Quasilinear singular elliptic equations

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Abstract

In this note we present a short review on known results about some elliptic equations having a lower order term $b(x, u, Du)$ growing quadratically in the Du-variable and singular in the u-variable. We will assume homogeneous Dirichlet boundary conditions. We also give an extension of the existence result given in [10] and discuss some applications to homogenization.

1 Introduction

Recently singular elliptic equations with homogeneous Dirichlet boundary conditions has attracted the attention of several authors. We refer to the model problem

$$(E)\begin{cases} -\Delta\, u = b(x, u, Du) + f(x) & \text{in} \quad \Omega \\ u = 0 \qquad \text{on} & \partial\Omega \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and $b(x, s, \xi) : \Omega \times \mathbb{R} - \{s_0\} \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function, singular in the s variable at a point s_0, growing at most quadratically in the ξ variable. The known results about existence of solutions belonging to $H_0^1(\Omega)$ or $H^1_{loc}(\Omega)$ depend on
• the fact that $s_0 = 0$ or $s_0 \neq 0$,
• the sign of the datum $f(x)$ in the case that $s_0 = 0$,
• the sign and the size of the lower order term $b(x, s, \xi)$,
• the growth of the function $b(x, s, \xi)$ near the singularity,
• the regularity of the datum $f(x)$.
The case where $b(x, s, \xi)$ is continuous in the s-variable has been widely studied both in the stationary and in the evolution case, beginning by the early papers [7],[8].

The interest in studying the case where $b(x,s,\xi)$ has a singular behaviour in s has several motivations.

It relies, first of all, on the fact that, confining to the case where $s_0 = 0$ and to non-negative data $f(x)$ and functions $b(x,s,\xi)$, the equation (E) looks like a simplified version of the Euler's equation for functional of the type

$$I[u] = \int_\Omega u^\alpha |Du|^2 - \int_\Omega f u\,,$$

with $\alpha \in (0,1)$.

Moreover, let us consider equations of the type

$$u_t - \Delta(u^m) = |Du|^q + f$$

with $m > 1$ and $1 < q \le 2$. which represents a model of growth in porous media. If we consider steady states solutions and we perform a change of unknown $u^m = v$, we get an equation with singular behaviour in v, growing as q in the Du-variable.

Another motivation in studying problems like (E) is their connection with existence of boundary blow-up solutions for semilinear problems.
Let us begin by some remark about the case where the singularity is placed at $s_0 = 0$.
Note that, in this case the term $b(x,s,\xi)$ is singular at each point of the boundary of Ω. If we are dealing with data $f(x) \ge 0$, the solution will lay at the right hand side of the singularity ($u \ge 0$) whatever is the sign of $b(x,s,\xi)$.
If $f(x) \ge 0$ and also $b(x,s,\xi) \ge 0$, the strong maximum principle guarantees that u is strictly positive inside of Ω and the lower order term is completely well define. The same holds true if $f(x) \ge 0$ and $b(x,s,\xi) \le 0$, by a deeper use of the strong maximum principle ([5]).
In the case where $f(x)$ or $b(x,s,\xi)$ can change sign, the solution u can vanishes inside Ω (and actually this occurs in some situation). Therefore, we have to define carefully the meaning of solution (see Theorem 1).
Referring to the case $|b(x,s,\xi)| \le B\frac{|\xi|^2}{|s|^k}$ with $k > 0$ and $f(x) \ge 0$, we have different results about existence and non existence of solutions in $H^1_0(\Omega)$or in $H^1_{loc}(\Omega)$ ([10],[5],[1], [3],[12] [6],[11],[2]), depending on the order k of the singularity near $s = 0$, combined with the sign of $b(x,s,\xi)$ and $f(x)$ and the regularity of f.

Recall that, if a lower order term appears in the form $g(u)|Du|^2$, test functions involving terms like $e^{\gamma(u)}$, where $\gamma(s)$ is a primitive function of $g(s)$, are often used in order to get a-priori estimates for solutions. The use of these test functions simulate the so called Cole-Hopf transformation which can be applied when the problem has a model structure and which allows to get rid of the term $b(x,u,Du)$ getting a semiliner problem. Note that, if $b(x,s,\xi) = B\frac{|\xi|}{|s|^k}$ and $k \in (0,1)$, the function $g(s) = \frac{1}{|s|^k}$ is an L^1-function near the singularity $s = 0$ so that $e^{\gamma(u)}$ is bounded near the singularity .

In the case $k \ge 1$, the function $g(s) = \frac{1}{|s|^k}$ is no more integrable near $s = 0$. Neverthenless, if $f \ge 0$, $e^{\gamma(u)}$ is still bounded for $s \in (0,\beta], \beta \in \mathbb{R}$, since $\gamma(s)$ is negative near $s = 0$ in such an interval. We refer to [10] for existence of solutions.

If we have a changing sign datum $f(x)$ and $k \geq 1$, the solution can cross the singularity and the function $e^{\gamma(u)}$ is unbounded near the singularity for $s < 0$, so that we are in trouble with a priori estimates near $s = 0$; this case is, up to now, an interesting open problem.
In the case where the singularity is placed at a point $s_0 \neq 0$ with homogeneous Dirichlet boundary conditions, no singularity appears on the boundary, but the sign of the datum f do not provide any information about the position of the possible solution with respect the singularity.
Very few results are available in this case (see [6], [11]).
In the following Section 2 we will present a list of results on the subject.
In Section 3 we will give an extension of the existence result given in [10].

2 Known results

2.1 Lower order term singular at $s_0 = 0$: existence results

Let us present as a first result on the subject the following theorems (see [10]), which deal with the case of lower order terms $b(x, u, Du)$ singular at $u = 0$.
We consider the problem

$$\begin{cases} -div(a(x, u, Du)) + \lambda u = b(x, u, Du) + f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is an open bounded set of $\mathbb{R}^N$, where

$$\lambda > 0, \tag{2.2}$$

$$f(x) \in L^\infty(\Omega), \quad f(x) \geq 0, \tag{2.3}$$

where the function

$$a(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

is a Carathéodory function which satisfies

$$a(x, s, \xi)\xi \geq \alpha|\xi|^2, \quad \alpha > 0, \tag{2.4}$$

$$|a(x, s, \xi)| \leq \nu|\xi|, \quad \nu > 0, \tag{2.5}$$

$$(a(x, s, \xi) - a(x, s, \eta))(\xi - \eta) > 0, \quad \forall \xi \neq \eta\ . \tag{2.6}$$

for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$, and the function

$$b(x, s, \xi) : \Omega \times (\mathbb{R} - \{0\}) \times \mathbb{R}^N \to \mathbb{R}$$

is a Carathéodory function on $\Omega \times (\mathbb{R} - \{0\}) \times \mathbb{R}^N$ satisfying for every $\xi \in \mathbb{R}^N$, for every s and for almost every $x \in \Omega$, **either**

$$|b(x, s, \xi)| \leq C_2 \frac{1}{|s|^k}|\xi|^2, \quad C_2 > 0, \quad 0 < k < 1 \tag{2.7}$$

or

$$C_1 \frac{1}{|s|^k}|\xi|^2 \le b(x,s,\xi) \le C_2 \frac{1}{|s|^k}|\xi|^2, \quad C_1 > 0, C_2 > 0, \quad k \ge 1. \tag{2.8}$$

Note that (2.8) is much more restrictive than (2.7), since (2.8) is a growth condition for $b(x,s,\xi)$ both from above and from below. In particular, when (2.7) holds true, $b(x,s,\xi)$ is not assumed to have a specified sign, while $b(x,s,\xi)$ has to be strictly positive (for $\xi \neq 0$) when (2.8) holds true.

Let $M > 0$ and $\beta : (0, M] \to \mathbb{R}$ be defined by

$$M = \frac{\|f\|_\infty}{\lambda}, \quad \beta(s) = \frac{1}{s^k}\,. \tag{2.9}$$

In the case where $0 < k < 1$, the function β belongs to $L^1(0, M)$, while in the case where $k \ge 1$, the function β is not integrable in 0.

Let us introduce the following function $\gamma(s)$, defined for $s \in (0, M]$, which is a primitive function of the function $\frac{C_2}{\alpha}\beta(s)$, defined by

$$\gamma(s) = \begin{cases} \frac{C_2}{\alpha}\frac{s^{1-k}}{1-k} & \text{if} \quad 0 < k < 1, \\[2ex] \frac{C_2}{\alpha}\ln(\frac{s}{M}) & \text{if} \quad k = 1, \\[2ex] \frac{C_2}{\alpha}\frac{M^{1-k}-s^{1-k}}{k-1} & \text{if} \quad k > 1. \end{cases} \tag{2.10}$$

Let us finally define, for $s \in [0, M]$, the function ψ by

$$\Psi(s) = \int_0^s e^{\gamma(\sigma)} d\sigma, \tag{2.11}$$

and, for $m > 0$ and $s \in \mathbb{R}$, the function S_m by

$$S_m(s) = \begin{cases} m & \text{if } s \le m, \\ s & \text{if } s \ge m. \end{cases}$$

Let us point out that, in the case where $0 < k < 1$, $\gamma(s)$ is an increasing, non negative bounded function on $[0, M]$, while in the case where $k \ge 1$, $\gamma(s)$ is an increasing, non positive function on $(0, M]$ with $\lim_{s\to 0^+} \gamma(s) = -\infty$.

In both cases $e^{\gamma(s)}$ is a bounded function on $[0, M]$ and, therefore, the function $\Psi(s)$ is well defined by (2.11).

Theorem 2.1

Suppose that (2.2)-(2.6) and (2.7) hold true. Then, there exists at least a function u such that

$$u \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad u \ge 0, \tag{2.12}$$

$$\Psi(u) \in H_0^1(\Omega), \quad \frac{|Du|^2}{u^k}\chi_{u>0} \in L^1(\Omega), \tag{2.13}$$

and

$$\int_\Omega a(x,u,Du)D\Phi + \lambda\int_\Omega u\Phi = \int_\Omega b(x,u,Du)\chi_{u>0}\Phi + \int_\Omega f\Phi, \ \ \forall\,\Phi \in C_c^\infty(\Omega). \tag{2.14}$$

Theorem 2.2

Suppose that (2.2)-(2.6) and (2.8) hold true. Then there exists at least a function u such that

$$u \in H^1_{\text{loc}}(\Omega) \cap L^\infty(\Omega), \quad u > 0, \tag{2.15}$$

$$S_m(u) \in H^1(\Omega), \quad \forall\, m > 0, \quad \Psi(u) \in H_0^1(\Omega), \quad \frac{|Du|^2}{u^k} \in L^1_{\text{loc}}(\Omega), \tag{2.16}$$

$$\int_\Omega a(x,u,Du)D\Phi + \lambda\int_\Omega u\Phi = \int_\Omega b(x,u,Du)\Phi + \int_\Omega f\Phi, \ \ \forall\,\Phi \in C_c^\infty(\Omega). \tag{2.17}$$

Sketch od the proofs of Theorem 2.1 and Theorem 2.2.

We recall that Theorem 2.1 deals with the case where the lower order term $b(x,s,\xi)$ behaves as $\frac{|Du|^2}{u^k}$ with $0 < k < 1$ and does not satisfy any sign condition, while Theorem 2.2 deals with the case where $b(x,s,\xi)$ has a stronger singularity, namely $k \geq 1$, but it has also a strict sign. We just give the main steps of the proofs, showing the test functions used in each step and trying to point out the differences between the two situations, $k < 1$ and $k \geq 1$.

Step 0. The approximating problems. They are defined by

$$\begin{cases} -div(a(x,u_n,Du_n)) + \lambda u_n = b(x, S_{\frac{1}{n}}(u_n), Du_n) + f(x) \ \text{ in}\,\Omega, \\ u_n = 0 \qquad \text{on } \ \partial\Omega, \end{cases} \tag{2.18}$$

where, for each $n \in \mathbb{N}$,

$$S_{\frac{1}{n}}(s) = \begin{cases} \frac{1}{n} & s \leq \frac{1}{n}, \\ \\ s & s \geq \frac{1}{n}. \end{cases}$$

There exists $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega), u_n \geq 0$ solution to (2.18).
We define $\beta_n(s) = \beta(S_{\frac{1}{n}}(s))$, $\gamma_n(s)$ its primitive and $\Psi_n(s) = \int_0^s e^{\gamma_n(\sigma)}d\sigma$.

Step 1. Uniform estimates on u_n in $L^\infty(\Omega)$. We use as test function in (2.18),

$$e^{\gamma_n(u_n)}(u_n - M)_+ \ , \quad M = \frac{||f||_\infty}{\lambda}.$$

Step 2. Uniform estimates on $\Psi_n(u_n)$ in $H_0^1(\Omega)$, i.e.

$$\int_\Omega |D\Psi_n(u_n))|^2 = \int_\Omega |Du_n|^2 e^{2\gamma_n(u_n)} \leq C.$$

We use as test function in (2.18),

$$v_n = e^{\gamma_n(u_n)}\Psi_n(u_n).$$

Step 3. Uniform estimates on $b(x, S_{\frac{1}{n}}(u_n), Du_n)$ in $L^1(\Omega)$ $(k<1)$ or in $L^1_{loc}(\Omega)$ $(k \geq 1)$. We use as test function in (2.18),

$$v_n = \begin{cases} e^{\gamma_n(u_n)} - 1 & k < 1, \\ (e^{\gamma_n(u_n)} - 1)\eta^2(x) & k \geq 1, \end{cases}$$

where $\eta(x)$ is a cut-off function.

Step 4. Uniform estimates on u_n in $H^1_0(\Omega)$ $(k<1)$ or in $H^1_{loc}(\Omega)$ $(k \geq 1)$.
This easily comes from Step 2 if $k<1$ and from Step 3 if $k \geq 1$ and implies

$$u_n \to u \quad \text{a.e.} \quad \text{in} \quad \Omega.$$

Step 5. For every $m > 0$, $\quad DS_m(u_n) \to DS_m(u)$ in $(L^2(\Omega))^N$ $(k<1)$, or in $(L^2_{loc}(\Omega))^N$ $(k \geq 1)$. We skip this point.

Step 6. One proves that

$$\lim_{m\to 0} \int_{C\cap\{u_n \leq m\}} b(x, S_{\frac{1}{n}}(u_n), Du_n) = 0$$

uniformly in n, for any compact set C in Ω. We use as test function in (2.18),

$$v_n = \begin{cases} -\eta^2(x)(e^{\gamma_n(m)-\gamma_n(u_n)} - 1)_+ & k < 1, \\ -\eta^2(x)(e^{\gamma_n(u_n)-\gamma_n(m)} - 1)_- & k \geq 1. \end{cases}$$

Step 7. Equi-integrability of $b(x, S_{\frac{1}{n}}(u_n), Du_n)$ on compact subsets C of Ω. To prove that, we write

$$\int_E b(x, S_{\frac{1}{n}}(u_n), Du_n) = \int_{E\cap\{u_n \leq m\}} + \int_{E\cap\{u_n > m\}}$$

for any subset E of C. The first term is small uniformly in n, for m sufficiently small while

$$\int_{E\cap\{u_n > m\}} b(x, S_{\frac{1}{n}}(u_n), Du_n) \leq \frac{1}{m^k}\int_E |DS_m(u_n)|^2$$

is small uniformly in n, for $|E|$ sufficiently small and fixed m (due to the strong convergence of $|DS_m(u_n)|$ in $L^2(\Omega)$).

Step 8. Passing to the limit.

Let us focus our attention on the term

$$\int_\Omega b(x, S_{\frac{1}{n}}(u_n), Du_n)\Phi = \int_{u>0} b(x, S_{\frac{1}{n}}(u_n), Du_n)\Phi + \int_{u=0} b(x, S_{\frac{1}{n}}(u_n), Du_n)\Phi.$$

We easily pass to the limit in the first integral (by a.e. convergence and equi-integrability) getting $\int_{u>0} b(x,u,Du)\Phi$.

Moreover we prove that, on the compact set C,

$$\lim_{n\to+\infty}\int_{C\cap\{u=0\}} b(x,S_{\frac{1}{n}}(u_n),Du_n)=0.$$

Indeed,

$$\int_{C\cap\{u=0\}} b(x,S_{\frac{1}{n}}(u_n),Du_n)=\int_{C^\epsilon\cap\{u=0\}} b(x,S_{\frac{1}{n}}(u_n),Du_n)$$
$$+\int_{(C-C^\epsilon)\cap\{u=0\}} b(x,S_{\frac{1}{n}}(u_n),Du_n).$$

Here C^ϵ is a subset of C such that in $C-C^\epsilon$ the sequence b_n converges uniformly and whose size is sufficiently small (using the Severini-Egoroff Theorem). Then, for fixed ϵ,the first integral is less than $\epsilon/2$ by equi-integrability.

The second one can be bounded by

$$\int_{(C-C^\epsilon)\cap\{u_n\le m\}} b(x,S_{\frac{1}{n}}(u_n),Du_n)\le\epsilon/2,$$

for every $n\ge n_o(m(\epsilon))=n_0(\epsilon)$, since we have proved that

$$\lim_{m\to 0}\int_{C\cap\{u_n\le m\}} b(x,S_{\frac{1}{n}}(u_n),Du_n)=0.$$

Note that, after passing to the limit, the function $\chi_{u>0}$ appears in the limit equation. In the case of Theorem 2.2 , due to the strong maximum principle, we have, a posteriori, $\chi_{u>0}=1$ a.e. in Ω and the proof is over. $\square$

Let us recall now some other existence results concerning this subject.
Let us first consider problems whose model is

$$-\alpha\Delta\,u+\frac{1}{u^k}|Du|^2=f$$

with $\alpha>0$, $f\ge 0$.
Note that the lower order term is placed here on the right hand side of the equation and it is non negative.
In [3] the authors prove that one has existence of finite energy solutions for every f bounded, strictly positive on compact sets if and only if $0<k<2$.
In the case that $f(x)$ simply satisfies $f(x)\ge 0$, in [5] the author proves existence of finite energy solutions for any coercivity constant $\alpha>0$ in the case where $0<k<1$ and $f\in L^m$, with $m\ge(\frac{2^*}{k})'$; existence is proved for $\alpha>2$, if $k=1$ and $f\in L^m$, with $m\ge\frac{2N}{N+2}$.
Let us now consider equations which look like

$$-\alpha\Delta\,u=\frac{B}{u}|Du|^2+f$$

with $\alpha > 0$, $f \geq 0$. The singular term appears at the right hand side of the equation and the order of the singularity is $k = 1$. In [1], the authors prove that, in general, solutions do not belong to $H_0^1(\Omega)$ unless $B < \alpha$. Existence results in different spaces are proved, depending on the ratio $\frac{B}{\alpha}$ and on the regularity of the datum $f(x)$.
Finally, the case where the datum $f(x)$ has a general sign and the lower order term grows like $\frac{1}{|u|^k}|Du|^2$ with $k < 1$ is studied in [12]. Note that in this case the possible solutions can cross the singularity so that we have to define carefully the meaning of solution in the same spirit of Theorem 2.1 , where the solutions could be zero somewhere.

2.2 More general growth of the lower order term in the gradient variable

A possible model of growth in a porous medium is given by the equation

$$v_t - \Delta(v^m) = |Dv|^q + \lambda f,$$

where $1 < q \leq 2$ and $m > 0$.
In [2] the authors study the corresponding stationary problem with a homogeneous Dirichlet boundary condition.

Performing the change of unknown $v^m = u$ the problem becomes

$$\begin{cases} -\Delta u = u^{q\alpha}|Du|^q + \lambda f(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.19}$$

Here $N \geq 3$, $\Omega \subset R^N$ is a bounded domain, $1 < q \leq 2$, $\alpha = \frac{1}{m} - 1 \in (-1, \infty)$ and $f \geq 0$. Note that the behaviour of the problem for $\alpha \in (-\infty, -1]$ has interest in itself, even if we are no more in the framework of porous media equations, so that in the following result the general case $\alpha \in (-\infty, +\infty)$ will be considered.

Theorem 2.3

1. If $q\alpha < -1$, then the above problem (2.19) has a distributional solution in $W_{loc}^{1,p}(\Omega), p < \frac{N}{N-1}$, for all $f \in L^1(\Omega)$, and all $\lambda > 0$.

2. If $-1 \leq q\alpha < 0$, then problem (2.19) has a solution for all $f \in L^{\frac{2N}{N+2}}(\Omega)$ and without any restriction on λ (infinitely many for $q = 2$).

3. If $0 \leq q\alpha$ the problem (2.19) has a positive solution for λ small and $f \in L^1(\Omega)$.

2.3 Homogenization problem

We are interested in the asymptotic behaviour, as $\varepsilon \to 0$, of the following problem:

$$\begin{cases} -div(A^\varepsilon Du_\epsilon) = \dfrac{b_\varepsilon(x, Du_\varepsilon)}{u_\varepsilon^k} + f(x) & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega, \end{cases} \tag{2.20}$$

where $0 < k < 1, f(x) \in L^\infty(\Omega), f(x) \geq 0$,

$$\begin{aligned} &i) \quad 0 \leq b_\epsilon(x,\xi) \leq c_1|\xi|^2, \\ &ii) \quad |b_\epsilon(x,\xi) - b_\epsilon(x,\xi_1)| \leq c_2(|\xi| + |\xi_1|)|\xi - \xi_1| \end{aligned}$$

and $\{A^\varepsilon\}$ is a sequence of matrices in the class

$$M(\alpha,\beta) = \{A \in (L^\infty(\Omega))^{n^2} : (A(x)\lambda,\lambda) \geq \alpha|\lambda|^2,\ |A(x)\lambda| \leq \beta|\lambda|\}$$

which H-converges to A^0 in the following sense ([16], [13], [17], [14])

Definition 1 *$\{A^\varepsilon\}$ H-converges to A^0 if for any g in $H^{-1}(\Omega)$, the solution v_ε of*

$$\begin{cases} -div(A^\varepsilon Dv_\varepsilon) = g \ on\ \Omega, \\ v_\varepsilon = 0 \ on\ \partial\Omega, \end{cases}$$

satisfies the weak convergences

$$\begin{cases} v_\varepsilon \rightharpoonup v \ weakly\ in\ H_0^1(\Omega), \\ A^\varepsilon Dv_\varepsilon \rightharpoonup A^0 Dv \ weakly\ in\ (L^2(\Omega))^n, \end{cases}$$

where $v \in H_0^1(\Omega)$ solves

$$\begin{cases} -div(A^0 Dv) = g \ on\ \Omega, \\ v = 0 \ on\ \partial\Omega. \end{cases}$$

Theorem 2.4

The solutions u_ϵ of (2.20) weakly converge in $H_0^1(\Omega)$ (up to a subsequence) to a function u_0 satisfying

$$\begin{cases} u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad u_0 > 0, \quad \dfrac{|Du_0|^2}{u_0^k} \in L^1(\Omega), \\ \displaystyle\int_\Omega A^0 Du_0 D\Phi = \int_\Omega \frac{b_0(x, Du_0)}{u_0^k}\,\Phi + \int_\Omega f\Phi, \\ \forall\, \Phi \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases}$$

As we can see, the limit problem consists of a principal part which is the H-limit of the principal parts and of a lower order term in which a function $b_0(x,\xi)$ appears: it is constructed from the corresponding terms b_ϵ in (2.20), using linear correctors C^ε.
The linear correctors C^ε are matrices satisfyng

$$\lim_{\epsilon\to 0} \|Dv_\varepsilon - C^\varepsilon Du_0\|_{(L^1(\Omega))^n} = 0$$

where v_ϵ solves

$$\begin{cases} -div(A^\varepsilon Dv_\varepsilon) = -div(A^0 Du_0) \text{ on } \Omega, \\ v_\varepsilon = 0 \text{ on } \partial\Omega. \end{cases}$$

Note that

$$v_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega),$$

The main point in the proof is to prove that actually C^ε is also a corrector for the nonlinear problem (2.20), i.e.

$$\lim_{\epsilon\to 0} \|Du_\varepsilon - C^\varepsilon Du_0\|_{(L^1(\Omega))^n} = 0$$

where u_0 is the weak limit of u_ϵ.

This situation applies in the framework of composite materials which are materials containing two or more finely mixed components: the parameter ϵ describes the heterogeneities of the material. In a good composite, the heterogeneities are very small compared with the global dimension of the sample. Smaller are the heterogeneities, better is the mixture, which appears then, at a first glance, as a homogeneous material.
From mathematical point of view, we try to find the limit problem, as $\epsilon \to 0$, which will be the model for the homogenized material.

2.4 Lower order term singular at a point $s_0 \neq 0$

We consider now the case where $b(x,s,\xi)$ is singular at a point $s_0 \neq 0$, for example at $s_0 = 1$. The model problem is:

$$\begin{cases} -\Delta u = \pm \dfrac{sign(u-1)}{|u-1|^k}|Du|^2 + f \text{ in } \Omega, \\ \\ u = 0 \text{ on } \partial\Omega\,. \end{cases} \tag{2.21}$$

and we assume $0 < k < 1$. Note that the problem is no more singular at the boundary, that sign hypotheses on the datum f does not help as in the case of $s_0 = 0$ and that we are considering a gradient term changing its sign at the singularity.
In [11] existence of solutions for every datum belonging to a suitable Lebesgue

space it has been proved. Furthermore, it is proved that the solution pass through the singularity when data are big enough at least in the case of " $-$ " sign.

In the following theorem the case of " $+$ " sign in (2.21) is considered.

Theorem 2.5

If $f(x) \in L^m(\Omega), m \geq \frac{N}{2}$, there exists $u \in H_0^1(\Omega)$: $\frac{|Du|^2}{|u-1|^k} \in L^1(\Omega)$ and $\forall\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega DuD\varphi = \int_\Omega \frac{sign(u-1)}{|u-1|^k}|Du|^2\varphi + \int_\Omega f\varphi$$

As far as the case of " $-$ " sign in (2.21) is concerned, we have the following result.

Theorem 2.6

If $f(x) \in L^m(\Omega), m \geq \frac{2N}{N+2}$, then there exists $u \in H_0^1(\Omega)$: $\frac{|Du|^2}{|u-1|^k} \in L^1(\Omega)$ and $\forall\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega DuD\varphi + \int_\Omega \frac{sign(u-1)}{|u-1|^k}|Du|^2\varphi = \int_\Omega f\varphi$$

The only other result we know in this case deals with problems that look like

$$\begin{cases} -\Delta u + \beta(u)|Du|^2 = f(x) \text{ in } \Omega \\ \\ u = 0\,, \text{ on } \partial\Omega\,. \end{cases}$$

with

$$\beta(s) = \begin{cases} \frac{1}{(1-s)^k}\,, & \text{if } 0 \leq s < 1\,; \\ +\infty\,, & \text{if } s \geq 1\,. \end{cases}$$

Existence and non existence results are proved in [6].
More precisely, if $0 \leq k < 2$, the authors prove existence of solutions in $H_0^1(\Omega) \cap L^\infty(\Omega)$ for $f(x)$ sufficiently small belonging to $L^q(\Omega),\ q > \frac{N}{2}$, and non existence for large $f(x)$. In the case $k \geq 2$ they get existence of solutions in $H_0^1(\Omega) \cap L^\infty(\Omega)$ whatever is the size of $f(x) \in L^1(\Omega),\ f \geq 0$.

3 Existence for problem (2.1) under more general assumptions

In this section we prove existence of bounded solutions to (2.1) improving the hypotheses on the regularity of the datum f and on the behaviour at infinity of

the lower order term $b(x,u,Du)$.
More precisely, let us assume, instead of (2.2), (2.3),

$$\lambda \geq 0 \tag{3.22}$$

$$f(x) \in L^m(\Omega),\ m > \frac{N}{2},\ f(x) \geq 0. \tag{3.23}$$

Moreover, if $g : s \in (0,+\infty) \to (0,+\infty)$ is a continuous function for which there exist $k > 0$ and $s_0 > 0$ such that

$$g(s) = \frac{1}{s^k} \quad \text{if } s \leq s_0 \quad \text{and} \quad \lim_{s\to+\infty} g(s) = 0, \tag{3.24}$$

we assume either

$$k < 1 \quad \text{and} \quad |b(x,s,\xi)| \leq C_2 g(s)|\xi|^2 \tag{3.25}$$

or

$$k \geq 1 \quad \text{and} \quad C_1 g(s)|\xi|^2 \leq b(x,s,\xi) \leq C_2 g(s)|\xi|^2. \tag{3.26}$$

Theorem 3.1

Assume (3.22), (3.23), (2.4)-(2.6), (3.25). Then, there exists u satisfying (2.12), (2.13), (2.14) and

$$\|u\|_{H_0^1(\Omega)} + \|u\|_{L^\infty(\Omega)} \leq C,$$

where $C = C(\alpha, \|f\|_{L^m(\Omega)}, \Omega, C_2)$.
If $\lambda > 0$ and $f \in L^\infty(\Omega)$, we can assume the function $g(s)$ just bounded at infinity, getting the same conclusion.

Theorem 3.2

Assume (3.22), (3.23), (2.4)-(2.6), (3.26). Then, thre exists u satisfying (2.15), (2.16), (2.17) and

$$\|\Psi(u)\|_{H_0^1(\Omega)} + \|u\|_{L^\infty(\Omega)} \leq C$$

where $C = C(\alpha, \|f\|_{L^m(\Omega)}, \Omega, C_2)$.
If $\lambda > 0$ and $f \in L^\infty(\Omega)$, we can assume the function $g(s)$ just bounded at infinity, getting the same conclusion.

Sketch of the proof.

We have just to modify Step 1 and Step 2 of the proof of Theorem 2.1 and Theorem 2.2 of Section 2. We need to prove them in the inverse order. Note that

the proof of the two steps is the same for Theorem 3.1 and Theorem 3.2.
We define the following auxiliary functions:

$$g_n(s) = \begin{cases} n^k & \text{if} \quad s \le \frac{1}{n} \\ \\ g(s) & \text{otherwise} \end{cases}$$

$$\gamma_n(s) = C_2 \int_0^s g_n(\sigma)\, d\sigma, \qquad \Psi_n(s) = \int_0^s e^{\gamma_n(\sigma)} d\sigma.$$

Since g vanishes at infinity, by the de l'Hôpital's rule we have

$$\lim_{s\to\infty} \frac{e^{\gamma_n(s)}}{\Psi_n(s)} = 0,$$

which implies that, for any $\varepsilon > 0$ there exists a constant C such that

$$e^{\gamma_n(s)} \le \varepsilon \Psi_n(s) + C, \quad \forall n \in \mathbb{N},\, s \in \mathbb{R}. \tag{3.27}$$

We will use this estimate in what follows.

Step 1. Uniform estimates on $\Psi_n(u_n)$ in $H_0^1(\Omega)$.

We take

$$e^{\gamma_n(u_n)} \Psi_n(u_n)$$

as test function in (2.18), getting

$$C_2 \int_\Omega g_n(u_n) e^{\gamma_n(u_n)} \Psi_n(u_n) |Du_n|^2 + \alpha \int_\Omega e^{2\gamma_n(u_n)} |Du_n|^2$$
$$\le C_2 \int_\Omega g_n(u_n) e^{\gamma_n(u_n)} \Psi_n(u_n) |Du_n|^2 + \int_\Omega f e^{\gamma_n(u_n)} \Psi_n(u_n),$$

that is, using (3.27),

$$\alpha \int_\Omega |D\Psi_n(u_n)|^2 \le \int_\Omega f |\Psi_n(u_n)|^2 + C \int_\Omega f \Psi_n(u_n).$$

The last inequality, thanks to both the summability of f and Sobolev's inequality, implies that there exists $C = C(\alpha, \|f\|_{L^{\frac{N}{2}}(\Omega)}, \Omega, C_2)$ such that, for all $n \in \mathbb{N}$,

$$\|\Psi_n(u_n)\|_{H_0^1(\Omega)} \le C. \tag{3.28}$$

Note that in the case of Theorem 7 (i.e. $k < 1$), since

$$|Du_n|^2 \le e^{2\gamma_n(u_n)} |Du_n|^2 = |D\Psi(u_n)|^2,$$

we also have, for all $n \in \mathbb{N}$,

$$\|u_n\|_{H_0^1(\Omega)} \le C. \tag{3.29}$$

Step 2. Uniform estimates on u_n in $L^\infty(\Omega)$.

We follows here the outline of [1], with some modifications due to the fact that in our case the parameter k can be any positive number while in [1] the only case $k = 1$ is considered.
We begin by proving that, if $f \in L^{\frac{N}{2}}(\Omega)$, the sequence $\{e^{u_n}\}$ is bounded in $L^p(\Omega)$ for any $p > 1$. Defining

$$A_{h,n} = \{x \in \Omega : \ u_n(x) > h\}$$

we have

$$\lim_{s \to \infty} |A_{h,n}| = 0, \quad \text{uniformly in } n. \tag{3.30}$$

This follows immediately by (3.29) if we are in the case of Theorem 3.1 (i.e. $k < 1$). If we are in the setting of Theorem 3.2, we cannot obtain anymore estimate (3.29) and we have just estimate (3.28).
In this case, we observe that, at any point x where $u_n(x) > h > 1$, we have

$$\Psi_n(u_n)(x) \geq \int_1^{u_n(x)} e^{\gamma_n(\sigma)} d\sigma \geq e^{\gamma(1)}(h-1),$$

so that

$$\{x \in \Omega : \ u_n(x) > h\} \subseteq \{x \in \Omega : \ \Psi_n(u_n(x)) > e^{\gamma(1)}(h-1)\}$$

and the measure of the last set goes to zero uniformly in n as h goes to $+\infty$, due to estimate (3.28). Therefore we proved (3.30).
We take now

$$e^{2\nu(u_n-h)_+} - 1$$

as test function in (2.18) with $\nu > 0$ and $h > 1$. By the assumptions, we easily get

$$2\nu\alpha \int_\Omega e^{2\nu(u_n-h)_+} |D(u_n-h)_+|^2 \leq C_2 \int_\Omega g_n(u_n)(e^{2\nu(u_n-h)_+} - 1)|Du_n|^2$$
$$+Q \int_\Omega f(e^{\nu(u_n-h)_+} - 1)^2 + \frac{1}{Q-1} \int_\Omega f \chi_{u_n \geq h},$$

where we used the inequality

$$t^2 - 1 \leq Q(t-1)^2 + \frac{1}{Q-1}, \quad \forall Q > 1, \quad \forall t \geq 1.$$

Since $\lim\limits_{s \to +\infty} g(s) = 0$, we have

$$g(s) < \frac{\nu\alpha}{C_2}, \quad \text{for every } s \geq h_0 \doteq h_0(\nu, \alpha),$$

so that, for $h \geq h_0$

$$\frac{\alpha}{\nu} \int_\Omega |D(e^{\nu(u_n-h)_+} - 1)|^2 \leq \nu\alpha \int_\Omega e^{2\nu(u_n-h)_+} |D(u_n-h)_+|^2$$

$$+Q\|f\|_{L^{\frac{N}{2}}(A_{h,n})} \|e^{\nu(u_n-h)_+} - 1\|^2_{L^{2^*}(\Omega)} + \tfrac{1}{Q-1}\|f\|_{L^1(A_{h,n})}.$$

This gives, denoting by S the constant of the Sobolev embedding,

$$(\frac{\alpha}{\nu}S^2 - Q\|f\|_{L^{\frac{N}{2}}(A_{h,n})})\|e^{\nu(u_n-h)_+} - 1\|^2_{L^{2^*}(\Omega)} \le \frac{1}{Q-1}\|f\|_{L^1(A_{h,n})}.$$

By (3.30), fixing h sufficiently large, we can make $\|f\|_{L^{\frac{N}{2}}(A_{h,n})}$ as small as we want uniformly in n, so that

$$\|e^{\nu(u_n-h)_+} - 1\|^2_{L^{2^*}(\Omega)} \le C\|f\|_{L^1(A_{h,n})} \quad \forall n,\ \forall \nu > 0, \tag{3.31}$$

where $C = C(\alpha, \nu, C_2, \|f\|_{L^{\frac{N}{2}}(\Omega)})$.
This implies also that the sequence $\{e^{\nu u_n}\}$ is bounded in $L^{2^*}(\Omega)$ for any $\nu > 0$ which gives $\{e^{u_n}\}$ bounded in $L^p(\Omega)$ for any $p > 1$.

Let us assume now $f \in L^m(\Omega)$, $m > \frac{N}{2}$. Recalling that $e^{\nu t} - 1 \ge \nu t$, $\forall t \ge 0$, by (3.31) we have, for h sufficiently large,

$$\Big(\int_\Omega (u_n - h)_+^{2^*}\Big)^{\frac{2}{2^*}} \le \frac{C}{Q-1}\|f\|_{L^m(\Omega)}|A_{h,n}|^{1-\frac{1}{m}}.$$

For $r \ge h$ we have

$$(r-h)^2|A_{r,n}|^{\frac{2}{2^*}} \le C|A_{h,n}|^{1-\frac{1}{m}}.$$

By classical results ([16]), it follows that

$$\|u_n\|_{L^\infty(\Omega)} \le C, \tag{3.32}$$

where $C = C(\alpha, \|f\|_{L^m(\Omega)}, \Omega, C_2)$.

Once we have proved Step 1 and Step 2, the proof of the two theorems follows the same outline of the proof of Theorem 2.1 and Theorem 2.2. The estimates on the solution u and on the function $\Psi(u)$ follow by semicontinuity from (3.29), (3.32),(3.28). For the case where $\lambda > 0$, $f(x) \in L^\infty(\Omega)$ and $g(s)$ is bounded at $+\infty$, we observe that the proof of Section 2 still works in the same way. □

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