1 & n \\
+ & n & n-1 & n-2 & n-3 & n-4 & & 2 & 1 \\
\hline
 & n+1 & n+1 & n+1 & n+1 & n+1 & & n+1 & n+1
\end{array}
$$

$$a = \sum_{k=1}^{n} k = 1+2+\dots+n$$

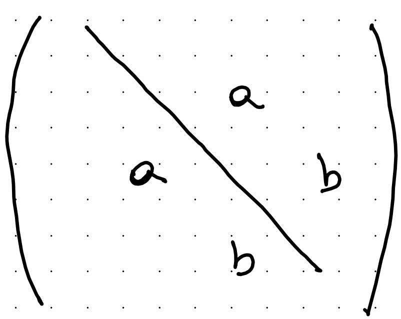
$$a+a = 1+2+\dots+n+1+2+\dots n-1+n$$

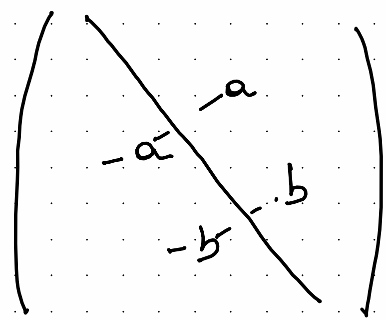
$$= n(n+1) \Rightarrow a = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

$$\dim \mathrm{Sym}_n(\mathbb{K}) = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n < \underbrace{n^2}_{=\dim \mathrm{Mat}_{n\times n}(\mathbb{K})}$$

Trovare un complementare in $V = \mathrm{Mat}_{n\times n}(\mathbb{K})$ di $\mathrm{Sym}_n(\mathbb{K})$.





Def: Una matrice $n \times n$ A si dice <u>anti-simmetrica</u> se

$$a_{ij} = -a_{ji} \quad \forall\, i,j = 1, \dots, n.$$

Es:

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

non è anti-simmetrica — è anti-simmetrica.

OSS: A anti-simmetrica $\Rightarrow$ $a_{ii} = 0 \ \forall i = 1, \dots, n$.

$$ASym_n(\mathbb{K}) = \{\text{Matrici } n \times n \text{ anti-simmetriche}\}.$$

Prop. :

$$\mathrm{Sym}_n(\mathbb{K}) \oplus \mathrm{ASym}_n(\mathbb{K}) = \mathrm{Mat}_{n\times n}(\mathbb{K})$$

dim :

Se $A \in \mathrm{Sym} \cap \mathrm{ASym}$ ~~allora~~ $A = 0_{n\times n}$.

$\Rightarrow \mathrm{Sym}_n(\mathbb{K}) \cap \mathrm{ASym}_n(\mathbb{K}) = \{0_{n\times n}\}$.

Sia $A \in \mathrm{Mat}_{n\times n}(\mathbb{K})$: considero

$A^t =$ "A trasposta" è la matrice data da

$$(A^t)_{ij} := A_{ji}$$

Es:
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$A^t_{12} = A_{21} \;\ldots$$

$\Rightarrow$ $A + A^t$ è simmetrica.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ c+b & 2d \end{pmatrix}$$

$A - A^t$ è anti-simmetrica

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}$$

$$2A = \overbrace{(A + A^t)}^{\text{simm.}} + \overbrace{(A - A^t)}^{\text{anti-simm.}}$$

$$\Rightarrow \quad A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$$

$$\Rightarrow \quad \boxed{\mathrm{Mat}_{n \times n} = \mathrm{Sym}_n \oplus \mathrm{Asym}_n.}$$

$$\dim \mathrm{ASym}(n) = (n-1) + (n-2) + \dots + 1 = \frac{n(n-1)}{2}$$

$$= \frac{1}{2} n^2 - \frac{1}{2} n$$

$$= n^2 - \left(\frac{1}{2} n^2 + \frac{1}{2} n \right) = n^2 - \dim \mathrm{Sym}(n)$$