JOURNAL OF THE COURSE PARTIAL DIFFERENTIAL EQUATIONS A.A. 2014/2015 CORSO DI DOTTORATO IN MODELLI MATEMATICI PER L'INGEGNERIA, ELETTROMAGNETISMO E NANOSCIENZE

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DIP. SCIENZE DI BASE E APPLICATE PER L'INGEGNERIA
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References:

- *DiBenedetto*: E.DiBenedetto, Partial Differential Equations, 2nd edition, Birkhauser 2010.
- *Evans*: L.Evans, Partial Differential Equations, AMS.

1. Monday 2015-03-02

Introduction to the course.
Partial differential equations.
Complete orthonormal systems in Hilbert spaces. Gram-Schmidt orthonormalization.
Galerkin's method for approximation of solutions to the problem

$$\begin{aligned} -(a_{ij}u_{x_i})_{x_j} &= \operatorname{div} f - F\,, \qquad && \text{in } \Omega, \\ u &= 0\,, && \text{on } \partial\Omega. \end{aligned}$$

Estimates of the approximating sequence.

Reference sections in textbooks: DiBenedetto 9.7.

2. Wednesday 2015-03-11

Convergence of Galerkin's approximating sequence. Existence, uniqueness and continuous dependence of the solution.
Non-homogeneous Dirichlet problem.

Homework 2.1. Study the problem

$$\begin{aligned} -(a_{ij}u_{x_i})_{x_j} &= f(u)\,, \qquad && \text{in } \Omega, \\ u &= 0\,, && \text{on } \partial\Omega. \end{aligned}$$

under suitable assumptions on f. □

Compact and symmetric operators in separable Hilbert spaces. Existence of a complete orthonormal system of eigenvectors.
Application to the second order differential operator

$$-\frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial}{\partial x_i}\Big)\,.$$

Properties of the first eigenvalue λ_1.

Lemma 2.2.

$$\lambda_1 = \min\Big\{ \int_\Omega a_{ij}u_{x_i}u_{x_j}\,\mathrm{d}x \mid u \in W^2_{o1}(\Omega)\,,\ \|u\|_{L^2(\Omega)} = 1 \Big\}\,.$$

Poincaré inequality.

Reference sections in textbooks: DiBenedetto 9.7, Evans 6.5.

3. Wednesday 2015-03-18

Properties of the first eigenvalue of elliptic operators.

Lemma 3.1. *A function* $u \in \overset{\circ}{W}{}^2_1(\Omega)$, $\|u\|_{L^2(\Omega)} = 1$ *is an eigenfunction with eigenvalue* λ_1 *if and only if*

$$\lambda_1 = \int_\Omega a_{ij} u_{x_i} u_{x_j} \, \mathrm{d}x \,.$$

Lemma 3.2. *Each* u *as in previous Lemma is a.e. positive (or a.e. negative) in* Ω.

Lemma 3.3. *Given* u_1, u_2 *as in previous lemma, there exists* $k \in \mathbb{R}$ *such that* $u_1 = k u_2$.

Example 3.4. Eigenvalues and eigenfunctions of the Laplacian in $\Omega = (0,1) \times (0,1)$:

$$w_{rs}(x,y) = \frac{2}{\pi} \sin(rx) \sin(sy) \,, \quad \lambda_{rs} = r^2 + s^2 \,, \qquad r, s = 1, 2, 3, \dots$$

□

Homework 3.5. (1) Find estimates (above and below) for λ_1 as a function of Ω.
(2) Study the behavior of any orthonormal system as $n \to \infty$.

□

Boundedness of weak solutions to the Dirichlet problem in dimension $N > 2$. Counterexample provided by

$$u(x) = \ln |\ln |x|| \,, \qquad \Omega = \left\{ x \mid |x| < \frac{1}{2} \right\} .$$

Bound of $\|\nabla (u-k)_+\|_{L^2(\Omega)}$ given by measures of level sets.

Reference sections in textbooks: Evans 6.5, DiBenedetto 9.15.

4. Wednesday 2015-03-25

Iterative lemma.
Conclusion of the proof of

Theorem 4.1. *The solution to the Dirichlet problem*

$$\begin{aligned} -(a_{ij}u_{x_i})_{x_j} &= \operatorname{div} f - F\,, && \text{in } \Omega, \\ u &= u_0\,, && \text{on } \partial\Omega. \end{aligned}$$

with $f \in L^{N+\varepsilon}(\Omega)$, $F \in L^{(N+\varepsilon)/2}(\Omega)$, $u_0 \in L^\infty(\partial\Omega)$ *is bounded in* $L^\infty(\Omega)$ *by a constant depending on the norms of the data,* $\varepsilon > 0$, *and* $|\Omega|$.

Uniform approximation with a complete orthonormal system in $L^2(\Omega)$ of functions in a compact set of $L^2(\Omega)$.
Definition of weak solutions to the parabolic Dirichlet problem

$$\begin{aligned} u_t - \operatorname{div} a(x,t,u,\nabla u) &= f(x,t,u)\,, && \text{in } \Omega \times (0,T), \\ u(x,t) &= 0\,, && (x,t) \in \partial\Omega \times (0,T)\,, \\ u(x,0) &= u_0(x)\,, && x \in \Omega\,. \end{aligned}$$

Approximation of solutions via Galerkin's approach. Integral bounds for the approximating functions and their spatial gradients.

Reference sections in textbooks: DiBenedetto 9.15, Notes.

5. Wednesday 2015-04-01

Homework 5.1. Prove that if the equation is linear, the estimates already proved are enough to infer existence of a solution. □

Strong convergence (up to subsequences) of the sequence of Galerkin approximations; weak convergence of the gradients.
Identification of the limit via Minty's lemma. Existence of a solution to the Dirichlet problem for nonlinear parabolic equations.
Self-similar solutions to the heat equation.

Homework 5.2. Find self-similar solutions to the equations:

$$\begin{aligned} u_t - \Delta u^m &= 0\,, && \text{porous media equation;} \\ u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) &= 0\,, && p\text{-Laplacian equation;} \\ u_t - \operatorname{div}(u^{m-1}|\nabla u|^{p-2}\nabla u) &= 0\,, && \text{doubly nonlinear equation.} \end{aligned}$$

□

Reference sections in textbooks: Notes.

6. Wednesday 2015-04-15

The Cauchy problem for the heat equation in $\mathbb{R}^N$.
Theorem of existence of solutions in the strip $\mathbb{R}^N \times (0, T)$ for initial data measures μ of type

$$\int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4T}}\,\mathrm{d}|\mu_y| < +\infty\,,$$

via the representation formula.

Theorem 6.1. (Widder) *Let* $u \in C^{2,1}(\mathbb{R}^N \times (0,T))$, $u \geq 0$ *be a solution to*

$$u_t - \Delta u = 0\,,$$

in $\mathbb{R}^N \times (0,T)$. *Then there exists a unique Radon measure* μ *in* $\mathbb{R}^N$ *such that*

$$u(x,t) = \int_{\mathbb{R}^N} \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x-y|^2}{4t}}\,\mathrm{d}\mu_y\,, \qquad (x,t) \in \mathbb{R}^N \times (0,T)\,.$$

Consequences of the representation formula: infinite speed of propagation, pointwise growth of $u(x,t)$ as $|x| \to \infty$.

Reference sections in textbooks: DiBenedetto 5.14.

7. Wednesday 2015-04-22

Completion of the proof of Widder's theorem:

- Uniqueness of solutions to the Cauchy problem with controlled growth as $|x| \to \infty$ and data in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$.
- Maximum principle for solutions with non-smooth initial data.

Characteristic scaling and self-similar solutions to

$$u_t - \operatorname{div}(u^{m-1} |\nabla u|^{p-2} \nabla u) = 0 \,,$$

in the case $p > 1$, $m + p - 3 \geq 0$.
The Cauchy problem with initial data the Dirac mass. Finite speed of propagation.
Solutions blowing up at a finite time. Optimal growth of the initial data as $|x| \to \infty$.

Reference sections in textbooks: DiBenedetto 5.14, Notes.

8. Wednesday 2015-04-29

Energy inequality for degenerate parabolic equations.

Lemma 8.1. *Let $u \in L^\infty(B_{\rho_2} \times (0,t))$ be a local solution of the p-laplacian equation, $p > 2$.*
Assume further that

$$\frac{\tau}{(\rho_2 - \rho_1)^p} \|u(\tau)\|_{\infty, B_{\rho_2}}^{p-2} \leq 1 \,, \qquad 0 < \tau < t \,,$$

where $0 < \rho_1 < \rho_2$, $0 < t$. Then for all $\sigma > 0$

$$\|u\|_{\infty, B_{\rho_1} \times (t/2, t)} \leq \gamma t^{-\frac{N}{N(p-2)+\sigma p}} \Bigg(\sup_{t/4 < \tau < t} \int\limits_{B_{\rho_2}(\tau)} u^\sigma \,\mathrm{d}x \Bigg)^{\frac{p}{N(p-2)+\sigma p}} .$$

Reference sections in textbooks: Notes.

9. Tuesday 2015-05-05

Case of the porous media equation.
Integral estimates of the gradient up to time $t = 0$ for the p-laplacian equation, $p > 2$.

Theorem 9.1 (Existence under optimal assumptions)**.** *If the initial data $u_0 \geq 0$ satisfies*

$$|||u_0|||_r := \sup_{\rho > r} \rho^{-\frac{p}{p-2}} \fint_{B_\rho(0)} u_0(x) \,\mathrm{d}x < +\infty \,,$$

the Cauchy problem for the p-laplacian equation has a solution in $\mathbb{R}^N \times (0, T_0)$, with $T_0 = \gamma_0 |||u_0|||_r^{-(p-2)}$.

Cases of global solvability.

Theorem 9.2. *The solution $u \geq 0$ to the Cauchy problem for the heat equation and initial data a Radon measure μ satisfies*

$$\|u(t)\|_{\infty, \mathbb{R}^N} < \infty \,, \quad t > 0 \,, \qquad \Leftrightarrow \qquad \sup_{x \in \mathbb{R}^N} \mu(B_1(x)) < \infty \,.$$

Homework 9.3. Prove that if

$$\sup_{x \in \mathbb{R}^N} \int_{B_1(x)} u_0(y) \,\mathrm{d}y < \infty \,,$$

then the Cauchy problem for the porous media equation has a solution satisfying $\|u(t)\|_{\infty, \mathbb{R}^N} < \infty$ for $t > 0$. □

Reference sections in textbooks: Notes.

10. Wednesday 2015-05-06

The Cauchy problem for the equation

$$u_t - \Delta u^m = u^p ,$$

with $p > m > 1$.
Local regularity conditions on the initial data necessary for existence (in terms of Morrey norms and of L^q norms).
The existence of global in time solutions is possible only if $p > m+2/N$.
Local in time existence for data satisfying

$$\sup_{x\in\mathbb{R}^N} \int_{B_1(x)} u_0(y)^q \, \mathrm{d}y < \infty ,$$

with $q > \frac{N}{2}(p-m)$ if $p \geq m + 2/N$, $q = 1$ otherwise.

Theorem 10.1. *The Cauchy problem for the equation above has a global solution in time if $p > m + 2/N$ and*

$$\|u_0\|_{1,\mathbb{R}^N} + \|u_0\|_{q,\mathbb{R}^N} < \delta ,$$

for a suitable $\delta > 0$ and $q > \frac{N}{2}(p-m)$.

11. Monday 2015-05-11

Theorem on global existence in time of the Cauchy problem for the equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 ,$$

for initial data in $L^1(\mathbb{R}^N)$.
Finite speed of propagation in the case of self-similar solutions.

Theorem 11.1. (Finite Speed of Propagation) *Let $u_0 \in L^1(\mathbb{R}^N)$, $\operatorname{supp} u_0$ be compact. Let u be the global solution of the theorem above.*
Then for large t

$$\operatorname{supp} u(t) \subset \left\{ |x| < \gamma t^{\frac{1}{\beta}} \|u_0\|_{1,\mathbb{R}^N}^{\frac{p-2}{\beta}} \right\} .$$

Here $\beta = N(p-2) + p$.

Consequences; optimality of the sup estimates.

12. Wednesday 2015-05-13

Short survey of other methods in PDE.
Free boundary problems; classical and weak formulation of the Stefan problem.