SVOLGIMENTI PROVA SCRITTA di ANALISI 2 del 24/6/2024 - COMPITO A

A1

1) $D_c = \mathbb{R}$. $\varphi_n(x) > 0$

Per $x=0$: $\varphi_n(0) = \dfrac{n}{n+1} \xrightarrow[n\to+\infty]{} 1$

Per $x \neq 0$: $e^{nx} \longrightarrow \begin{cases} +\infty & \text{se } x>0 \\ -\infty & \text{se } x<0 \end{cases}$

Pertanto $\lim_{n\to+\infty} \varphi_n(x) = \begin{cases} 0 & \text{se } x>0 \\ 1 & \text{se } x=0 \\ 1 & \text{se } x<0 \end{cases} = \varphi(x)$

$D_{c\varphi} = \mathbb{R}$.

Non può esserci convergenza uniforme in $\mathbb{R}$, perché $\varphi_n(x) \in C^0(\mathbb{R})$, ma $\varphi(x) \notin C^0(\mathbb{R})$.

In $(0,+\infty)$:

$$|\varphi_n(x) - \varphi(x)| = \varphi_n(x) = \frac{n}{e^{nx}+n}$$

$$\sup_{(0,+\infty)} |\varphi_n(x) - \varphi(x)| \geq \varphi_n\left(\frac{1}{n}\right) = \frac{n}{n+e} \longrightarrow 1 \neq 0$$

NO CONV. UNIF. in $(0,+\infty)$.

$$\sup_{\substack{[\alpha,+\infty)\\ \alpha>0}} |\varphi_n - \varphi| = \sup_{\substack{[\alpha,+\infty)\\ \alpha>0}} \varphi_n \leq \sup_{\substack{[\alpha,+\infty)\\ \alpha>0}} \frac{n}{e^{nx}} = n e^{-n\alpha} \xrightarrow[n\to+\infty]{} 0$$

decrescente

In $(-\infty, 0]$: $\quad$ A2

$$|f_n(x) - f(x)| = \Big| \underbrace{\frac{n}{e^{nx}+n}}_{\leq 1} - 1 \Big| = 1 - \frac{n}{e^{nx}+n}$$

$$|f_n - f|' = \frac{n^2 e^{nx}}{(e^{nx}+n)^2} > 0 \qquad \forall x \in (-\infty, 0]$$

$$\Rightarrow \sup_{(-\infty,0]} |f_n - f| = \cancel{\frac{n}{e^{0}+n}}\ 1 - \frac{n}{n+1} \xrightarrow[n\to+\infty]{} 0$$

$\Rightarrow$ CONV. UNIF. in $(-\infty, 0]$.

Poiché $\varphi_n(x) \not\to 0$ in $(-\infty, 0]$ $\Rightarrow$ NO CONV. PUNTUALE

In $(0, +\infty)$: $\qquad \varphi_n(x) \sim \dfrac{n}{e^{nx}}$

La serie $\sum \dfrac{n}{e^{nx}}$ converge (ad es. per il criterio della radice: $\sqrt[n]{\varphi_n} = \dfrac{\sqrt[n]{n}}{e^x} \to \dfrac{1}{e^x} < 1 \quad \forall x \in (0, +\infty)$)

$D_{cp} = D_{ca} = (0, +\infty)$

CONV. TOTALE: $\qquad \varphi_n(x) = \dfrac{n}{e^{nx}+n}$ decrescente in $(0, +\infty)$

$$\Rightarrow \sup_{(0,+\infty)} |\varphi_n| = \lim_{x\to 0^+} \frac{n}{e^{nx}+n} = \frac{n}{n+1}.$$

$$= \varphi_n(0)$$

$\sum \varphi_n(0) = \sum \frac{n}{n+1}$ diverge $\Rightarrow$ NO CONV. TOT. in $(0,+\infty)$.

$$\sup_{\substack{[\alpha,+\infty) \\ \alpha>0}} |\varphi_n| = \varphi_n(\alpha) = \frac{n}{e^{n\alpha}+n} \leq \frac{n}{e^{n\alpha}}$$

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e $\sum \frac{n}{e^{n\alpha}}$ (come già visto) converge $\forall \alpha>0$

$\Rightarrow$ CONV. TOT. in ogni $[\alpha,+\infty)$, $\alpha>0$.

2) La $f(x,y)$ è sicuramente di classe C^∞ in $\mathbb{R}^2-\{(0,0)\}$, quindi è di classe $C^1(\mathbb{R}-\{(0,0)\})$.

In $(0,0)$:

$$\lim_{(x,y)\to(0,0)} |f(x,y)| = \lim_{\rho\to0} \left| \frac{\cos(\rho^2)-e^{\rho^2\cos^2\vartheta}}{\rho} \right|$$

$$= \lim_{\rho\to0} \left| \frac{\cancel{1}-\frac{\rho^4}{2}+o(\rho^4)-\cancel{1}-\rho^2\cos^2\vartheta+o(\rho^2)}{\rho} \right|$$

$$= \lim_{\rho\to0} \left| \frac{-\rho^2\cos^2\vartheta+o(\rho^2)}{\rho} \right| \leq \lim_{\rho\to0} \rho = 0$$

f CONTINUA in $(0,0)$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to0} \frac{\cos(h^2)-e^{h^2}}{h|h|} = \lim_{h\to0} \frac{-h^2}{h|h|} \quad \nexists$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to0} \frac{\cos(k^2)-1}{k|k|} = \lim_{k\to0} \frac{\left(-\frac{k^4}{2}\right)}{k|k|}$$

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$$= -\frac{1}{2}\lim_{k\to 0} \frac{k^{\cancel{2}}|k|^{\cancel{2}}}{\cancel{|k||k|}} = -\frac{1}{2}\lim_{k\to 0} k|k| = 0.$$

Poichè $\nexists\ f_x(0,0)$, f NON è differenziabile in $(0,0)$, quindi NON è C^1 in $(0,0)$.

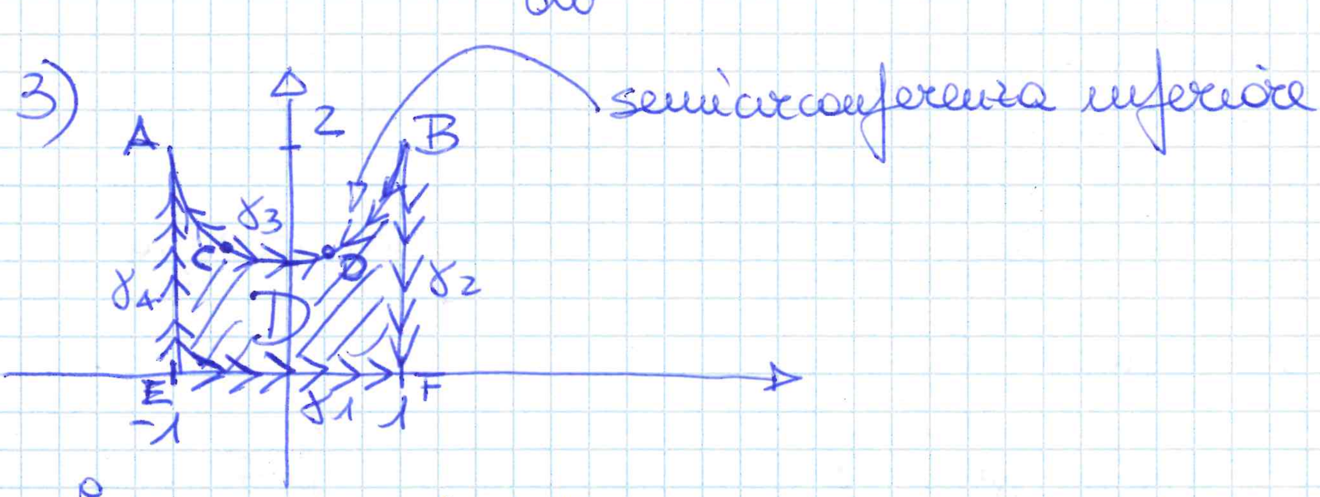
$$\frac{df}{d\vec{v}}(0,0) = \lim_{t\to 0} \frac{\cos(t^2) - e^{\alpha^2 t^2}}{t|t|}$$

$$= \lim_{t\to 0} \frac{-\alpha^2 t^2}{t|t|}$$

Se $\alpha = 0$ $\quad \frac{df}{d\vec{v}}(0,0) = 0$ $\quad$ (UNICA DER. DIR: infatti $f_y(0,0)=0$)

Se $\alpha \neq 0$ $\quad \nexists\ \frac{df}{d\vec{v}}(0,0)$.

3)



In $\mathring{D}$: $\begin{cases} f_x = (y-2)^2 = 0 \\ f_y = 2x(y-2) = 0 \end{cases} \Rightarrow \begin{cases} y = 2 \\ x \in \mathbb{R} \end{cases}$

Pertanto gli unici punti stazionari si trovano sulla frontiera di D.

Anche se non richiesto, si osservi che il segno di f è

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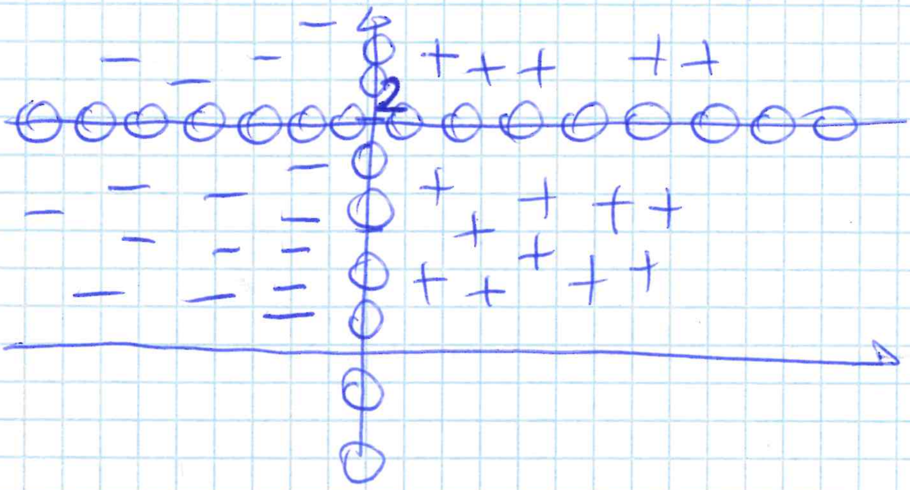

Quindi i punti stazionari liberi $(x,2)$ sono di MAX. REL. per $x<0$; di MIN. REL. per $x>0$; di SELLA $x=0$.

Dobbiamo pertanto aspettarci che, anche sulla ∂D, $B\equiv(1,2)$ sia di MIN. REL. e $A\equiv(-1,2)$ sia di MAX. REL.

Inoltre, si osservi che D è SIMMETRICO rispetto all'asse y, mentre f è ANTISIMMETRICA:

$$f(-x,y) = -x(y-2)^2 = -f(x,y).$$

Quindi nei punti simmetrici rispetto all'asse y il comportamento di f è opposto.

$$\partial D = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4.$$

$\gamma_1 : \begin{cases} x\in[-1,1] \\ y=0 \end{cases}$; $\gamma_2 : \begin{cases} x=1 \\ y\in[0,2] \end{cases}$; $\cancel{\gamma_3 :}$

$$f\Big|_{\gamma_3} = f(x, 2-\sqrt{1-x^2}) = x\left(-\sqrt{1-x^2}\right)^2 = x(1-x^2)$$
$$= x - x^3$$

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$$\left(f\Big|_{\gamma_3}\right)' = 1 - 3x^2 \geq 0 \iff -\frac{1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{3}}$$

Se $x = \pm\frac{1}{\sqrt{3}} \Rightarrow y = 2 - \sqrt{\frac{2}{3}} > 0$

$$f\Big|_{\gamma_4} = -(y-2)^2$$ (parabola con vertice in $y=2$ e rivolta verso il basso)

$A \equiv (-1,2)$ di MAX. REL $\Rightarrow B \equiv (1,2)$ di MIN. REL.
(per antisimmetria)

$f(-1,2)=0$ $\quad f(1,2)=0$

$C \equiv \left(-\frac{1}{\sqrt{3}}, 2-\sqrt{\frac{2}{3}}\right)$ di MIN. REL.

$\Rightarrow D = \left(\frac{1}{\sqrt{3}}, 2-\sqrt{\frac{2}{3}}\right)$ di MAX. REL.

$$f\left(\pm\frac{1}{\sqrt{3}}, 2-\sqrt{\frac{2}{3}}\right) = \pm\frac{1}{\sqrt{3}}\cdot\frac{2}{3}$$

$E = (-1,0)$ di MIN. REL. $\Rightarrow F \equiv (1,0)$ di MAX. REL.

$f(\pm 1, 0) = \pm 4$ $\quad$ (si osservi che $0 < \frac{2}{3\sqrt{3}} < 2$)

$\Rightarrow$ MAX. ASS. in $(1,0)$: $f(1,0) = 4$

MIN. ASS. in $(-1,0)$: $f(-1,0) = -4$.

4) Ⓐ7

$$\vec{\nabla}\wedge\vec{F}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ x-y & y+z & x^2z\end{vmatrix}=-\hat{\imath}-2xz\,\hat{\jmath}+\hat{k}$$

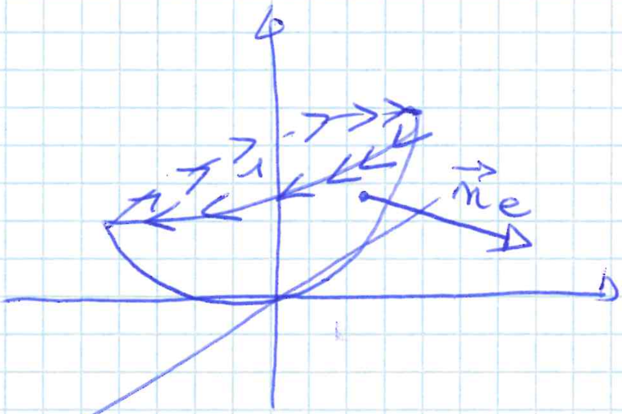
Se calcoliamo direttamente il flusso, allora

$$S:\begin{cases}z=x^2+y^2\\ x^2+y^2\le 1\end{cases}\quad;\quad J=\begin{pmatrix}1 & 0 & 2x\\ 0 & 1 & 2y\end{pmatrix}$$

$$(L,M,N)=\pm(-2x,-2y,1)$$

Serve $N<0$

$$\Rightarrow (L,M,N)=(2x,2y,-1)$$



$$(\vec{\nabla}\wedge\vec{F})\cdot\vec{n}_e=-2x-4xyz-1$$

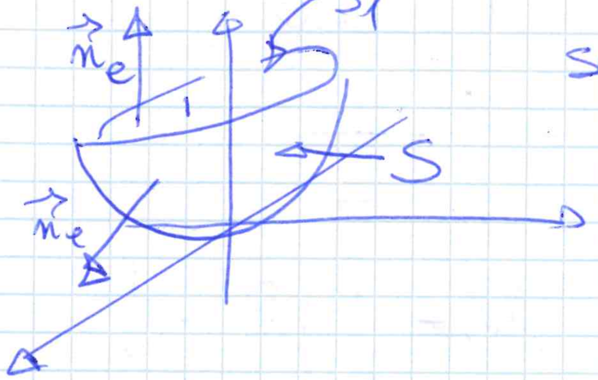
$$\Phi_S(\vec{\nabla}\wedge\vec{F})=-\iint_{D(x,y)}\left(+2x+4xy(x^2+y^2)+1\right)dx\,dy$$

$$=-\int_0^1\rho\,d\rho\int_0^{2\pi}\left[2\rho\cos\vartheta+4\rho^4\cos\vartheta\sin\vartheta+1\right]d\vartheta$$

$$=-\int_0^1 d\rho\left[\rho\left(\cancel{2\rho\sin\vartheta}+\cancel{4\rho^4\frac{\sin^2\vartheta}{2}}+\vartheta\right)_0^{2\pi}\right]$$

$$=-2\pi\int_0^1\rho\,d\rho=-2\pi\frac{\rho^2}{2}\Big|_0^1=-\pi$$

In alternativa, considerando la superficie CHIUSA $\Sigma = S \cup S_1$,



sappiamo che

(A8)

$$0 = \Phi_\Sigma = \Phi_{S \cup S_1} = \Phi_S + \Phi_{S_1}$$

$$\Rightarrow \underbrace{\Phi_S = -\Phi_{S_1}}$$

Ma $S_1 : \begin{cases} x^2+y^2 \le 1 \\ z = 1 \end{cases} \Rightarrow \cancel{\begin{cases} z=1 \\ x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad \begin{matrix} \rho \in [0,1] \\ \theta \in [0,2\pi] \end{matrix}}$

e $\vec{n}_e = (0,0,1)$

$$\Rightarrow (\vec{\nabla} \wedge \vec{F}) \wedge \vec{n}_e \Big|_{S_1} = (-1, -2\rho\cos\theta, 1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1$$

$$\Rightarrow \Phi_S = -\Phi_{S_1} = -\iint_{D(x,y)} 1 \, dx\,dy = -\text{Area}(S_1) = -\pi.$$

Applicando il Teorema del Rotore, invece,

$$\Phi_S(\vec{\nabla} \wedge \vec{F}) = \oint_{BS^+} \vec{F} \cdot \vec{ds} =$$

$$BS: \begin{cases} z=1 \\ x=\cos\vartheta \\ y=\sin\vartheta \end{cases} \qquad \vartheta\in[0,2\pi]$$

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BS^+ è CONTROVERSO rispetto al verso di crescita di ϑ

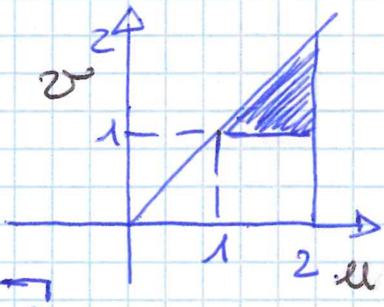
$$\Rightarrow \Phi_S(\vec{\nabla}\wedge\vec{F}) = -\int_0^{2\pi}\Big[(\cos\vartheta-\sin\vartheta)(-\sin\vartheta)$$

$$+(1+\sin\vartheta)\cos\vartheta\Big]d\vartheta$$

$$= -\int_0^{2\pi}\left[2\sin\vartheta\cos\vartheta+\sin^2\vartheta+\cos\vartheta\right]d\vartheta$$

$$= -\left[\cancel{\frac{-\cos 2\vartheta}{2}}+\frac{1}{2}\left[\vartheta-\cancel{\frac{1}{2}\cos 2\vartheta}\right]\cancel{+\sin\vartheta}\right]_0^{2\pi} = -\pi.$$

5) $$J=\begin{pmatrix} 1 & 1 & v \\ 2v & -2v & u \end{pmatrix}$$



$$W(u,v)=\sqrt{(u+2v^2)^2+(2v^2-u)^2+(4v)^2}$$

$$=\sqrt{2u^2+8v^4+16v^2}$$

$$f\big(x(u,v),y(u,v),z(u,v)\big)=\frac{uv}{\sqrt{\frac{(2u)^2}{2}+2\left(2v^2\right)^2+8\left(2v^2\right)}}$$

$$=\frac{uv}{\sqrt{2u^2+8v^4+16v^2}}$$

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$$\Rightarrow \int f\,dS=\int_1^2 du\int_1^u dv\;\cancel{\frac{uv}{\sqrt{2u^2+8v^4+16v^2}}\cdot\sqrt{2u^2+8v^4+16v^2}}$$

$$=\int_1^2 du\cdot u\left[\frac{v^2}{2}\right]_1^u=\frac{1}{2}\int_1^2\left[u^3-u\right]du$$

$$=\frac{1}{2}\left[\frac{u^4}{4}-\frac{u^2}{2}\right]_1^2=\frac{1}{2}\left[\left(\frac{16}{4}-\frac{4}{2}\right)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]$$

$$=\frac{1}{2}\left[2+\frac{1}{4}\right]=\frac{9}{8}$$