Vectorial Ingham–Beurling type estimates

Vilmos Komornik
Département de mathématique
Université de Strasbourg
7 rue René Descartes
67084 Strasbourg Cedex, France

E-mail: komornik@math.unistra.fr

Abstract

We discuss a vectorial variant of Ingham's and Beurling's classical therems.

1 Introduction

We consider the coupled string–beam system

$$\begin{cases} u_{tt} - u_{xx} + au + bw = 0, \\ w_{tt} + w_{xxxx} + cu + dw = 0 \end{cases}$$

with usual initial conditions and with Dirichlet–hinged boundary conditions on a bounded interval $(0, \ell)$, where a, b, c, d are given coupling constants.

Given $T > 0$, we investigate the validity of the estimates

$$c_1 E(0) \le \int_0^T |u_x(t,0)|^2 + |w_x(t,0)|^2 \ dt \le c_2 E(0)$$

with suitable positive constants c_1, c_2 where $E(0)$ denotes the usual initial energy ($\mathcal{H} = H_0^1 \times L^2 \times H_0^1 \times H^{-1}$).

Following [9] we may write these estimates in the abstract form

$$c_1 \sum_{k\in\mathbb{Z}} |x_k|^2 \le \int_0^T |x(t)|^2 \ dt \le c_2 \sum_{k\in\mathbb{Z}} |x_k|^2$$

where

$$x(t) := (u, u_t, w, w_t)(t) = \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t}$$

with square-summable complex coefficients x_k. Here (U_k) is a given sequence of unit vectors in $\mathbb{C}^4$ and (ω_k) is a given sequence of real numbers, depending on the parameters of the problem (eigenvector traces and eigenvalues).

2 Statement of the results

We make the following assumptions:

(i) Let $\Omega := (\omega_k)_{k\in\mathbb{Z}}$ be a family of real numbers satisfying the *gap condition*

$$\gamma := \inf_{k\neq n} |\omega_k - \omega_n| > 0.$$

(ii) Let $(U_k)_{k\in\mathbb{Z}}$ be a corresponding family of unit vectors in some finite-dimensional complex Hilbert space H and consider the sums

$$x(t) = \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t}$$

with square summable complex coefficients x_k.

(iii) By the gap condition Ω has a finite upper density defined by

$$D^+ = D^+(\Omega) := \lim_{r\to\infty} n^+(r)/r$$

where $n^+(r)$ denotes the maximum number of terms ω_k contained in an interval of length r. We have $D^+ \leq 1/\gamma$.

We are going to discuss the followign theorem obtained in [3] in collaboration with A. Barhoumi and M. Mehrenberger:

Theorem 2.1

(a) If $T > 2\pi D^+$, then the estimates

$$c_1 \sum_{k\in\mathbb{Z}} |x_k|^2 \leq \int_0^T \|x(t)\|_H^2 \, dt \leq c_2 \sum_{k\in\mathbb{Z}} |x_k|^2$$

hold with suitable $c_1, c_2 > 0$.

(b) Conversely, if the above estimates hold true and $\dim H = d$, then $T \geq 2\pi D^+/d$.

Let us discuss this result.

Remark First we consider the scalar case $d = 1$. In this case the critical length is $T = 2\pi D^+$ and our result reduces to a theorem of Beurling [4].

(i) For $\omega_k = k$ we have $D^+ = 1$ and the critical length is 2π in correspondence with Parseval's equality:

$$\int_0^{2\pi} \Big|\sum_{k\in\mathbb{Z}} x_k e^{ikt}\Big|^2 \, dt = 2\pi \sum_{k\in\mathbb{Z}} |x_k|^2.$$

(ii) For $\omega_k = k^3$ we have $D^+ = 0$, so that

$$c_1 \sum_{k\in\mathbb{Z}} |x_k|^2 \le \int_0^T |x(t)|^2 \, dt \le c_2 \sum_{k\in\mathbb{Z}} |x_k|^2$$

for any $T > 0$ (the constants $c_1, c_2 > 0$ depend on T).

(iii) Ingham's earlier sufficient condition ensured the preceding estimates for $T > 2\pi/\gamma = 2\pi$. (We recall that $D^+ \le 1/\gamma$.)

□

Remark Next we give higher-dimensional examples.

(i) If the vectors U_k are identical, then the critical length is $2\pi D^+$ like in the one-dimensional case.

(ii) If $d > 1$, (U_k) is d-periodical and $U_1, \dots, U_d$ is an orthonormal basis of H, then the critical length is $T = 2\pi D^+/d$. Indeed,

$$\int_0^T \Big| \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t} \Big|_H^2 \, dt = \sum_{j=1}^d \int_0^T \Big| \sum_{k\in\mathbb{Z}} x_{kd+j} e^{i\omega_{kd+j} t} \Big|^2 \, dt$$

and we may apply the scalar case to each sum on the right side.

(iii) We show later that the critical length can be anything between $2\pi D^+/d$ and $2\pi D^+$.

□

In what follows we explain the proof of Theorem 2.1. It is based on our previous works in collaboration with C. Baiocchi and P. Loreti [1], [2].

3 Sufficiency of the condition $T > 2\pi D^+$

We begin with the scalat case. We recall Ingham's following classical theorem [7]:

Theorem 3.1

If

$$\gamma := \inf_{k\ne n} |\omega_k - \omega_n| > 0$$

and $T > 2\pi/\gamma$, then we have

$$c_1 \sum_{k\in\mathbb{Z}} |x_k|^2 \le \int_0^T \Big| \sum_{k\in\mathbb{Z}} x_k e^{i\omega_k t} \Big|^2 \, dt \le c_2 \sum_{k\in\mathbb{Z}} |x_k|^2.$$

Idea of the proof. By introducing suitable orthogonalizing weight functions we imitate the proof of Parseval's equality.

There are infinitely many suitable weight functions but only a very particular choice yields the theorem under the condition $T > 2\pi/\gamma$. During the extension of Ingham's theorem to higher dimension, this optimal weight function turned to be intimately related to the first eigenfunction of the Laplacian operator in a ball; see [1] or [9]. □

Ingham's condition $T > 2\pi/\gamma$ was weakened in [2] as follows:

Theorem 3.2

If $\Omega_1 \cup \dots \cup \Omega_M$ be a finite partition of $\Omega = \{\omega_k\}$ and

$$T > \frac{2\pi}{\gamma(\Omega_1)} + \dots + \frac{2\pi}{\gamma(\Omega_M)}$$

then we have

$$c_1 \sum_{k\in\mathbb{Z}} |x_k|^2 \le \int_0^T \Big|\sum_{k\in\mathbb{Z}} x_k e^{i\omega_k t}\Big|^2 \, dt \le c_2 \sum_{k\in\mathbb{Z}} |x_k|^2. \tag{3.1}$$

We note that for $M = 1$ this reduces to Ingham's theorem.

Idea of the proof. We combine a Fourier transform method of Kahane [8], by replacing an implicit estimate by a constructive one, based on a constructive method of Haraux [6]. □

Example 3.3:

For $\omega_k = k^3$ and $\Omega_j := \{\omega_{kM+j} \ : \ k \in \mathbb{Z}\}$, $j = 1, \dots, M$ we have $\gamma = 1$ but

$$\frac{2\pi}{\gamma(\Omega_1)} + \dots + \frac{2\pi}{\gamma(\Omega_M)} \le M \frac{2\pi}{M^3/4} \to 0, \quad M \to \infty.$$

Hence in this case the estimates (3.1) hold for all $T > 0$ instead of Ingham's assumption

The upper density is related to the partitions via the following result proved in [2]:

Proposition 3.4

For every $T > 2\pi D^+$ there exists a finite partition of Ω such that

$$\frac{2\pi}{\gamma(\Omega_1)} + \dots + \frac{2\pi}{\gamma(\Omega_M)} < T.$$

Proof. We choose $\gamma' > 0$ such that $T > \frac{2\pi}{\gamma'} > 2\pi D^+$, and then a large integer M such that $\frac{2\pi}{\gamma'} > 2\pi\frac{n^+(M\gamma')}{M\gamma'}$, i.e., $n^+(M\gamma') < M$.

Arranging the exponents into an increasing sequence $(\omega_k)_{k\in K}$ we have $\omega_{k+M} - \omega_k > M\gamma'$ for all k, so that the sets $\Omega_j := \{\omega_{Mk+j} \ : \ k \in K\}$ satisfy

$$\sum_{j=1}^{M} \frac{2\pi}{\gamma(\Omega_j)} \le \sum_{j=1}^{M} \frac{2\pi}{M\gamma'} = \frac{2\pi}{\gamma'} < T.$$

$\square$

The sufficiency of assumption $T > 2\pi D^+$ in the vectorial case follows from the scalar case. Indeed, we fix an orthonormal basis $(E_n)_{n\in N}$ of H and we develop each U_k into Fourier series:

$$U_k = \sum_{n\in N} u_{kn} E_n.$$

If $T > 2\pi D^+$, then using the scalar case we have

$$\begin{aligned}\int_0^T \Big\|\sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t}\Big\|_H^2 \, dt &= \sum_{n\in N} \int_0^T \Big|\sum_{k\in\mathbb{Z}} x_k u_{kn} e^{i\omega_k t}\Big|^2 \, dt \\ &\asymp \sum_{n\in N} \sum_{k\in\mathbb{Z}} |x_k u_{kn}|^2 \\ &= \sum_{k\in\mathbb{Z}} |x_k|^2\end{aligned}$$

with $\asymp$ meaning equivalence in the sense of (3.1).

4 Necessity of the condition $T \ge 2\pi D^+/d$

We may assume by scaling that

$$\int_0^{2\pi} \Big|\sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t}\Big|^2 \, dt \asymp \sum_{k\in\mathbb{Z}} |x_k|^2.$$

We need to show that $D^+ \le d$. Following Mehrenberger [10] we adapt a method of Gröchenig and Razafinjatovo.

Step 1. Fix $R > 0$, $y \in \mathbb{R}$, $r > 0$ and set

$$\begin{aligned}V = V_{y,r} &:= \mathrm{Vect}\{U_k e^{i\omega_k t} \ : \ |\omega_k - y| < r\},\\ W = W_{y,r+R} &:= \mathrm{Vect}\{U e^{ikt} \ : \ U \in H, |k - y| < r + R\}.\end{aligned}$$

Note that

$$n^+(2r) = \sup_y \dim V \quad \text{and} \quad \dim W \le (2r + 2R)d.$$

We will prove that

$$\dim V \le (1 + o_R(1)) \dim W \quad \text{as} \quad R \to \infty.$$

This will imply that

$$n^+(2r) = \sup_y \dim V \le (2r + 2R)d(1 + o_R(1))$$

and hence that

$$D^+ = \lim_{r\to\infty} \frac{n^+(2r)}{2r} \le d(1 + o_R(1))$$

for all $R > 0$. Letting $R \to \infty$ this yields $D^+ \le d$.

Step 2. Let P, Q be the orthogonal projections of $L^2(0, 2\pi; H)$ onto V and W. Then

$$S := P \circ Q|_V \in L(V, V)$$

has norm ≤ 1 and rank $\le \dim W$, so that

$$\operatorname{tr} S \le \dim W.$$

Hence the estimate $\dim V \le (1 + o_R(1)) \dim W$ will follow if we prove that

$$\operatorname{tr} S \ge (1 - o_R(1)) \dim V.$$

Step 3. Let (f_k) be a bounded biorthogonal sequence to $e_k := U_k e^{i\omega_k t}$ in $L^2(0, 2\pi; H)$. Since

$$\operatorname{tr} S = \sum_{|\omega_k - y| < r} (Se_k, f_k)_{L^2(0,2\pi;H)} = \sum_{|\omega_k - y| < r} (Qe_k, Pf_k)_{L^2(0,2\pi;H)},$$

we have

$$\begin{aligned} \dim V - \operatorname{tr} S &= - \sum_{|\omega_k - y| < r} ((Q - I)e_k, Pf_k)_{L^2(0,2\pi;H)} \\ &\le (\sup \|f_k\|)(\dim V) \sup_{|\omega_k - y| < r} \|(Q - I)e_k\|_{L^2(0,2\pi;H)} \\ &= o_R(1) \dim V \end{aligned}$$

by a direct computation, where we have assumed for a moment that

$$\|(Q - I)e_k\|_{L^2(0,2\pi;H)} = o_R(1). \tag{4.2}$$

Under this assumption we have thus proved the required estimate $\operatorname{tr} S \ge (1 - o_R(1)) \dim V$.

Step 4. For the proof of (4.2) first have, using the Fourier expansion

$$e_k = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \sum_{j=1}^{d} (e_k, E_j e^{int}) E_j e^{int}$$

where (E_j) is an orthonormal basis of H, the following estimates:

$$\begin{aligned}
\|(Q-I)e_k\|^2_{L^2(0,2\pi;H)} &= \frac{1}{2\pi} \sum_{|n-y|\geq r+R} \sum_{j=1}^{d} \int_0^{2\pi} |(e_k, E_j e^{int})|^2 \, dt \\
&= \frac{1}{2\pi} \sum_{|n-y|\geq r+R} \sum_{j=1}^{d} |(e_k, E_j)|^2 \Big| \int_0^{2\pi} e^{i(\omega_k - n)t} \, dt \Big|^2 \\
&\leq \frac{2d}{\pi} \sum_{|n-y|\geq r+R} \frac{1}{|\omega_k - n|^2}.
\end{aligned}$$

Now, since $|n-y| \geq r+R$ and $|\omega_k - y| < r$ imply $|n-\omega_k| > R$, it follows that

$$\begin{aligned}
\|(Q-I)e_k\|^2_{L^2(0,2\pi;H)} &\leq \frac{2d}{\pi} \sum_{|n-y|\geq r+R} \frac{1}{|\omega_k - n|^2} \\
&\leq \frac{4d}{\pi} \sum_{n=0}^{\infty} \frac{1}{(R+n)^2} \\
&\leq \frac{4d}{\pi} \left(\frac{1}{R^2} + \int_R^{\infty} \frac{1}{x^2} \, dx \right) \\
&= \frac{4d}{\pi R^2} + \frac{4d}{\pi R}.
\end{aligned}$$

This implies (4.2) and the proof of the theorem is completed.

5 Partitions and upper density

In order to show that the critical value of T may be anything between $2\pi D^+/d$ and $2\pi D^+$, we use the following combinatorial result obtained in [3]:

Theorem 5.1

Let Ω be a set of real numbers with a finite upper density D^+ and let $\alpha_1, \alpha_2, \ldots$ be a finite or infinite sequence of numbers in $[0, 1]$ satisfying

$$\alpha_1 + \alpha_2 + \cdots \geq 1.$$

Then there exists a partition

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots$$

such that the upper density of Ω_j is equal to $\alpha_j D^+$ for every j.

Now, given $1/d \le \alpha \le 1$ arbitrarily we choose $\alpha_1, \dots, \alpha_d \ge 0$ such that

$$\alpha_1 + \cdots + \alpha_d = 1 \quad \text{and} \quad \max\{\alpha_1, \dots, \alpha_d\} = \alpha.$$

Applying the above theorem we obtain a partition $\Omega = \Omega_1 \cup \cdots \cup \Omega_d$ such that $D^+(\Omega_j) = \alpha_j D^+$ for all j. Fix an orthonormal basis $E_1, \dots, E_d$ of H and set $U_k = E_j$ if $\omega_k \in \Omega_j$. Then using the identity

$$\int_0^T \Big\| \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t} \Big\|_H^2 \, dt = \sum_{j=1}^d \int_0^T \Big| \sum_{\omega_k \in \Omega_j} x_k e^{i\omega_k t} \Big|^2 \, dt$$

and applying the scalar case of the theorem we conclude that the required estimates hold if $T > 2\pi\alpha D^+$, and they fail if $T < 2\pi\alpha D^+$.

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The author thanks the organizers for their kind invitation to this Conference.