Homogenization models with fractal strings

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Abstract

We describe homogenization models for reiforcement problems of plane domains with fractal boundaries and for elastic membranes reinforced by the inclusion of a fractal string. We follow a variational approach consisting in proving the convergence of certain energy functionals. This leads to the spectral convergence of a sequence of weighted second order elliptic partial differential operators to an elliptic operator with a fractal term.

1 Introduction

Fractals are geometric objects with highly non Euclidean characteristics: despite their tricky geometry there are large classes of fractals which possess a very rich analytic structure. Then we are able to study fractals both as intrinsic bodies, in which it is possible to give a notion of Laplacian and as boundaries of Euclidean domains supporting traces of functions belonging to classical spaces as Sobolev spaces. Hence fractal boundaries and fractal layers provide new, interesting settings to study boundary value problems with "large boundaries and small volumes" (see f.i. [29]). This interest emerges naturally in models of transmission problems of absorption or irrigation type where surface effects are enhanced. Fractal analysis could provide appropriate frameworks to study physical and biological phenomena (irrigation models, oxygen diffusion towards and across alveolar tissue pulmonary acinus, root infiltration, tree foliage) and technical applications in electrochemistry (electric current through metallic electrodes into electrolyte) and petrochemistry (diffusion of reactive molecules towards catalytic surface) with dominant surface effects (see f.i. [11] and [12]). In this talk we see two homogenization models for reiforcement problems: the first example involving insulating fractal layers and the second one highly conductive fractal layers.

2 An homogenization result for insulating fractal layers

Reiforcement problems for smooth domains have been largely studied in connection with various applications. Let me mention only the works of H. Brezis, L.A.

Caffarelli and A. Friedman ([5]), of E. Acerbi and G. Buttazzo ([1]), of L.A. Caffarelli and A. Friedman ([7]), of G. Buttazzo, G. Dal Maso and U.Mosco ([6]) and let me refer to the reference quoted therein.

The reinforcement problem across a regular layer is chosen, in the book of H. Attouch (see [3]), as an interesting example of homogenization. Let me recall the classical homogenization result in the simplest geometry. We denote

$\Omega = \{z \in \mathbb{R}^2 : \ |z| < 1\}$, $\Omega_\varepsilon = \{z \in \mathbb{R}^2 \ : \ |z| < 1+\varepsilon\}$, $\Sigma_\varepsilon = \{z \in \mathbb{R}^2 \ : \ 1 < |z| < 1+\varepsilon\}$,

$z = (x, y) \in \mathbb{R}^2$ and we put

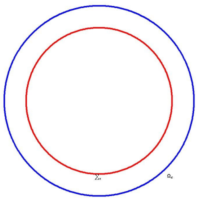

Figure 1: *The reinforced smooth domain*

$$a_\varepsilon(x,y) = \begin{cases} \lambda(\varepsilon) & \text{on } \Sigma_\varepsilon \\ 1 & \text{on } \Omega. \end{cases}$$

The classical homogenization result is: if $\frac{\lambda(\varepsilon)}{\varepsilon} \to c_0$ as $\varepsilon \to 0$, $c_0 > 0$, then

$$F_\varepsilon[u] = \int_{\Omega_\varepsilon} a_\varepsilon |\nabla u|^2 dxdy \to F_0[u] = \int_\Omega |\nabla u|^2 dxdy + c_0 \int_{\partial\Omega} u^2 ds,$$

where the domains are respectively

$$D_0[F_\varepsilon] = H_0^1(\Omega_\varepsilon), \ \ D[F_0] = H^1(\Omega).$$

Moreover, for every$f \in L^2(\Omega)$, the function u that minimizes on $H^1(\Omega)$ the functional

$$F_0[u] - 2\int_\Omega f u dxdy,$$

satisfies the following Robin problem on Ω

$$\begin{cases} i) & -\Delta u = f \quad \text{on} \quad \Omega \\ ii) & \frac{\partial u}{\partial \nu} + c_0 u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

From the point of view of the applications, the main interest is not in "regular" fractals but in irregular objects which exhibit some fractal properties. Hence more general structures have been studied, mostly from the probabilistic point of view, the so called "mixtures of fractals" or "fluctuating fractals" . These structures are locally spatially homogeneous but they do not satisfy any exact scaling relation. The mixtures of fractals are generated by different families of Euclidean similarities operating in a deterministic or random way that mimics the influence of the environment. For an exaustive discussion on this topic and for properties of irregular scale fractals we refer to the works of M.T. Barlow and B.M. Hambly (see [4] and also [30], [31]).

In this talk we consider a domain Ω whose boundary is a deterministic or random "mixture" of self-similar Koch curves. The fractal boundary of Ω is locally spatially homogeneous but it does not satisfy any exact scaling relation. More precisely: let $\mathcal{A} = \{1, 2\}$: for $a \in \mathcal{A}$, let $2 < \ell_a < 4$, and, for each $a \in \mathcal{A}$, let

$$\Psi^{(a)} = \{\psi_1^{(a)}, \ldots, \psi_4^{(a)}\} \tag{2.1}$$

be the family of contractive similitudes $\psi_i^{(a)} : \mathbb{C} \to \mathbb{C},\ i = 1, \ldots, 4$, with contraction factor ${\ell_a}^{-1}$:

$$\psi_1^{(a)}(z) = \frac{z}{\ell_a}, \qquad \psi_2^{(a)}(z) = \frac{z}{\ell_a} e^{i\theta(\ell_a)} + \frac{1}{\ell_a},$$

$$\psi_3^{(a)}(z) = \frac{z}{\ell_a} e^{-i\theta(\ell_a)} + \frac{1}{2} + i\sqrt{\frac{1}{\ell_a} - \frac{1}{4}}, \quad \psi_4^{(a)}(z) = \frac{z-1}{\ell_a} + 1,$$

where

$$\theta(\ell_a) = \arcsin\left(\frac{\sqrt{\ell_a(4-\ell_a)}}{2}\right). \tag{2.2}$$

Let $\Xi = \mathcal{A}^{\mathbb{N}}$; we call $\xi \in \Xi$ an environment. We define a left shift S on Ξ such that if $\xi = (\xi_1, \xi_2, \xi_3, \ldots)$, then $S\xi = (\xi_2, \xi_3, \ldots)$. For $\mathcal{O} \subset \mathbb{R}^2$ set

$$\Phi^{(a)}(\mathcal{O}) = \bigcup_{i=1}^{4} \psi_i^{(a)}(\mathcal{O})$$

and

$$\Phi_n^{(\xi)}(\mathcal{O}) = \Phi^{(\xi_1)} \circ \cdots \circ \Phi^{(\xi_n)}(\mathcal{O}) .$$

The fractal $K^{(\xi)}$ associated with the environment sequence ξ is defined by

$$K^{(\xi)} = \overline{\bigcup_{n=1}^{+\infty} \Phi_n^{(\xi)}(\Gamma)}$$

where $\Gamma = \{P_1, P_2\}$ with $P_1 = (0, 0)$ and $P_2 = (1, 0)$. We stress the fact that these fractals do not have any exact self-similarity, that is, there is no scaling factor which leaves the set invariant: however, the family $\{K^{(\xi)}, \xi \in \Xi\}$ satisfies the following relation

$$K^{(\xi)} = \Phi^{(\xi_1)}(K^{(S\xi)}). \tag{2.3}$$

Moreover, the spatial symmetry is preserved and the set $K^{(\xi)}$ is locally spatially homogeneous, that is, the volume measure $\mu^{(\xi)}$ on $K^{(\xi)}$ satisfies the locally spatially homogeneous condition (2.4). Before describing this measure, we introduce some notations. For $\xi \in \Xi$, we set $i|n = (1_1, ..., i_n)$ and $\psi_{i|n} = \psi_{i_1}^{(\xi_1)} \circ \cdots \circ \psi_{i_n}^{(\xi_n)}$.

The volume measure $\mu^{(\xi)}$ is the unique Radon measure on $K^{(\xi)}$ such that

$$\mu^{(\xi)}(\psi_{i|n}(K^{(S^n\xi)})) = \frac{1}{4^n} \tag{2.4}$$

(see Section 2 in [4]) as, for each $a \in \mathcal{A}$, the family $\Psi^{(a)}$ has 4 contractive similitudes.

The fractal set $K^{(\xi)}$ and the volume measure $\mu^{(\xi)}$ depend on the structural constants of the families and on the asymptotic frequency of the occurrence of each family. We denote by $h_a^{(\xi)}(n)$ the frequency of the occurrence of a in the finite sequence $\xi|n,\ n \geqslant 1$:

$$h_a^{(\xi)}(n) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{\xi_i = a\}},\ \ a = 1, 2.$$

Let p_a be a probability distribution on $\mathcal{A}$ and suppose that ξ satisfies

$$h_a^{(\xi)}(n) \to p_a,\ \ n \to +\infty,$$

where $0 \leq p_a \leq 1,\ \ p_1 + p_2 = 1$ and

$$|h_a^{(\xi)}(n) - p_a| \leq \frac{g(n)}{n},\ \ a = 1, 2,\ \ (n \geq 1),$$

where g is a regular increasing function on the real line, $g(0) = 1$, $g(n) \leq g_o n^{1-\eta}$, $g_o > 1, 0 < \eta < 1$. If $\eta = 1$, that is, if we consider the case of the fastest convergence of the occurrence factors, the measure $\mu^{(\xi)}$ has the property that there exist two positive constants C_1, C_2, such that,

$$C_1 r^{d^{(\xi)}} \leq \mu^{(\xi)}(B(P, r) \cap K^{(\xi)}) \leq C_2 r^{d^{(\xi)}}\ ,\quad \forall\, P \in K^{(\xi)}, \tag{2.5}$$

with

$$d^{(\xi)} = \frac{\ln 4}{p_1 \ln \ell_1 + p_2 \ln \ell_2}, \tag{2.6}$$

where $B(P, r)$ denotes the Euclidean ball with center in P and radius $0 < r \leq 1$ (see [4], [30] and [31]). According to Jonsson and Wallin (see [19]), we say that $K^{(\xi)}$ is a d-set with respect to the measure $\mu^{(\xi)}$ that is the restriction to the fractal

$K^{(\xi)}$ of the d-dimensional Hausdorff measure $\mathcal{H}^d$ in $\mathbb{R}^2$, normalized to be of total mass $= 1$,

$$\mu^{(\xi)} = \frac{\mathcal{H}^d_{|K^{(\xi)}}}{\mathcal{H}^d(K^{(\xi)})} \tag{2.7}$$

with $d = d^{(\xi)}$.

If instead $\eta < 1$ then

$$C_1 r^{d^{(\xi)}+\iota} \leq \mu^{(\xi)}(B(P,r) \cap K^{\xi}) \leq C_2 r^{d^{(\xi)}-\iota}\ , \quad \forall\, P \in K^{(\xi)} \tag{2.8}$$

with $\iota > 0$.

Let Ω^0 be the square $\{(x,y) : 0 < x < 1, -1 < y < 0\}$ with vertices $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (1,-1)$, and $P_4 = (0,-1)$. On each of the 4 sides we construct either a scale irregular Koch curve $K^{(\xi)}$ or the corresponding approximating pre-fractal curve $K^{(\xi),n}$.

We construct an $\varepsilon-$thin, polygonal, $2-$dimensional fiber Σ^n_ε, $n \in \mathbb{N}$, $0 < \varepsilon < 1$, around pre-fractal approximating domains Ω^n. The geometry of the fiber is regulated by the families of contractive similarities, whose $n-$iterations in the plane generate the Koch mixture as $n \to +\infty$.

More precisely, we consider $\Omega^{(\xi)}$ the set bounded by the 4 scale irregular Koch curves $K^{(\xi)}_j$ and the set $\Omega^n = \Omega^{(\xi),n}$ bounded by approximating the prefractal curves $K^{(\xi),n}_j$ j=1,2,3,4 with endpoints P_1 and P_2, P_2 and P_3, P_3 and P_4, P_4 and P_1 respectively.

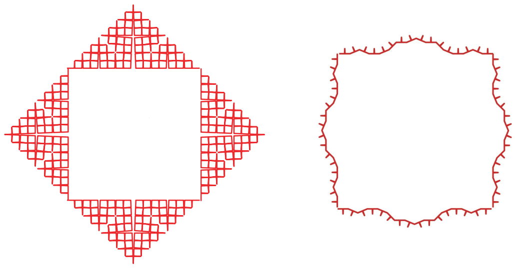

Figure 2: *Prefractal domains*

We start by constructing a suitable ε-neighborhood around Ω^0 that is the square introduced before.

Let K^0_1 be the interval with end-points P_1 and P_2. For every $0 < \varepsilon \leq \varepsilon_0 \leq c_1/2$, where $c_1 = \tan(\beta/4)$, $\beta < \pi$ we define the "ε-neighborhood" of K^0_1, denoted Σ_ε, to be the polygon whose vertices are the points P_1, P_2, P_5, P_6, where

$$P_5 = (1 - \frac{\varepsilon}{c_1}, \varepsilon), P_6 = (\frac{\varepsilon}{c_1}, \varepsilon).$$

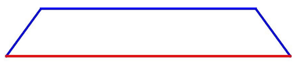

Figure 3: *The initial fiber*

For every n and ε as above, we define the "ε-neighborhood", $\Sigma^n_{1,\varepsilon}$, of K^n_1 to be the (open) set

$$\Sigma^n_{1,\varepsilon} = \bigcup_{i|n} \Sigma^{i|n}_{1,\varepsilon}.$$

$$\Sigma^{i|n}_{1,\varepsilon} = \psi_{i|n}(\Sigma_{1,\varepsilon}),$$

see Figures 3 and 4.

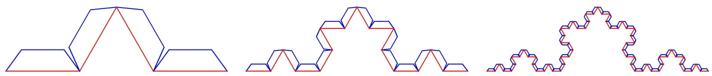

Figure 4: *The layers*

We proceed in a similar way in order to construct the "ε-neighborhood", $\Sigma^n_{j,\varepsilon}$, of K^n_j $(j = 2, 3, 4)$ and define the "ε-neighborhood", Σ^n_ε, of Ω^n

$$\Sigma^n_\varepsilon = \sum_j \Sigma^n_{j,\varepsilon}$$

and

$$\Omega^n_\varepsilon = \overline{\Omega^n} \bigcup \Sigma^n_\varepsilon.$$

We define a *weight*, w^n_ε, as follows.

Let P – for some $i|n$ – belong to the boundary $\partial(\Sigma^{i|n}_{1,\varepsilon})$ of $\Sigma^{i|n}_{1,\varepsilon}$ and let $P^\perp$ be the orthogonal projection of P on $(K^0_1)^{i|n}$. If (x, y) belongs to the segment with end-points P and $P^\perp$, we set, in our current notation,

$$w^n_{1,\varepsilon}(x, y) = c_0|P - P^\perp| \qquad if \quad (x, y) \in \Sigma^{i|n}_{1,\varepsilon}$$

where c_0 is a fixed positive constant, $|P - P^\perp|$ is the (Euclidean) distance between P and $P^\perp$ in $\mathbb{R}^2$.

We proceed in a similar way in order to construct the weights $w^n_{j,\varepsilon}$ on $\Sigma^n_{j,\varepsilon}$, of K^n_j $(j = 2, 3, 4)$ and we put

$$w^n_\varepsilon(x, y) = w^n_{j,\varepsilon}(x, y) \qquad \text{if} \quad (x, y) \in \Sigma^n_{j,\varepsilon}.$$

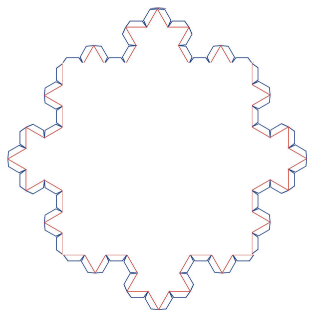

Figure 5: *The reinforced prefractal domain*

Moreover, we set

$$w_\varepsilon^n(x,y) = 1 \qquad\qquad \text{if} \quad (x,y) \notin \Sigma_\varepsilon^n \,. \tag{2.9}$$

Associated with the weight w_ε^n, there are the Sobolev spaces

$$H^1(\Omega_\varepsilon^n; w_\varepsilon^n) = \{u \in L^2(\Omega_\varepsilon^n) : \int_{\Omega_\varepsilon^n} |\nabla u|^2 w_\varepsilon^n dxdy < +\infty\} \tag{2.10}$$

and $H_0^1(\Omega_\varepsilon^n; w_\varepsilon^n)$, the latter being the completion of $C_0^\infty(\Omega_\varepsilon^n)$ in the norm

$$\|u\|_{H^1(\Omega_\varepsilon^n; w_\varepsilon^n)} = \{\int_{\Omega_\varepsilon^n} |u|^2 dxdy + \int_{\Omega_\varepsilon^n} |\nabla u|^2 w_\varepsilon^n dxdy\}^{\frac{1}{2}}\} \,.$$

Let Ω^* denote a smooth open set containing all the sets $\bar{\Omega}_\varepsilon^n$, (for every n and ε), we define the "weighted" energy functionals in $L^2(\Omega^*)$

$$F_\varepsilon^n[u] = \begin{cases} \int_{\Omega_\varepsilon^n} a_\varepsilon^n(x,y)|\nabla u|^2 dxdy & \text{if} \quad u \in H_0^1(\Omega_\varepsilon^n, w_\varepsilon^n) \\ +\infty & \text{if} \quad u \in L^2(\Omega^*) \setminus H_0^1(\Omega_\varepsilon^n, w_\varepsilon^n) \end{cases} , \tag{2.11}$$

where

$$a_\varepsilon^n(x,y) = \begin{cases} \sigma_n \, w_\varepsilon^n(x,y) & \text{if} \quad (x,y) \in \Sigma_\varepsilon^n \\ 1 & \text{if} \quad (x,y) \notin \Sigma_\varepsilon^n. \end{cases} \tag{2.12}$$

and

$$F[u] = \begin{cases} \int_{\Omega^{(\xi)}} |\nabla u|^2 dxdy + c_0 \int_{\partial\Omega^{(\xi)}} |\gamma_0 u|^2 d\mu^{(\xi)} & if \quad u|_{\Omega^{(\xi)}} \in H^1(\Omega^{(\xi)}) \\ +\infty & if \quad u \in L^2(\Omega^*) \setminus H^1(\Omega^{(\xi)}) \end{cases} \tag{2.13}$$

where $\gamma_0 u$ denotes the trace of u on the boundary of $\Omega^{(\xi)}$.

In order to state our first result, we need also to recall the notion of $M-convergence$ of functionals, introduced in [26], see also [28].

Definition 2.1. *: A sequence of functionals $F^n : H \to (-\infty, +\infty]$ is said to $M-$converge to a functional $F : H \to (-\infty, +\infty]$ in a Hilbert space H, if*

(a) For every $u \in H$ there exists u_n converging strongly in H such that

$$\limsup F^n[u_n] \leq F[u], \quad as \quad n \to +\infty. \tag{2.14}$$

(b) For every v_n converging weakly to u in H

$$\liminf F^n[v_n] \geq F[u], \quad as \quad n \to +\infty. \tag{2.15}$$

In this setting our homogenization result is the following

Theorem 2.1. *Let $\sigma_n = \frac{\ell^{(\xi)}(n)}{4^n}$ and let $\varepsilon = \varepsilon(n)$ be an arbitrary sequence such that $\varepsilon(n) \to 0$ as $n \to +\infty$. Then the sequence of functionals $F^n_{\varepsilon(n)}$, defined in (2.11), $M-$converges to the functional F defined in (2.13) as $n \to +\infty$.*

To prove Theorem 2.1 we combine and extend some delicate and deep results of P.W. Jones ([16]), A. Jonsson and H. Wallin ([19]) and L.G. Roger ([36]) hence we refer to the joint work with R. Capitanelli ([9]) for the proof, comments and details. In that paper ([9]) we show, in particular, that as in the classical model, for any choice of the datum $f \in L^2(\Omega^\varepsilon_n)$ the functions u^ε_n, minimizers of the complete energy forms, converge to the solution u of the Robin Problem in $\Omega^{(\xi)}$.

The Mosco-convergence of the functionals F^n_ε can be characterized in terms of the convergence of the resolvent operators, semigroups and spectral families associated with the forms allowing developments and applications (see [28]). However we will not deal here with these consequences.

3 An homogenization result for highly conductive fractal layers

This section concerns a singular homogenization result: a sequence of weighted volume energy functionals converges to a limit functional sum of a volume energy and a layer energy supported in a fractal set.

Singular homogenization results have been largely studied, in the classical setting of smooth domains and regular layers from the seventies: we refer to the works of J.R. Cannon and G.H. Meyer [8], H.Pham Huy and E.Sanchez-Palencia [35], to the already mentioned contributions [5] and [3] and to the references quoted there.

Let me recall the classical homogenization result, according to [3]; a smooth manifold Σ is located in a median position in a regular domain Ω, Σ_ε is an $\varepsilon-$neighbourhood of Σ. In the simplest geometry we denote

$$\Omega = \{(x, y) : \ 0 < x < 1, -1/2 < y < 1/2\}, \Sigma = \{(x, 0) \ : \ 0 < x < 1\},$$

$$\Sigma_\varepsilon = \{(x, y) \in \Omega \ : \ |y| < \frac{\varepsilon}{2}\}.$$

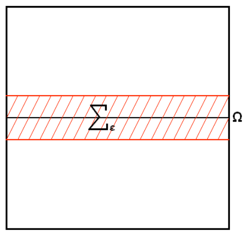

Figure 6: *The smooth layer*

The conductivity coefficients $a = a_{\varepsilon,\lambda}$ are assumed to be

$$a_{\varepsilon,\lambda} = \begin{cases} \lambda & \text{in } \Sigma_\varepsilon \\ 1 & \text{in } \Omega_\varepsilon = \Omega \setminus \Sigma_\varepsilon. \end{cases} \tag{3.16}$$

The corresponding energy functionals are

$$F^{\varepsilon,\lambda}[u] = \int_\Omega a_{\varepsilon,\lambda} |\nabla u|^2 dxdy \tag{3.17}$$

and the limit behaviour of the sequence $F^{\varepsilon,\lambda}$ when the thickness of the layer Σ_ε vanishes and the conductivity $a_{\varepsilon,\lambda}$ of the layer becomes infinite depends on the limit:

$$\lim_{(\varepsilon,\lambda)\to(0,+\infty)} \varepsilon \cdot \lambda.$$

In particular if

$$\lim_{(\varepsilon,\lambda)\to(0,+\infty)} \varepsilon \cdot \lambda = c^*$$

where $c^* \in (0, +\infty)$ then the limit functional is

$$F[u] = \int_\Omega |\nabla u|^2 dxdy + c^* \int_\Sigma |\nabla_\Sigma u|^2 dx \tag{3.18}$$

and the domain is the subspace of $H_0^1(\Omega)$ of the functions having trace on Σ belonging to the space $H_0^1(\Sigma)$.

The limit layer Σ divides the domain Ω in two adjacent sub-domains Ω^j and for every $f \in L^2(\Omega)$, the function u that minimizes on $H_0^1(\Omega)$ the functional

$$F_0[u] - 2\int_\Omega f u dxdy$$

satisfies the boundary value problem

$$(T.P.) \begin{cases} \quad -\Delta u^j = f & \text{on} \quad \Omega^j \\ u = 0 & \text{on} \quad \partial\Omega \\ u^1 = u^2 & \text{on} \quad \Sigma \\ u_{|\Sigma} = 0 & \text{on} \quad \partial\Sigma \\ \quad \frac{\partial u^1}{\partial \nu^1} + \frac{\partial u^2}{\partial \nu^2} = c^* \Delta_\Sigma u & \text{on} \quad \Sigma \end{cases} \tag{3.19}$$

$u^j = u|_{\Omega^j}$.

From the point of view of the boundary value problems we can remark that second order transmission problem (T.P.) is an unusual boundary value problem that is a Venttsel problem (see the survey of D.E. Apushkinskaya, A.I. Nazarov [2])

The study of second order transmission problems on domains with fractal (or prefractal) layers is recent and, to our knowledge, the first papers have been [23] and [24] and they concern the Koch curve. In this talk a singular homogenization result for the Sierpiński curve is presented (see [32],[33] and [34]) .

Let Ω be the triangle with vertices D, E, F,

$$D = (1/2, -\sqrt{3}/2),\ E = (3/2, \sqrt{3}/2),\ F = (-1/2, \sqrt{3}/2).$$

We construct the Sierpiński curve, $\mathcal{G}$ starting from the set $\Gamma = \{A, B, C\}$ (the midle-points A, B, C of the sides of Ω):

$$A = (0, 0),\ B = (1, 0),\ C = (1/2, \sqrt{3}/2)\,.$$

by iteration of the family of 3 contractive similarities $\Psi = \{\psi_1, \psi_2, \psi_3\}$ (in $\mathbb{R}^2$):

$$\psi_1(z) = \frac{z}{2}, \quad \psi_2(z) = \frac{z}{2} + \frac{1}{2}, \quad \psi_3(z) = \frac{z}{2} + \frac{1}{4} + i\frac{\sqrt{3}}{4}$$

where $z = x + iy$.

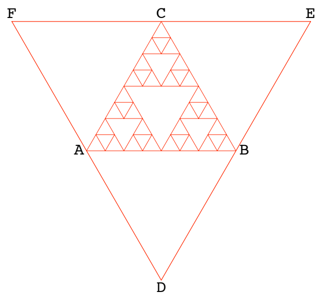

Figure 7: *Geometry of the domain*

For each integer $n > 0$, we consider arbitrary $n-$tuples of indices $i|n = (i_1, i_2, \dots, i_n)$. We define $\psi_{i|n} = \psi_{i_1} \circ \psi_{i_2} \circ \cdots \circ \psi_{i_n}$,$\psi_{i_j} \in \Psi$ and, for every set $\mathcal{O}(\subseteq \mathbb{R}^2)$, $\mathcal{O}^{i|n} = \psi_{i|n}(\mathcal{O})$. Occasionally, the index $i|n$ will be referred to as the $n-$ address of the set $\mathcal{O}^{i|n}$.

Let $V_0 = \Gamma = \{A, B, C\}$. For every integer $n > 0$ we put

$$V^n = \bigcup_{i|n} V_0^{i|n},$$

where $V_0^{i|n} = \psi_{i|n}(V_0)$, and then

$$V^\infty = \bigcup_{n=1}^{+\infty} V^n.$$

The fractal set $\mathcal{G}$ is obtained by taking the closure $\mathcal{G} = \overline{V}^\infty$ of the set V^∞ in $\mathbb{R}^2$.

The fractal set $\mathcal{G}$ has Hausdorff dimension $d^\mathcal{G} = \ln N / \ln \alpha$ where N denote the number of similarities of the family Ψ and α the contraction factor: in our setting $d^\mathcal{G} = \ln 3 / \ln 2$.

The measure μ, restriction to the fractal $\mathcal{G}$ of the $d^\mathcal{G}$-dimensional Hausdorff measure in $\mathbb{R}^2$, normalized to be of total mass $= 1$, is the (unique) invariant measure defined by the family Ψ of similarities of the fractal $\mathcal{G}$:

$$\mu = \frac{\mathcal{H}^{d^\mathcal{G}}_{|\mathcal{G}}}{\mathcal{H}^{d^\mathcal{G}}(\mathcal{G})} \tag{3.20}$$

(see Hutchinson [15]). From now on, we will denote the Hausdorff dimension $d^\mathcal{G}$ simply by d. According to Jonsson and Wallin (see [19]), we say that $\mathcal{G}$ is a d-set with respect to the measure μ.

An energy form $\mathcal{E}[u]$ is also defined on the fractal which is the limit of an increasing sequence of quadratic forms constructed by finite difference schemes. Namely,

$$\mathcal{E}[u] = \lim_{n \to +\infty} \mathcal{E}_n[u] \tag{3.21}$$

with domain

$$D^\infty[\mathcal{E}] = \{u : V^\infty \mapsto \mathbb{R} | \sup_{n \geq 0} \mathcal{E}_n[u|_{V^n}] < +\infty\}.$$

Where

$$\mathcal{E}_n[u] = \rho_n/2 \sum_{p \in V^n} \sum_{q \sim_n p} (u(p) - u(q))^2, \tag{3.22}$$

(two neighboring points $p \sim_n q$ in V^n being any pair (p,q) belonging to the same set $\psi^{i|n}(\Gamma)$) here

$$\psi^{(\xi)}_{i|n}(\Gamma) = \psi^{(\xi_1)}_{i_1} \circ \cdots \circ \psi^{(\xi_n)}_{i_n}(\Gamma) \tag{3.23}$$

for some finite sequence $i|n = (i_1, \cdots, i_n), \psi_{i_j} \in \Psi$. The functions $u \in D^\infty[\mathcal{E}]$ satisfy the estimate

$$|u(P) - u(Q)| \leq c\sqrt{\sup_{n\geq 0} \mathcal{E}_n[u|_{V^n}]}\ |P - Q|^\beta$$

for every $P, Q \in V^\infty$, with $\beta = \ln(5/3)/\ln 4$. Therefore, the form $\mathcal{E}$ can be uniquely extended to the domain

$$D[\mathcal{E}] = \{u \in C(\mathcal{G})| \sup_{n\geq 0} \mathcal{E}_n[u|_{V^n}] < +\infty\}.$$

The extended form is still denoted by $\mathcal{E}$. By the previous inequality, the estimate

$$|u(p) - u(q)| \leq c\sqrt{\mathcal{E}[u]}\ |p - q|^\beta \tag{3.24}$$

holds for every $p, q \in \mathcal{G}$.

Therefore, $D[\mathcal{E}] \subset C^\beta(\mathcal{G})$, where $C^\beta(\mathcal{G})$ is the space of Hölder continuous functions on the fractal $\mathcal{G}$ with exponent β. For these Hölder estimates we refer to Kozlov [21] (see also [27], where Kozlov's result is interpreted as an intrinsic Morrey's imbedding).

By $\mathcal{E}(u, v)$ we denote the bilinear form

$$\mathcal{E}(u, v) = \frac{1}{2}\{\mathcal{E}[u + v] - \mathcal{E}[u] - \mathcal{E}[v]\} \tag{3.25}$$

with domain $D[\mathcal{E}]$ dense in $L^2(\mathcal{G}, \mu)$. The form $\mathcal{E}$ is a regular Dirichlet form in $L^2(\mathcal{G}, \mu)$. By the Hölder estimate, $D[\mathcal{E}]$ has a compact imbedding in $L^2(\Omega)$ (see also Fukushima-Shima [14]). By $D_0[\mathcal{E}]$ we denote the subspace of $D[\mathcal{E}]$ of all functions $u \in D[\mathcal{E}]$ that vanish on Γ, that is on the points A, B and C.

We recall the value, in our setting, of the scaling factor of the energies (see(3.22))

$$\rho_n = \rho^n = \frac{5^n}{3^n}.$$

Now we construct the "initial fiber" denoted $\Sigma_{0,\varepsilon}$: for every $0 < \varepsilon \leq b_0/2$, where $b_0 = \tan(\pi/12)$, we define $\Sigma_{0,\varepsilon}$ to be the polygon whose vertices are the points A, P_1, P_2, B, P_3, P_4, where

$$P_1 = (\frac{\varepsilon}{b_0}, \frac{\varepsilon}{2}), \quad P_2 = (1 - \frac{\varepsilon}{b_0}, \frac{\varepsilon}{2}), \quad P_3 = (1 - \frac{\varepsilon}{b_0}, -\frac{\varepsilon}{2}), \quad P_4 = (\frac{\varepsilon}{b_0}, -\frac{\varepsilon}{2})\,.$$

We then subdivide $\Sigma_{0,\varepsilon}$ as the union of the rectangle $\mathcal{R}_{0,\varepsilon}$ and the two triangles $\mathcal{T}_{0,j,\varepsilon}, j = 1, 2$. Here, $\mathcal{R}_{0,\varepsilon}$ is the rectangle with vertices P_1, P_2, P_3, P_4; $\mathcal{T}_{0,1,\varepsilon}$ is the triangle with vertices A, P_1, P_4 and $\mathcal{T}_{0,2,\varepsilon}$ is the triangle with vertices P_2, B, P_3.

$\Sigma_{0,\varepsilon}$ is a "ε-neighborhood" of K_0, the segment with end-points A and B and can be considered as the "thin layer" at the step number zero of the iteration procedure.

In the "initial fiber "$\Sigma_{0,\varepsilon}$ we define the "initial weight" $w^0_{0,\varepsilon}$. Let P

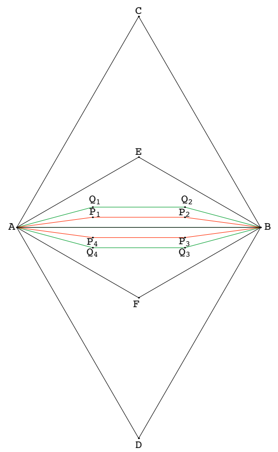

Figure 8: *Geometry of the initial fiber*

belong to the boundary $\partial\Sigma_{0,\varepsilon}$ of $\Sigma_{0,\varepsilon}$. Let $P^{\perp}$ be the orthogonal projection of P on K_0. If (x,y) belongs to the segment with end-points P and $P^{\perp}$, we set

$$w^0_\varepsilon(x,y) = \begin{cases} \frac{2+b_0^2}{4|P-P^{\perp}|} & \text{if} \quad (x,y) \in \overset{\circ}{\mathcal{T}}_{0,j,\varepsilon} \quad \text{and } j=1,2 \\ \frac{1}{2|P-P^{\perp}|} & \text{if} \quad (x,y) \in \mathcal{R}_{0,\varepsilon} \end{cases} \tag{3.26}$$

where $|P-P^{\perp}|$ is the (Euclidean) distance between P and $P^{\perp}$ in $\mathbb{R}^2$. We denote by K_l, $l=0,1,2$ the segments with end-points A and B, B and C, C and A, respectively and for $l=1,2$, we construct the "ε-neighborhood" $\Sigma_{l,\varepsilon}$ of K_l, as before, we decompose $\Sigma_{l,\varepsilon}$, in the union of the rectangle $\mathcal{R}_{l,\varepsilon}$ and the two triangles $\mathcal{T}_{l,j,\varepsilon}$, $j=1,2$, and we define the " weight" $w^0_{l,\varepsilon}$ as above (see 3.26).

We define the set

$$G_0 = \bigcup_{l=0,1,2} K_l$$

and the "ε-neighborhood"

$$\Sigma_\varepsilon = \bigcup_{l=0,1,2} \Sigma_{l,\varepsilon}$$

of G_0.

Now we apply the iteration procedure with respect to the family Ψ and we obtain both the "thin layer" and the "weight" at the step number n:

$$\Sigma^n_\varepsilon = \bigcup_{i|n} \Sigma^{i|n}_\varepsilon,$$

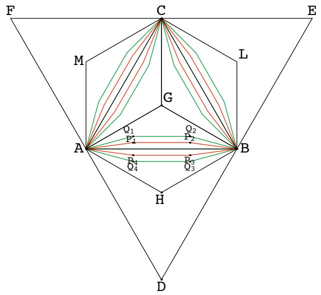

Figure 9: *Geometry of the layers*

where $\Sigma_\varepsilon^{i|n} = \psi_{i|n}(\Sigma_\varepsilon)$, (see Figure 10). Let P – for some l and $i|n$ – belong to the boundary $\partial\Sigma_{l,\varepsilon}^{i|n}$ of $\Sigma_{l,\varepsilon}^{i|n}$. Let $P^\perp$ be the orthogonal projection of P on $K_l^{i|n}$. If (x, y) belongs to the segment with end-points P and $P^\perp$, we set

$$w_\varepsilon^n(x,y) = \begin{cases} \frac{2+b_0^2}{4|P-P^\perp|} & \text{if} \quad (x,y) \in \overset{\circ}{\mathcal{T}}{}_{l,j,\varepsilon}^{i|n} \text{ and } j = 1,2 \\ \frac{1}{2|P-P^\perp|} & \text{if} \quad (x,y) \in \mathcal{R}_{l,\varepsilon}^{i|n} \end{cases} \tag{3.27}$$

where $|P - P^\perp|$ is the (Euclidean) distance between P and $P^\perp$ in $\mathbb{R}^2$, $\mathcal{T}_{l,j,\varepsilon}^{i|n} = \psi_{i|n}(\mathcal{T}_{l,j,\varepsilon})$, $\mathcal{R}_{l,\varepsilon}^{i|n} = \psi_{i|n}(\mathcal{R}_{l,\varepsilon})$.

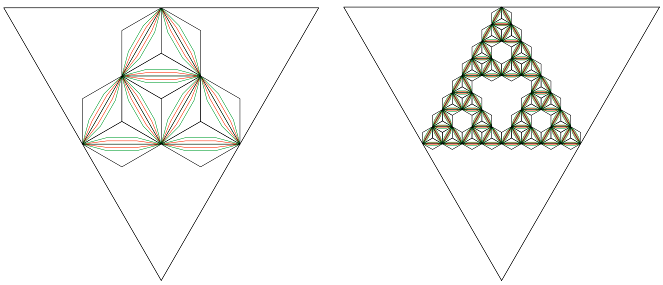

Figure 10: *The iteration procedure*

Then, in the domain Ω, taken together with the embedded layer Σ_ε^n, for given n and ε we define the " *unbounded conductivity coefficient*"

$$a_\varepsilon^n(x,y) = \begin{cases} \sigma_n\, w_\varepsilon^n(x,y) & \text{if} \quad (x,y) \in \Sigma_\varepsilon^n \\ \zeta_n & \text{if} \quad (x,y) \notin \Sigma_\varepsilon^n \end{cases}$$

where σ_n denotes (positive) renormalizing factor and 1 ¿ $\zeta_n > 0$.
For given n and ϵ, we define the weighted Sobolev space

$$H^1(\Omega; a_\varepsilon^n) = \{u \in L^2(\Omega) : \int_\Omega |\nabla u|^2 a_\varepsilon^n dxdy < +\infty\} \tag{3.28}$$

as the completion of $C^1(\overline{\Omega})$ with the Hilbert norm

$$\|u\|_{H^1(\Omega;a_\varepsilon^n)} = \{\int_\Omega |u|^2 dxdy + \int_\Omega |\nabla u|^2 a_\varepsilon^n dxdy\}^{\frac{1}{2}}\,.$$

The space $H_0^1(\Omega; a_\varepsilon^n)$ is the closure of $C_0^1(\Omega)$ in $H^1(\Omega; a_\varepsilon^n)$.
By F_ϵ^n we denote the (quadratic) functional defined in the Hilbert space $L^2(\Omega)$ with extended real values:

$$F_\varepsilon^n([u]) = \begin{cases} \int_\Omega a_\varepsilon^n(x,y)|\nabla u|^2 dxdy & \text{if} \quad u \in H_0^1(\Omega; a_\varepsilon^n) \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus H_0^1(\Omega; a_\varepsilon^n) \end{cases} \tag{3.29}$$

where σ_n and ζ_n are positive constants that will be specified later. By E_ε^n we denote the associate bilinear form:

$$E_\varepsilon^n(u,v) = \int_\Omega a_\varepsilon^n(x,y) \nabla u \nabla v \; dxdy$$

with domain $D_0[E_\epsilon^n] = H_0^1(\Omega; a_\varepsilon^n)$.

The form E_ϵ^n is a regular, Dirichlet form with dense domain in $L^2(\Omega)$. In order to state our main result, we need also to recall the notion of $M-$*convergence* of functionals, in the sense of Kuwae and Shioya [22] (see also [20] and [37]) that extends to the case of different Hilbert spaces the notion of $M-$*convergence* introduced in [26], see also [28].

Definition 3.1. *Let H^n and H be Hilbert spaces. A sequence of Hilbert spaces H^n is said to $M-$converge to the Hilbert space H (i.e. $H^n \to H$), if there exists a dense subspace $C \subset H$ and a sequence of linear maps $\Phi_n : C \to H^n$ such that*

$$||\Phi_n u||_{H^n} \to ||u||_H, \tag{3.30}$$

as $n \to +\infty$, for every $u \in C$.

In our setting we will choose $H^n = L^2(\Omega;\, \mu_\epsilon^n)$ and $H = L^2(\Omega;\, \mu^*)$.

$L^2(\Omega, \mu^*)$ denotes the completion of the space $C(\bar{\Omega})$ in the norm $\left\{ \int_\Omega \varphi^2 \, d\mu^* \right\}^{1/2}$, $\mu^* = \zeta^* \mathcal{L} + \mu_{\mathcal{G}}, \mathcal{L}$ is the 2- dimensional Lebesgue measure in $\mathbb{R}^2$.

More precisely with notation as given above we have:

Theorem 3.1. *Let the sequence ζ_n converge to the number $\zeta^* \geq 0$ then the Hilbert spaces $H^n = L^2(\Omega, \mu^n_{\varepsilon_n})$ converge to the Hilbert space $H = L^2(\Omega; \mu^*)$ in the sense of Definition 3.1 as $n \to +\infty$ where $\mu^* = \zeta^* \mathcal{L} + \mu$*

$$\mu^n_{\varepsilon_n} = \begin{cases} \tau_n w^n_\varepsilon & if \quad (x,y) \in \Sigma^n_\varepsilon \\ \zeta_n \mathcal{L} & if \quad (x,y) \notin \Sigma^n_\varepsilon, \end{cases}$$

$\tau_n = \frac{\alpha^n}{N^n} = \frac{2^n}{3^n}$ *and $\mathcal{L}$ denotes the 2- dimensional Lebesgue measure in $\mathbb{R}^2$.*

See ([34]) for the proof.

Definition 3.2. *Let $u_n \in H^n$ and $u \in H$. We say that: $u_n \to u$* strongly *if there exists a sequence $\tilde{u}_m \in C$ such that*

$$||\tilde{u}_m - u||_H \to 0 \tag{3.31}$$

as $m \to +\infty$ and

$$\lim_m \limsup_n ||\Phi_n \tilde{u}_m - u_n||_{H^n} = 0. \tag{3.32}$$

Definition 3.3. *Let $u_n \in H^n$ and $u \in H$. We say that: $u_n \to u$* weakly *if*

$$(u_n, v_n)_{H^n} \to (u, v)_H \tag{3.33}$$

for every $v_n \to v$ strongly*, as $n \to +\infty$.*

Definition 3.4. *Let E_n and E be Dirichlet forms in on H^n and H respectively. We say that the sequence E_n* K-S-M-converges *to E if:*

(a) For every $u \in H$ there exists $u_n \in H^n$ converging strongly to u such that

$$\limsup E^n[u_n] \leq E[u], \quad as \quad n \to +\infty. \tag{3.34}$$

(b) For every $v_n \in H^n$ converging weakly to $u \in H$

$$\liminf E^n[v_n] \geq E[u], \quad as \quad n \to +\infty. \tag{3.35}$$

We can now state our singular homogenization result with notation as given above:

Theorem 3.2. *Let $\zeta_n \to \zeta^* \geq 0$, $\sigma_n = \frac{c_0 \rho^n}{\alpha^n}$, $\varepsilon = \varepsilon_n = \frac{\rho^n}{N^n}\omega_n$, with $\omega_n > 0$, $\omega_n \to 0$. As $n \to +\infty$ the sequence of the functionals F^n_ε defined in (3.29) $K-S-M-$ converges to the functional E:*

$$E[u] = \zeta^* \int_\Omega |\nabla u|^2 dxdy + \mathcal{E}[u_{|\mathcal{G}}]$$

where

$$D[E] = \{v \in L^2(\Omega, \mu^*) : \zeta^* \int_\Omega |\nabla v|^2 dxdy < +\infty \,, v_{|_\mathcal{G}} \in D[\mathcal{E}]\}.$$

This result shows that, in the case with $0 < \zeta^* \leq 1$, the asymptotic energy splits into a residual two-dimensional bulk term and a lower dimensional fractal term, both acting as a coupled system in governing the spectral behavior of the asymptotic composite medium. In the case with $\zeta^* = 0$ we can fully absorb the "bulk" energy of a composite two-dimensional membrane into a lower dimensional fractal manifold, by progressively driving bulk energy into thin highly conductive two-dimensional manifolds, which asymptotically collapse into the lower dimensional set. In particular, the vanishing viscosity approach to the construction of dynamical fractals of the so-called nested type [25] in the plane, like the Sierpiński set, as collapsing thin two-dimensional manifolds, presented in the above theorem (see [34]) is new.

To prove Theorem 3.2 we combine and extend some delicate and deep results of A.Jonsson ([17]),([18]) and of A.Jonsson and H.Wallin ([19]) hence we refer to the joint work with U. Mosco ([34]) for the proof, comments and details.

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