Qualitative properties to magnetoelastic plates

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Abstract

In this paper one proves that the semigroup associated to a class of magnetoelastic plate models is analytic.

1 Introduction

Let us suppose that a magnetoelastic plate is configured over an open bounded and simply connected set $\Omega \subset \mathbb{R}^2$, with boundary Γ, and consider the model given by

$$\omega_{tt} + \mu\Delta^2\omega + \gamma\nabla\times[\nabla\times\omega_t\mathbb{H}_1]\cdot\mathbb{H}_1 - \alpha\nabla\times[\nabla\times \eqsim]\cdot\mathbb{H}_2 = 0 \quad \text{in} \quad \Omega\times]0,T[, \tag{1.1}$$

$$\eqsim_t + \nabla\times[\nabla\times\eqsim] + \beta\nabla\times[\nabla\times\omega_t\mathbb{H}_2] = 0 \quad \text{in} \quad \Omega\times]0,T[, \tag{1.2}$$

$$\operatorname{div}\eqsim = 0 \quad \text{in} \quad \Omega\times]0,T[, \tag{1.3}$$

with boundary conditions

$$\eqsim\cdot\nu = \nu\times\nabla\times\eqsim = \mathbf{0}, \qquad \omega = \frac{\partial\omega}{\partial\nu} = 0 \quad \text{on} \quad \Gamma\times]0,T[, \tag{1.4}$$

and initial data

$$\omega(0) = \omega_0 \quad \omega_t(0) = \omega_1 \quad \text{and} \quad \eqsim(0) = \eqsim_0 \quad \text{in} \quad \Omega. \tag{1.5}$$

Here, ω denotes the transverse displacement of the plate, $\eqsim\ =\ (h^1, h^2)$ is the electromagnetic field, $\mathbb{H}_i = (H_i^1, H_i^2)$, $i = 1, 2$, are two constant magnetic fields, α, μ, β are positive real numbers. The physical motivation of the problem can be founded, for instance, in [2, 12]. This problem is closely related to the linear thermoelastic plate model. In this direction, Renardy and Liu [7] showed that the corresponding semigroup is analytic.

Concerning three-dimensional magnetoelastic materials, one has the work of Andreou and Dassios [1], who showed that the solutions decays polynomially to zero provided the material is configured in the whole $\mathbb{R}^3$ space. See also [9, 10]. On the other hand, Duyckaerts [3], using micro-local analysis, showed the lack of exponential stability for three-dimensional magnetoelastic model and gave a complete description of the uniform rate of decay of the solutions in bounded domains.

The main purpose of the present paper is to show the analyticity to the magnetoelastic plate model (1.1)-(1.5) in the case $\mathbf{H}_1$ and $\mathbf{H}_2$ are linearly independent vector fields. In particular our result implies the exponential stability.

2 The main result

Let us begin with some notations and remarks. For $\eqsim : \mathbb{R}^2 \ \rightarrow\ \mathbb{R}^2$ we define

$$\nabla \times \eqsim = \nabla \times \varepsilon(h^1, h^2)^T \ :=\ \partial_1 \, h^2 - \partial_2 \, h^1, \tag{2.6}$$

where $\partial_i = \dfrac{\partial}{\partial\, x_i}$. Similarly, for $\omega : \mathbb{R}^2 \ \rightarrow\ \mathbb{R}$, we define

$$\nabla \times \omega := \begin{pmatrix} \partial_2\omega \\ -\partial_1\omega \end{pmatrix}. \tag{2.7}$$

Note that $\nabla \times [\nabla \times \omega] = \ -\Delta\, \omega$. Besides, for $\varepsilon u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ we have

$$\Delta\, \varepsilon u \ = \ \nabla \mathrm{div}\, \varepsilon u - \nabla \times [\nabla \times \varepsilon u]. \tag{2.8}$$

Let us consider

$$\mathbb{Y} \ := \{\eqsim \in L^2(\Omega) \times L^2(\Omega);\ \mathrm{div} \eqsim = 0 \text{ in } \Omega \quad \text{and} \quad \varepsilon\nu \cdot \eqsim = 0 \text{ on } \Gamma\, \},$$

which is a Hilbert space when equipped with the inner-product

$$\langle \mathbf{h}_1, \eqsim_2 \rangle_{\mathbb{Y}} = \frac{\alpha}{\beta} \int_\Omega \ \eqsim_1 \eqsim_2 \, dx.$$

Then we introduce the operator $\mathcal{B}$, defined by

$$\mathcal{B}\varepsilon g = \ \nabla \times [\nabla \times \varepsilon g], \tag{2.9}$$

with domain

$$\mathcal{D}(\mathcal{B}) = \{\, \varepsilon g \in \mathbb{Y} \cap (H^2)^2; \quad \varepsilon\nu \times [\nabla \times \varepsilon g] = 0 \quad \text{on} \quad \Gamma\}.$$

Note that $\mathcal{D}(\mathcal{B})$ is dense in $\mathbb{Y}$. Next, we denote by $\mathcal{H}$ the space

$$\mathcal{H} \;=\; H_0^2(\Omega) \times\, L^2(\Omega) \times \mathbb{Y},$$

with inner-product

$$\langle \varepsilon U_1, \varepsilon U_2 \rangle_{\mathcal{H}} = \int_\Omega \mu \Delta \omega_1 \Delta \overline{\omega_2} + v_1 \overline{v_2} + \frac{\alpha}{\beta} \eqsim_1 \cdot \overline{\eqsim_2} \; dx, \tag{2.10}$$

where $\varepsilon U_i = \; (\omega_i, v_i, \eqsim_i)^T \in \mathcal{H}\,,\; i = \; 1, 2$. Then it is easy to see that $\mathcal{H}$ is a Hilbert space.

Finally we define the unbounded operator $\mathcal{A} : \mathcal{H} \;\rightarrow\; \mathcal{H}\,,$

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ -\mu\Delta^2(\cdot) & -\gamma\nabla\times[\nabla\times\mathbb{H}_1(\cdot)] & \alpha\nabla\times[\nabla\times\mathbb{H}_2(\cdot)] \\ 0 & -\beta\nabla\times[\nabla\times\varepsilon H_2(.)] & -\nabla\times[\nabla\times(\cdot)] \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}) = \; H^4(\Omega) \cap H_0^2(\Omega) \times H_0^2(\Omega) \times \mathcal{D}(\mathcal{B}).$$

It is not difficult to see that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$, $0 \in \rho(\mathcal{A})$, and that

$$\langle \mathcal{A}\,\varepsilon U, \varepsilon U \rangle_{\mathcal{H}} = -\gamma ||\nabla \times \varepsilon H_1\, v||^2_{L^2(\Omega)} - \frac{\alpha}{\beta} ||\nabla \times \eqsim||^2_{L^2(\Omega)} \;\leq\; 0. \tag{2.11}$$

Therefore we have:

Theorem 2.1. *The operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contraction. Furthermore, this semigroup is the one associated to the system (1.1)-(1.5).*

Our main result is the following.

Theorem 2.2. *Let εH_1 and εH_2 be two linearly independent magnetic fields. Then the semigroup associated to the system (1.1)-(1.5) is analytic.*

3 Proof of main result

We use the following characterization of analytic semigroups, as in [8, 11].

Theorem 3.1. *A semigroup of contractions $\{\, e^{t\,\mathcal{A}}\,\}_{t\geq 0}$ is analytic if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \limsup_{|\eta|\to\infty} ||\eta(i\eta\, I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} < \infty, \tag{3.12}$$

where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$.

We note that the condition (3.12) is equivalent to show that the solution εU of the spectral equation

$$(i\eta\, I - \mathcal{A})\varepsilon U \;=\; \varepsilon F \tag{3.13}$$

is uniformly bounded by εF with respect to the norm of $\mathcal{H}$, over the whole imaginary axis.

Lemma 3.1. *If* $\mathbb{H}_1$ *and* $\mathbb{H}_2$ *are linearly independent, then* $i\mathbb{R} \subset \rho(\mathcal{A})$.

Proof. Suppose $i\mathbb{R} \subset \rho(\mathcal{A})$ does not hold. Then there exist eigenvectors εU such that $\eqsim = \nabla \times \mathbb{H}_1 v = 0$, which imply that $\nabla \times [\nabla \times \mathbb{H}_2 v] = 0$. Since $v = \frac{\partial v}{\partial \nu} = 0$, we have that $\nabla \times \mathbb{H}_2 v = 0$. Because of the linear independency of $\mathbb{H}_1$ and $\mathbb{H}_2$ we conclude that $v = 0$. Therefore $\omega = 0$, which is a contradiction.

The equation (3.13), in terms of the components, can be written as

$$\begin{aligned} i\eta\, \omega - v \;&=\; f_1 \quad \text{in } H_0^2(\Omega), && (3.14)\\ i\eta\, v + \mu\Delta^2\omega + \gamma\nabla \times [\nabla \times \varepsilon H_1 v] - \alpha\nabla \times [\nabla \times \eqsim] \cdot \varepsilon H_2 \;&=\; f_2 \quad \text{in } L^2(\Omega), && (3.15)\\ i\eta \eqsim + \beta\nabla \times [\nabla \times \varepsilon H_2 v] + \nabla \times [\nabla \times \eqsim] \;&=\; \varepsilon f_3 \quad \text{in } \mathbb{Y}, && (3.16) \end{aligned}$$

where

$$\varepsilon U = (\omega,\, v,\, \eqsim)^T \in \mathcal{D}(\mathcal{A}), \quad \varepsilon F = (f_1,\, f_2,\, \varepsilon f_3)^T \in \mathcal{H}.$$

Lemma 3.2. *The solution* εU *of the spectral equation (3.13) satisfies*

$$\gamma||\nabla \times \varepsilon H_1\, v||^2_{L^2(\Omega)} + \frac{\alpha}{\beta}||\nabla \times \eqsim||^2_{L^2(\Omega)} \leq\; ||\varepsilon U||_{\mathcal{H}}||\varepsilon F||_{\mathcal{H}}. \tag{3.17}$$

Proof. Taking inner-product of equation (3.13) with εU in $\mathcal{H}$ and using equation (3.20) our conclusion follows.

Our next step is to estimate the term $\eta\eqsim$.

Lemma 3.3. *For any* $\epsilon > 0$ *there exists* $C^0_\epsilon > 0$ *such that the solution* εU *of (3.13) verifies*

$$|\eta|^2\, ||\eqsim||^2_{\mathbb{Y}} \leq\, \epsilon\, |\eta|^2 C\, ||\varepsilon U||^2_{\mathcal{H}} + C^0_\epsilon ||\varepsilon F||^2_{\mathcal{H}}, \tag{3.18}$$

where $C > 0$ *is a constant not depending on* ϵ.

Proof. Multiplying equation (3.16) by $\eta\frac{\alpha}{\beta}\overline{\eqsim}$ and integrating over Ω we get

$$\begin{aligned} &i\eta^2\,\frac{\alpha}{\beta}\int_\Omega |\eqsim|^2\, dx - \alpha\eta\int_\Omega \nabla \times \varepsilon H_2 v \cdot \nabla \times \eqsim\, dx + \eta\frac{\alpha}{\beta}\int_\Omega |\nabla \times \eqsim|^2\, dx \\ &\quad = \eta\frac{\alpha}{\beta}\int_\Omega \varepsilon f_3 \cdot \eqsim\, dx, \end{aligned} \tag{3.19}$$

from where it follows that

$$|\eta|^2 \frac{\alpha}{\beta}\int_\Omega |\approx|^2\, dx \le \epsilon \|v\|^2_{H_0^1(\Omega)} + C_\epsilon ||\varepsilon U||_{\mathcal{H}} ||\varepsilon F||_{\mathcal{H}}. \tag{3.20}$$

Using interpolation we get

$$||v||_{H_0^1(\Omega)} \le C||v||^{1/2}_{H_0^2(\Omega)} ||v||^{1/2}_{L^2(\Omega)}. \tag{3.21}$$

From (3.14) we see that

$$\begin{aligned} ||\Delta\, v||_{L^2(\Omega)} &\le |\eta|||\Delta\omega||_{L^2(\Omega)} + ||\Delta f_1||_{L^2(\Omega)} \\ &\le C|\eta|||\varepsilon U||_{\mathcal{H}} + C||\varepsilon F||_{\mathcal{H}}. \end{aligned} \tag{3.22}$$

Combining (3.21) and (3.22) it follows that

$$\begin{aligned} \|v\|_{H_0^1(\Omega)} &\le C(|\eta|^{1/2}||\varepsilon U||^{1/2}_{\mathcal{H}} + ||\varepsilon F||^{1/2}_{\mathcal{H}})||v||^{1/2}_{L^2(\Omega)} \\ &\le C|\eta|^{1/2}||\varepsilon U||_{\mathcal{H}} + C||\varepsilon F||^{1/2}_{\mathcal{H}}||\varepsilon U||^{1/2}_{\mathcal{H}}. \end{aligned}$$

Then putting this last inequality into (3.20) yields (3.18).

Lemma 3.4. *Let* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *be linearly independent, and let* Ω *be a simply connected bounded set of* $\mathbb{R}^2$. *Then for any* $\epsilon > 0$ *there exists* $C^1_\epsilon > 0$ *such that the solution* εU *of (3.13) verifies*

$$|\eta|^2||v||^2_{L^2(\Omega)} \le \epsilon\, C|\eta|^2||\varepsilon U||^2_{\mathcal{H}} + C^1_\epsilon||\varepsilon F||^2_{\mathcal{H}}, \tag{3.23}$$

where $C > 0$ *is a constant not depending on* ϵ.

Proof. Multiplying equation (3.16) by $\mathbf{H}_2 v$ we get

$$\begin{aligned} |\eta|\int_\Omega |\nabla\times\mathbf{H}_2 v|^2\, dx &\le c|\eta|^2\|\mathbf{h}\|_{\mathbb{Y}}\|v\|_{H_0^1(\Omega)} + c|\eta|||\varepsilon F||_{\mathcal{H}}||\varepsilon U||_{\mathcal{H}} \\ &\le c_\delta|\eta|^2\|\mathbf{h}\|_{\mathbb{Y}} + \frac{\delta}{2}|\eta|^2||\varepsilon U||_{\mathcal{H}} + c|\eta|||\varepsilon F||_{\mathcal{H}}||\varepsilon U||_{\mathcal{H}}. \end{aligned}$$

Using Lemma 3.3 with $\epsilon = \delta/4c_\delta$,

$$|\eta|\int_\Omega |\nabla\times\mathbf{H}_2 v|^2\, dx \le \delta|\eta|^2||\varepsilon U||_{\mathcal{H}} + c_\delta||\varepsilon F||^2_{\mathcal{H}}.$$

From Lemma 3.2 we get

$$|\eta|\int_\Omega |\nabla\times\mathbf{H}_1 v|^2\, dx \le \delta|\eta|^2||\varepsilon U||_{\mathcal{H}} + c_\delta||\varepsilon F||^2_{\mathcal{H}}.$$

Denoting

$$\begin{aligned} H_1^1 \frac{\partial v}{\partial x_2} - H_1^2 \frac{\partial v}{\partial x_1} &= \nabla \times \mathbf{H}_1 v = G_1, \\ H_2^1 \frac{\partial v}{\partial x_2} - H_2^2 \frac{\partial v}{\partial x_1} &= \nabla \times \mathbf{H}_2 v = G_2, \end{aligned}$$

and using the fact that $\mathbf{H}_1$ and $\mathbf{H}_2$ are linearly independent, we conclude that

$$|\eta|^{1/2} \|\nabla v\|_{L^2(\Omega)} \leq C|\eta|^{1/2} \|G_1\|_{L^2(\Omega)} + C|\eta|^{1/2} \|G_2\|_{L^2(\Omega)}.$$

From Lemma 3.3, we conclude that for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\eta| \|\nabla v\|_{L^2(\Omega)}^2 \leq \epsilon C |\eta|^2 \|\varepsilon U\|_{\mathcal{H}}^2 + C_\epsilon^1 ||\varepsilon F||_{\mathcal{H}}^2. \tag{3.24}$$

Now, fixing v given by εU, let us consider v_1, v_2 the solution of

$$i\eta\, v_1 - \Delta v_1 = f_2, \tag{3.25}$$

$$i\eta\, v_2 + \mu \Delta^2 \omega + \gamma \nabla \times [\, \nabla \times \varepsilon H_1 v\,] + \Delta v_1 - \alpha \nabla \times [\, \nabla \times \eqsim\,] \cdot \varepsilon H_2 = 0, \tag{3.26}$$

with

$$v_1 = v_2 = 0 \quad \text{on} \quad \Gamma.$$

Then we have $v = v_1 + v_2$. It is not difficult to see that

$$|\eta| \|v_1\|_{L^2(\Omega)} + |\eta|^{1/2} \|\nabla v_1\|_{L^2(\Omega)} + \|v_1\|_{H_0^2(\Omega)} \leq c||\varepsilon F||_{\mathcal{H}}. \tag{3.27}$$

From (3.26) we get

$$\begin{aligned} |\eta| \|v_2\|_{H^{-2}(\Omega)} &\leq C\|\varepsilon U\|_{\mathcal{H}} + C\|v_1\|_{L^2(\Omega)} \\ &\leq C\|\varepsilon U\|_{\mathcal{H}} + \frac{C}{|\eta|}\|\varepsilon F\|_{\mathcal{H}}. \end{aligned}$$

Using interpolation, inequality (3.24) and that $v_2 = v - v_1$, we get

$$\begin{aligned} \|v_2\|_{L^2(\Omega)} &\leq C\|v_2\|_{H^{-2}(\Omega)}^{1/3} \|v_2\|_{H_0^1(\Omega)}^{2/3} \\ &\leq C\left(\frac{1}{|\eta|}\|\varepsilon U\|_{\mathcal{H}} + \frac{1}{|\eta|^2}\|\varepsilon F\|_{\mathcal{H}}\right)^{1/3} \left(\epsilon C|\eta| \|\varepsilon U\|_{\mathcal{H}}^2 + \frac{C_\epsilon}{|\eta|}||\varepsilon F||_{\mathcal{H}}^2\right)^{2/3} \\ &\leq \epsilon C\|\varepsilon U\|_{\mathcal{H}} + \frac{C_\epsilon}{|\eta|}\|\varepsilon F\|_{\mathcal{H}}, \end{aligned}$$

which implies that

$$\|v_2\|_{L^2(\Omega)} \leq \epsilon C\|\varepsilon U\|_{\mathcal{H}}^2 + \frac{C_\epsilon}{|\eta|}\|\varepsilon F\|_{\mathcal{H}}^2.$$

Since $v = v_1 + v_2$, using the above inequality and (3.27) we have

$$\|v\|_{L^2(\Omega)} \le \epsilon C \|\varepsilon U\|^2_{\mathcal{H}} + \frac{C_\epsilon}{|\eta|} \|\varepsilon F\|^2_{\mathcal{H}}.$$

Then (3.23) follows.

Proof of Theorem 2.2: We apply Theorem 3.1. From Lemma 3.1 we know that $i\mathbb{R} \subset \rho(\mathcal{A})$. It remains to show that solutions εU of (3.13) are uniformly bounded with respect to η.

Multiplying equation (3.15) by $\overline{v}$ and using (3.14) we get

$$i\eta \int_\Omega |v|^2\, dx + \mu \int_\Omega \Delta\omega(-i\eta\overline{\Delta\omega} - \overline{\Delta f_1})\, dx - \alpha \int_\Omega \nabla \times \nabla \approx \varepsilon H_2 \overline{v}\, dx$$
$$+ \int_\Omega |\nabla \times \varepsilon H_1\, v|^2\, dx = \int_\Omega f_2.\overline{v}\, dx.$$

Therefore

$$\begin{aligned} |\eta|^2 \int_\Omega |\Delta\omega|^2\, dx &\le |\eta|^2 \int_\Omega |v|^2\, dx + C|\eta|\, ||\varepsilon U||_{\mathcal{H}} ||\varepsilon F||_{\mathcal{H}} \\ &\le \epsilon |\eta|^2 \|\varepsilon U\|^2_{\mathcal{H}} + C_\epsilon \|\varepsilon F\|^2_{\mathcal{H}}. \end{aligned}$$

Using the above inequality together with Lemmas 3.3 and 3.4, we infer that

$$|\eta|^2 \|\varepsilon U\|^2_{\mathcal{H}} \le \epsilon\, C\, |\eta|^2\, \|\varepsilon U\|^2_{\mathcal{H}} + C_\epsilon \|\varepsilon F\|^2_{\mathcal{H}}. \tag{3.28}$$

Taking ϵ small such that $\epsilon C < 1$ we conclude that

$$|\eta|^2 \|\varepsilon U\|^2_{\mathcal{H}} \le C \|\varepsilon F\|^2_{\mathcal{H}}, \quad \forall\, \eta \in \mathbb{R}.$$

This implies the analyticity of $e^{\mathcal{A}t}$.

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