HOPF-LAX FORMULAS AND RELATED PROBLEMS

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ABSTRACT. This note summarizes several talks I gave on this subject in the last two years (IHP Paris 2007, Waseda Tokyo 2007, Sapienza Rome 2008, Paul Sebatier Toulouse 2008, Arlington Texas 2008, Imperial College London 2009), and it is based on several works in collaboration with A. Avantaggiati. I will give a complete reference in the bibliography.

1. A BRIEF REMAIND ON HOPF-LAX FORMULA

We assume H smooth, convex, coercive, $u_0 \in$Lip($\mathbb{R}^N$), $u_0 \in B(\mathbb{R}^N)$ (B reads bounded).

$$H^*(x) = \max_y \{xy - H(y)\}$$

As a reference to this part we may refer to L.C. Evans's book [11], in which the problem is split in three parts

- **Variational Approach.**

$$\tilde{u}(x,t) = \inf\left\{ \int_0^t H^*(\dot{\zeta}(s))ds + u_0(y) : \zeta(0) = y, \zeta(t) = x \right\}$$

- **PDEs.**
  Consider the Cauchy problem for the Hamilton-Jacobi equation

$$\begin{cases} v_t + H(Dv) = 0 & \text{in} \quad (0,+\infty) \times \mathbb{R}^N \\ v(0,x) = u_0 \end{cases} \tag{1}$$

- **Hopf-Lax formula.**

$$u(x,t) = \min_{y \in R^N} \left\{ tH^*\left(\frac{x-y}{t}\right) + u_0(y) \right\} \qquad (Hopf - Lax\ formula)$$

1.1. **Reminds on viscosity solutions.** The notion of viscosity solution was introduced by M. G. Crandall and P. L. Lions [7]. Let us recall their notion using test function, as introduced in M.G. Crandall, L.C. Evans, and P.-L. Lions [8], as a book reference we may give [2] and as an explaining paper we refer to [16], although here we use, to simplify, the vanishing viscosity method. The key point is that the notion gives a meaning to the solution of the equation also if the solution has very weak property of regularity (for example u is just a continuous function or even less).

The equation to consider is

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon) - \epsilon\Delta u^\epsilon = 0 \qquad \text{in} \quad (0,+\infty)\times\mathbb{R}^{\mathrm{N}} \\ u(0,x) = u_0 \end{cases}$$

Fix $\epsilon > 0$ and we consider a subsequence u^{ϵ_j}, such that

$$u^{\epsilon_j} \to u,$$

next, we consider $\phi \in C^1$ such that $u - \phi$ has a strict maximum at x_0. Assume that u is continuous and differerentiable at x_0. Then there exists $\phi \in C^1$ such that

$$u(x_0) = \phi(x_0)$$

and $u - \phi$ has a strict maximum at x_0. Since ϵ_j is small $u_{\epsilon_j} - \phi$ has a max in $(x_{\epsilon_j}, t_{\epsilon_j})$ with

$$(x_{\epsilon_j}, t_{\epsilon_j}) \to (x_0, t_0).$$

Moreover,

$$Du^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) = D\phi(x_{\epsilon_j}, t_{\epsilon_j})$$

$$u_t(x_{\epsilon_j}, t_{\epsilon_j}) = \phi_t(x_{\epsilon_j}, t_{\epsilon_j})$$

$$-\Delta u^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) \geq -\Delta\phi(x_{\epsilon_j}, t_{\epsilon_j})$$

$$\phi_t(x_{\epsilon_j}, t_{\epsilon_j}) + H(D\phi_t(x_{\epsilon_j}, t_{\epsilon_j}) = u_t(x_{\epsilon_j}, t_{\epsilon_j}) + H(Du_t(x_{\epsilon_j}, t_{\epsilon_j}) =$$

$$\epsilon\Delta u^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) \leq \epsilon\Delta\phi(x_{\epsilon_j}, t_{\epsilon_j})$$

$\epsilon_j \to 0$, $\phi \in C^1$, H continuous...

$$\phi_t(x_0, t_0) + H(D\phi_t(x_0, t_0) \leq 0$$

Then, we are now ready to recall the definition using test function. We say that u is a (viscosity) subsolution of (2.3) if for every $\phi \in C^1$. such that $u - \phi$has a max in x

$$\phi_t + H(x, u(x), D\phi(x)) \leq 0$$

We say that u is a viscosity supersolution if for every $\phi \in C^1$. such that $u - \phi$ has a min in x

$$\phi_t + H(x, u(x), D\phi(x)) \geq 0$$

A viscosity solution of (2.3) is a viscosity subsolution and a viscosity supersolution (of (2.3))

1.2. **Equivalence of the three problems.** The three problems are equivalent, that is the $u = \tilde{u} = v$ (v being the unique viscosity solution of (2.3)). We give the reference of the complete proof to [11], however here the rewrite the equivalence between u and $\tilde{u}$ to give an idea how to argue in this topic. We define the trajectory

$$\zeta(s) = y + \frac{s}{t}(x - y), \quad 0 \le s \le t, \quad \dot{\zeta}(s) = \frac{x - y}{t}$$

By definition, for this trajectory

$$\inf\left\{ \int_0^t H^*(\zeta(s))ds + u_0(y) : \zeta(0) = y, \zeta(t) = x \right\} \le \int_0^t H^*(\dot{\zeta}(s))ds + u_0(y) =$$

$$\int_0^t H^*\left(\frac{x-y}{t}\right)ds + u_0(y),$$

which immediately shows

(2) $$\tilde{u}(x,t) \le u(x,t)$$

Jensen's inequality gives (H^*convex)

$$H^*\left(\frac{1}{t}\int_0^t \dot{\zeta}(s)ds\right) \le \frac{1}{t}\int_0^t H^*(\dot{\zeta}(s))ds$$

Since

$$\int_0^t \dot{\zeta}(s)ds = \zeta(t) - \zeta(0) = x - y$$

$$tH^*\left(\frac{x-y}{t}\right) \le \int_0^t H^*(\dot{\zeta}(s))ds$$

$$tH^*\left(\frac{x-y}{t}\right) + u_0(y) \le \int_0^t H^*(\dot{\zeta}(s))ds + u_0(y)$$

Passing to the inf

(3) $$u(x,t) \le \tilde{u}(x,t),$$

hence

(4) $$u(x,t) = \tilde{u}(x,t)$$

To give just an idea how to pass the equation, it is relevant to show a property of u, showing a semigroup property.

(5) $$u(x,t) = \min_{y \in R^N}\left\{(t-s)H^*\left(\frac{x-y}{t-s}\right) + u(y,s)\right\}$$

Select $\hat{x}$ such that

$$u(x,t) = tH^*\left(\frac{x-\hat{x}}{t}\right) + u_0(\hat{x})$$

$$y = \frac{s}{t}x + (1 - \frac{s}{t})\hat{x}$$

$$\frac{x-y}{t-s}=\frac{x-\hat{x}}{t}=\frac{y-\hat{x}}{s}$$

$$(t-s)H^*\left(\frac{x-y}{t-s}\right)+u(y,s)=(t-s)H^*\left(\frac{x-\hat{x}}{t}\right)+u(y,s)\leq$$

$$(t-s)H^*\left(\frac{x-\hat{x}}{t}\right)+sH^*\left(\frac{y-\hat{x}}{s}\right)+u_0(\hat{x})=$$

$$tH^*\left(\frac{x-\hat{x}}{t}\right)+u_0(\hat{x})=u(x,t)$$

Passing to the min

$$\min_{y\in R^N}\left\{(t-s)H^*\left(\frac{x-y}{t-s}\right)+u(y,s)\right\}\leq u(x,t) \tag{6}$$

Next, choose $z\in R^N$

$$u(y,s)=sH^*\left(\frac{y-z}{s}\right)+u_0(z)$$

$$\frac{x-z}{t}=(1-\frac{s}{t})\frac{x-y}{t-s}+\frac{s}{t}\frac{y-z}{s}$$

By the convexity of H^*

$$H^*\left(\frac{x-z}{t}\right)\leq(1-\frac{s}{t})H^*\left(\frac{x-y}{t-s}\right)+\frac{s}{t}H^*\left(\frac{y-z}{s}\right)$$

Then

$$u(x,t)\leq$$

$$tH^*\left(\frac{x-z}{t}\right)+u_0(z)\leq(t-s)H^*\left(\frac{x-y}{t-s}\right)+sH^*\left(\frac{y-z}{s}\right)+u_0(z)=$$

$$(t-s)H^*\left(\frac{x-y}{t-s}\right)+u(y,s)$$

The result follows since y can be chosen in arbitrary way.
Now the check how it is possibile to connect the problem to the PDEs, assuming regularity for the function u and using the semigroup formula.
Fix $q\in\mathbb{R}^N$ $h>0$

$$u(x+hq,t+h)=\min_{y\in R^N}\left\{(t-s)H^*\left(\frac{x+hq-y}{h}\right)+u(y,t)\right\}\leq$$

$$hH^*(q)+u(x,t)$$

From which we deduce that

$$\frac{u(x+hq,t+h)-u(x,t)}{h}\leq H^*(q)$$

$h\to 0^+$

$$qDu+u_t-H^*(q)\leq 0,$$

the inequality being true also for the max yields

$$u_t + H(Du) \leq 0.$$

We will not show the other inequality, refering to [11]

2. New Results

In two papers, jointly with Y. Fujita and H. Ishii [9], [10] we worked on asymptotic questions for large time for the Cauchy problem

$$\begin{cases} u_t(x,t) + \alpha x Du(x,t) + H(Du(x,t)) = f(x) & \text{in} \quad \mathbb{R}^{\mathrm{N}} \times (0,+\infty) \\ u(x,0) = u_0, \end{cases} \tag{7}$$

and jointly with A. Avantaggiati, we considered the case $f = 0$, analyzing other aspects as Hopf-Lax type formulas, hypercontractivity, entropy-energy inequality, logarithmic-Sobolev inequalities. From now, we focus on this subject:

$$\begin{cases} u_t(x,t) + \alpha x Du(x,t) + H(Du(x,t)) = 0 & \text{in} \quad \mathbb{R}^{\mathrm{N}} \times (0,+\infty) \\ u(x,0) = u_0, \end{cases} \tag{8}$$

As in [3] we begin our analysis considering the one dimensional case

$$H(p) = \frac{1}{2}|p|^2 \qquad \alpha \in \mathbb{R}_+$$

and we study

$$\begin{cases} u_t(x,t) + \boxed{\alpha x u_x(x,t)} + \frac{1}{2}|u_x|^2 = 0 & \text{in} \quad \mathbb{R} \times (0,+\infty) \\ u(0,x) = u_0, \end{cases}$$

constructing the associate semigroup: this is the first step for the formula which generalizes the well-known Hopf-Lax formula. Indeed the Hopf-Lax formula can be obtained by the formula given here by limit as $\alpha \to 0^+$. Also we give extension of known results by showing hypercontractivity, ultracontractivity for the semigroup and by obtaining a class of Sobolev logarithimic inequalities. In [4] we extend our analysis by considering the problem

$$\begin{cases} u_t(x,t) + \boxed{\displaystyle\sum_{i=1}^{N} \alpha_i x_i u_{x_i}(x,t)} + \frac{1}{2}\sum_{i=1}^{N} |u_{x_i}(x,t)|^2 = 0 \,\text{in}\, \mathbb{R}^{\mathrm{N}} \times (0,+\infty) \\ u(0,x) = u_0, \end{cases}$$

We study also the case in which some α_i $(1 \leq i \leq N)$ could vanish, and this will give a mixed behaviour.

$$\begin{cases} u_t(x,t) + \boxed{\alpha x Du(x,t)} + H(Du(x,t)) = 0 & \text{in} \quad \mathbb{R}^{\mathrm{N}} \times (0,+\infty) \\ u(x,0) = u_0, \end{cases}$$

As a model of the latter case we may consider

$$H(s) = \frac{1}{p}|s|^p$$

Our work is based on several papers by I. Gentil. A reference to this arguments is S.G. Bobkov, I. Gentil, M. Ledoux

Three motivations

- **Idempotent setting**

  A very recent theory of S. Maslov describes a new approach to analysis, known as *idempotent analysis*, and the theory has recently interested a large number of mathematicians.
  This type of arguments are often used in the minimization process in the analysis of Hopf-Lax type formulas ([**?**]).
  We consider the the semiring $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$ with the operations

$$\oplus := \min, \qquad \odot := +$$

where

$$\mathbf{0} = +\infty, \qquad \mathbf{1} = 0.$$

In $\mathbb{R}_{\min}$ the idempotent analog of integration on $\mathbb{R}$ is defined by the formula

$$I(\phi) = \int_{\mathbb{R}}^{\oplus} \phi(x)dx = \inf_{x\in\mathbb{R}} \phi(x),$$

An idempotent measure on $\mathbb{R}_{\min}$ is defined by the formula

$$m_\zeta(Y) = \inf_{x\in Y} \zeta(x)$$

where $\zeta \in B(\mathbb{R}, \mathbb{R}_{\min})$, and $Y \subset \mathbb{R}$. Here $B(\mathbb{R}, \mathbb{R}_{\min})$ means the set of functions which are usually bounded (from below) in $\mathbb{R}_{\min}$, and

$$\int_{\mathbb{R}}^{\oplus} \phi(x)dm_\zeta = \int_{\mathbb{R}}^{\oplus} \phi(x) \odot \zeta(x)dx = \inf_{x\in\mathbb{R}} (\phi(x) \odot \zeta(x))$$

Let us recall the semigroup associated to Ornstein-Uhlenbeck operator

$$T_t u_0(x) = \int_{\mathbb{R}} u_0(e^{-\alpha t}x + \sqrt{1-e^{-2\alpha t}}y)d\mu(y)$$

where the measure μ is given by

$$\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha x^2}{2}} dx$$

$$u_t - u_{xx} + \alpha x u_x(x) = 0, \qquad u_0(x) = u_0$$

What is the idempotent analog ? Which measure?
Candidate of the form

$$\int_{\mathbb{R}}^{\oplus} u_0(e^{-\alpha t}x + \sqrt{1-e^{-2\alpha t}}y)dm_\psi$$

dm_ψ *invariant measure*

$$\int_{\mathbb{R}}^{\oplus} u_0(e^{-\alpha t}x + \sqrt{1-e^{-2\alpha t}}y)dm_{\psi} =$$
$$\int_{\mathbb{R}}^{\oplus} (u_0(e^{-\alpha t}x + \sqrt{1-e^{-2\alpha t}}y) \odot \psi(y)) =$$
$$\min_{y\in\mathbb{R}}(u_0(e^{-\alpha t}x + \sqrt{1-e^{-2\alpha t}}y) + \psi(y))$$

Definition. $d\psi_\alpha(x)$ is idempotent invariant with respect to the semigroup Q_t if

$$\int_{\mathbb{R}}^{\oplus} v(x) \odot Q_t u_0(x) d\psi_\alpha(x) = \int_{\mathbb{R}}^{\oplus} u_0(x) \odot Q_t v(x) d\psi_\alpha(x)$$

Theorem 2.1. The measure $d\psi_\alpha(x) = \alpha x^2$ is idempotent invariant with respect to the semigroup Q_t.

$$\int_{\mathbb{R}}^{\oplus} v(x) \odot Q_t u_0(x) d\psi_\alpha(x) =$$
$$\int_{\mathbb{R}}^{\oplus} v(x) \odot \min_y \{u_0(e^{-\alpha t}x + \sqrt{1-e^{-2\alpha t}}y) + \alpha y^2\} d\psi_\alpha(x) =$$
$$\min_x \left\{ v(x) + \min_y \{u_0(e^{-\alpha t}x + \sqrt{1-e^{-2\alpha t}}y + \alpha y^2\} + \alpha x^2 \right\} =$$
$$\min_{x,y}\{v(x) + u_0(e^{-\alpha t}x + \sqrt{1-e^{-2\alpha t}}y + \alpha(y^2+x^2)\} =$$

We consider the change of variable

$$\begin{cases} \sqrt{1-e^{-2\alpha t}}x - e^{-\alpha t}y = \zeta \\ e^{-\alpha t}x + \sqrt{1-e^{-2\alpha t}}y = \eta \end{cases}$$

This is a linear invertible trasformation

$$\begin{cases} \sqrt{1-e^{-2\alpha t}}\zeta + e^{-\alpha t}\eta = x \\ e^{-\alpha t}\zeta + \sqrt{1-e^{-2\alpha t}}\eta = y \end{cases}$$

Squaring and adding,

$$\zeta^2 + \eta^2 = x^2 + y^2,$$
$$\min_{\zeta,\eta}\{v(e^{-\alpha t}\eta + \sqrt{1-e^{-2\alpha t}}\zeta) + u_0(\eta) + \alpha(\zeta^2+\eta^2)\} =$$
$$\min_{\eta}\{u_0(\eta) + \min_{\zeta}\{v(e^{-\alpha t}\eta + \sqrt{1-e^{-2\alpha t}}\zeta)\} + \alpha\eta^2\} =$$
$$= \min_{\eta} \int_{\mathbb{R}}^{\oplus} v(e^{-\alpha t}\eta + \sqrt{1-e^{-2\alpha t}}\zeta) d\psi_\alpha(\zeta) + u_0(\eta) + \alpha\eta^2\} =$$
$$\int_{\mathbb{R}}^{\oplus} u(\eta) \odot \int_{\mathbb{R}}^{\oplus} v(e^{-\alpha t}\eta + \sqrt{1-e^{-2\alpha t}}\zeta) d\psi_\alpha(\zeta) d\psi_\alpha(\eta) =$$

$$\int_{\mathbb{R}}^{\oplus} u_0(y) \odot Q_t v(y) d\psi_\alpha(y).$$

- **Ornstein-Ulhenbeck operator** We recall the definition of the Ornstein-Uhlenbeck operator $\mathcal{L}$.

  Definition. Given $Q = (a_{i,j})_{i,j=1,\ldots,n}$ a symmetric and positive definite matrix, and $B = (b_{i,j})_{i,j=1\ldots n}$ a non null matrix, the Ornstein-Uhlenbeck operator is $\mathcal{L}f(x) = \sum_{i,j} a_{i,j} D_i D_j f(x) + BxDf$,

  The semigroup associated is given by

$$T_t u_0(x) = \int_{\mathbb{R}^n} k_t(e^{tB}x - y) u_0(y) dy,$$

  where

$$k_t(x) = \frac{1}{(4\pi)^{\frac{1}{2}} (\det Q_t)^{\frac{1}{2}}} \exp(-\frac{1}{4} Q_t^{-1} x, x),$$

  and

$$Q_t = \int_0^t e^{sB} Q e^{sB^*} ds$$

$$\begin{cases} u_t - \epsilon(\Delta u + x\nabla u) + \alpha x \nabla u + \frac{1}{2}|\nabla u|^2 = 0 & \text{in} \quad (0, +\infty) \times \mathbb{R} \\ u(0, x) = u_0 = e^{-\frac{u_0(x)}{2\epsilon}} \end{cases}$$

  Using the Hopf-Cole trasform, we arrive to the solution of

$$\begin{cases} v_t - \epsilon(\Delta v + x\nabla v) + \alpha x \nabla v = 0 & \text{in} \quad (0, +\infty) \times \mathbb{R} \\ v(0, x) = v_0 = e^{-\frac{u_0(x)}{2\epsilon}} \end{cases}$$

  ϵ is a small, positive parameter. Solution in the form

$$v(x, t, \epsilon) = C \int_{\mathbb{R}} \left[w(z, x, t, \epsilon)\right]^{\frac{1}{2\epsilon}} dz,$$

  where C does not modify the value of the limit as $\epsilon \to 0$ and

$$w(z, x, t, \epsilon) = \exp\left[-u_0(z) - \frac{(\alpha - \epsilon)}{(1 - e^{-2(\alpha-\epsilon)t})}(z - e^{-(\alpha-\epsilon)t}x)^2\right]$$

  The limit to compute is

$$\lim_{\epsilon \to 0} 2\epsilon \log \int_{\mathbb{R}} [w(z, x, t, \epsilon]^{\frac{1}{2\epsilon}} dz = \lim_{\epsilon \to 0} \log\big[\int_{\mathbb{R}} w(z, x, t, \epsilon)^{\frac{1}{2\epsilon}} dz\big]^{2\epsilon}$$

$$w_0(z, x, t) = \lim_{\epsilon \to 0} w(z, x, t, \epsilon) = \exp\big[-u_0(z) - \frac{\alpha}{1 - e^{-2\alpha t}}(e^{-\alpha t}x - z)^2\big].$$

  Theorem 2.2.

$$\lim_{\epsilon \to 0} 2\epsilon \log \int_{\mathbb{R}} [w(z, x, t, \epsilon]^{\frac{1}{2\epsilon}} dz = \log \|w_0(z, x, t)\|_{L^\infty}$$

  where

$$\|w_0(z, x, t)\|_{L^\infty} = \sup_{z \in \mathbb{R}} \exp\big[-u_0(z) - \frac{\alpha}{1 - e^{-2\alpha t}}(e^{-\alpha t}x - z)^2\big] =$$

$$\exp\big[-\min_{z\in\mathbb{R}}[u_0(z)+\frac{\alpha}{1-e^{-2\alpha t}}(e^{-\alpha t}x-z)^2]\big]$$

For the proof we refer to [3]

• **1-d: Hopf-Lax type formulas** We take $u_0 \in \mathrm{Lip}(\mathbb{R})$ and we denote by L_{u_0} the Lipschitz constant of u_0. Generalization of the Hopf-Lax formula

$$u(x,t)=\min_{z\in\mathbb{R}}\big[u_0(z)+\frac{\alpha}{1-e^{-2\alpha t}}(e^{-\alpha t}x-z)^2\big]=Q_t u_0(x),$$

which is the Lipschitz solution, in the viscosity sense, to the Hamilton-Jacobi problem

$$\begin{cases} u_t+\alpha x\nabla u+\frac{1}{2}|\nabla u|^2=0 & \text{in} \quad (0,+\infty)\times\mathbb{R} \\ u(0,x)=u_0 \end{cases}$$

Let us observe that there are equivalent representations. For instance, we can rewrite as

$$u(x,t)=\min_{z\in\mathbb{R}}\left[u_0(z)+\alpha\frac{1-e^{-\alpha t}}{1+e^{-\alpha t}}\left(\frac{e^{-\alpha t}x-z}{1-e^{-\alpha t}}\right)^2\right],$$

Moreover, if we set

$$y=\frac{z-e^{-\alpha t}x}{\sqrt{1-e^{-\alpha t}}},$$

we have the following

$$u(x,t)=\min_{z\in\mathbb{R}}\left[u_0(e^{-\alpha t}x+\sqrt{1-e^{-2\alpha t}}z)+\alpha z^2\right].$$

We set

$$Q_t u_0(x)=\min_{z\in\mathbb{R}}\left[u_0(e^{-\alpha t}x+\sqrt{1-e^{-2\alpha t}}z)+\alpha z^2\right],$$

Corollary 2.3. The formula holds

$$u(x,t)=\min_{y\in\mathbb{R}}\left\{u(y,s)+\alpha\frac{1-e^{-\alpha(t-s)}}{1+e^{-\alpha(t-s)}}\left(\frac{y-e^{-\alpha(t-s)}x}{1-e^{-\alpha(t-s)}}\right)^2\right\}$$

and we wish to show

Theorem 2.4. For any s and $t\in(0,+\infty)$

$$Q_{t+s}=Q_t(Q_s), \qquad \lim_{t\to 0}Q_t=I.$$

Since

$$Q_s u_0(x)=\min_{w\in\mathbb{R}}\left[u_0(e^{-\alpha s}x+\sqrt{1-e^{-2\alpha s}}z)+\alpha z^2\right],$$

we have

$$Q_t(Q_s f)(x)=\min_{z\in\mathbb{R}}\left\{\min_{w\in\mathbb{R}}\left[f(e^{-\alpha s}(xe^{-\alpha t}+\right.\right.$$

$$\left.\left.\sqrt{1-e^{-2\alpha t}}z)+\sqrt{1-e^{-2\alpha s}}w)+\alpha w^2\right]+\alpha z^2\right\}=$$

$$\min_{z\in\mathbb{R}}\left\{\min_{w\in\mathbb{R}}\left[f(e^{-\alpha(s+t)}x+e^{-\alpha s}\sqrt{1-e^{-2\alpha t}}x+\sqrt{1-e^{-2\alpha s}}w)+\alpha w^2\right]+\alpha z^2\right\}$$

We consider the change of variable

$$\begin{cases} e^{-\alpha s}\sqrt{1-e^{-2\alpha t}}z+\sqrt{1-e^{-2\alpha s}}w=\sqrt{1-e^{-2\alpha(s+t)}}u \\ -\sqrt{1-e^{-2\alpha s}}z+e^{-\alpha s}\sqrt{1-e^{-2\alpha t}}w=v \end{cases}$$

This is a linear invertible trasformation of $\mathbb{R}^2$: $(z,w)\to(u,v)$ whose coefficients determinant is

$$\begin{vmatrix} e^{-\alpha s}\sqrt{1-e^{-2\alpha t}} & \sqrt{1-e^{-2\alpha s}} \\ -\sqrt{1-e^{-2\alpha s}} & e^{-\alpha s}\sqrt{1-e^{-2\alpha t}} \end{vmatrix} = 1-e^{-2\alpha(t+s)}$$

Squaring and adding,

$$(1-e^{-2\alpha(t+s)})(z^2+w^2)=(1-e^{-2\alpha(s+t)})u^2+v^2,$$

which gives

$$z^2+w^2=u^2+\frac{1}{1-e^{-2\alpha(s+t)}}v^2$$

Since

$$\min_{z\in\mathbb{R}}\left\{\min_{w\in\mathbb{R}}\left[f(e^{-\alpha(s+t)}x+e^{-\alpha s}\sqrt{1-e^{-2\alpha t}}x+\sqrt{1-e^{-2\alpha s}}w)+\alpha w^2\right]+\alpha z^2\right\}=$$

$$\min_{u\in\mathbb{R}}\left\{\min_{v\in\mathbb{R}}\left[f(e^{-\alpha(s+t)}x+\sqrt{1-e^{-2\alpha(t+s)}}u)+\alpha u^2\right]+\frac{\alpha}{1-e^{-2\alpha(t+s)}}v^2\right\}$$

Now, we observe that the sum of the first two terms is constant with respect to v, hence the minimum with respect to v is attained at $v=0$, and, in conclusion,

$$\min_{z\in\mathbb{R}}\left\{\min_{w\in\mathbb{R}}\left[f(e^{-\alpha(s+t)}x+e^{-\alpha s}\sqrt{1-e^{-2\alpha t}}x+\sqrt{1-e^{-2\alpha s}}w)+\alpha w^2\right]+\alpha z^2\right\}=$$

$$\min_{u\in\mathbb{R}}\left[f(e^{-\alpha(s+t)}x+\sqrt{1-e^{-2\alpha(t+s)}}u)+\alpha u^2\right]$$

which ends the first part of the proof.

To conclude we consider the representation formula

$$u(x,t)=Q_t u_0(x)=\min_{z\in\mathbb{R}}\left[u_0(z)+\alpha\frac{1-e^{-\alpha t}}{1+e^{-\alpha t}}\left(\frac{e^{-\alpha t}x-z}{1-e^{-\alpha t}}\right)^2\right],$$

We take $z=x$, then

$$u(x,t)\le u_0(x)+\alpha\frac{1-e^{-\alpha t}}{1+e^{-\alpha t}}x^2$$

$$u(x,t)-u_0(x)\le\alpha(1-e^{-\alpha t})M^2 \qquad \forall x\in[-M,M],$$

where M is any fixed positive number. On the other hand,

$$\min_{z\in\mathbb{R}}\left[u_0(z)+\alpha\frac{1-e^{-\alpha t}}{1+e^{-\alpha t}}\left(\frac{e^{-\alpha t}x-z}{1-e^{-\alpha t}}\right)^2\right]=$$

$$u_0(e^{-\alpha t}x)+\min_{z\in\mathbb{R}}\left[u_0(z)-u_0(e^{-\alpha t}x)+\alpha\frac{1-e^{-\alpha t}}{1+e^{-\alpha t}}\left(\frac{e^{-\alpha t}x-z}{1-e^{-\alpha t}}\right)^2\right]\geq$$

$$u_0(e^{-\alpha t}x)+\min_{z\in\mathbb{R}}\left[-|u_0(z)-u_0(e^{-\alpha t}x)|+\alpha\frac{1-e^{-\alpha t}}{1+e^{-\alpha t}}\left(\frac{e^{-\alpha t}x-z}{1-e^{-\alpha t}}\right)^2\right]=$$

$$u_0(e^{-\alpha t}x)-\max_{z\in\mathbb{R}}\left[|u_0(z)-u_0(e^{-\alpha t}x)|-\alpha\frac{1-e^{-\alpha t}}{1+e^{-\alpha t}}\left(\frac{e^{-\alpha t}x-z}{1-e^{-\alpha t}}\right)^2\right]=$$

We set

$$y=\frac{e^{-\alpha t}x-z}{1-e^{-\alpha t}},$$

then

$$u(x,t)\geq u_0(e^{-\alpha t}x)-\max_{y\in\mathbb{R}}\left\{L_{u_0}|y|-\frac{\alpha}{1+e^{-\alpha t}}|y|^2\right\}(1-e^{-\alpha t})$$

Next, we set

$$C=\max_{|y|}\left\{L_{u_0}|y|-\frac{\alpha}{1+e^{-\alpha t}}|y|^2\right\},$$

from which

$$u(x,t)-u_0(x)\geq u_0(e^{-\alpha t}x)-u_0(x)-C(1-e^{-\alpha t})\geq-(1-e^{-\alpha t})[L_{u_0}|x|-C]$$

Hence in any bounded interval $[-M,M]$ there exists a constant K such that

$$|u(x,t)-u_0(x)|\leq K(1-e^{-\alpha t}),$$

which ends the proof.

2.1. **Mixed case.** We consider the following Cauchy problem

$$\begin{cases} u_t(x,t)+\frac{1}{2}|Du(x,t)|^2+\sum_{i=1}^N \alpha_i x_i u_{x_i}(x,t)=0 \qquad \text{in} \quad \mathbb{R}^{\mathrm{N}}\times(0,+\infty) \\ u(0,x)=u_0, \qquad \text{in}\, \mathbb{R}^{\mathrm{N}} \end{cases}$$

We shall use the following assumptions:

$$u_0 \in \mathrm{Lip}(\mathbb{R}^{\mathrm{N}}) \text{ i.e. } |\mathrm{u_0(x)-u_0(y)}| \le \mathrm{L_{u_0}|x-y|} \qquad \forall \mathrm{x,y} \in \mathbb{R}^{\mathrm{N}}$$

$$\alpha_1,\alpha_2,\dots,\alpha_N \in \mathbb{R}_+.$$

We introduce the operator applied to a function f of one variable, x_j, by

$$(Q_t^{\alpha_j} f)(x_j)=\min_{y_j\in\mathbb{R}}\big[f(y_j)+\frac{\alpha_j}{1-e^{-2\alpha_j t}}(y_j-e^{-\alpha_j t}x_j)^2\big].$$

We proved that $t\to Q_t f$ has the semigroup properties. Hence $Q_t^{\alpha_1}(x_1), Q_t^{\alpha_2}(x_2)$, $Q_t^{\alpha_3}(x_3)\dots, Q_t^{\alpha_N}(x_N)$ are semigroups. Then we define (here $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_N)$)

$$(Q_t^{\alpha}u_0)(x_1,\dots,x_N):=(Q_t^{\alpha_1}\dots,Q_t^{\alpha_N})u_0(x_1,\dots,x_N)$$

Remark. By the permutability between $Q_t^{\alpha_1}$, $Q_t^{\alpha_2}$, ... Q_t^{α} does not depend on the order where $Q_t^{\alpha_j}$ appears in the formula.

Theorem 2.5. Then the following properties hold true

$$Q_t^{\alpha}(Q_s^{\alpha}u_0)(x_1,\dots,x_N)=(Q_{s+t}^{\alpha}u_0)(x_1,\dots,x_N)$$

For every compact set K of $\mathbb{R}^N$

$$\lim_{t\to 0^+} Q_t^{\alpha}u_0(x_1,\dots,x_N)=u_0(x_1,\dots,x_N)$$

uniformly on K

$$Q_t^{\alpha}(Q_s^{\alpha}u_0)(x_1,\dots,x_N)=Q_t^{\alpha_1}\dots,Q_t^{\alpha_N}Q_s^{\alpha_1},\dots,Q_s^{\alpha_N}(u_0)=$$
$$(Q_t^{\alpha_1}Q_s^{\alpha_1})(Q_t^{\alpha_2}Q_s^{\alpha_2})\dots(Q_t^{\alpha_N}Q_s^{\alpha_N})(u_0)=Q_{s+t}^{\alpha_1}Q_{s+t}^{\alpha_2}\dots Q_{s+t}^{\alpha_N}=Q_{s+t}^{\alpha}u_0.$$

The property is proved. Neverthless it may be useful to give the following direct proof:

$$(Q_s^{\alpha}u_0)(x_1,\dots,,s_N)=$$
$$\min_{y\in\mathbb{R}^N}\big[u_0(e^{-\alpha_1 s}x_1+\sqrt{1-e^{-2\alpha_1 s}}y_1,\dots,$$
$$e^{-\alpha_N s}x_N+\sqrt{1-e^{-2\alpha_N s}}y_N)+\sum_{i=1}^N \alpha_i y_i^2]$$

Then

$$Q_t^{\alpha}(Q_s^{\alpha}u_0)(x_1,\dots,,x_N)=$$
$$\min_{z\in\mathbb{R}^N}\big\{\min_{y\in\mathbb{R}^N}[u_0(e^{-\alpha_1 s}[e^{-\alpha_1 t}x_1+\sqrt{1-e^{-2\alpha_1 t}}z_1]+\sqrt{1-e^{-2\alpha_1 s}}y_1,\dots,$$
$$e^{-\alpha_N s}[e^{-\alpha_N t}x_N+\sqrt{1-e^{-2\alpha_N t}}z_N]+\sqrt{1-e^{-2\alpha_N s}}y_N)+\sum_{i=1}^N \alpha_i y_i^2]+\sum_{i=1}^N \alpha_i z_i^2\big\}=$$

$$\min_{(z,y)\in\mathbb{R}^{2N}}\big\{u_0(e^{-\alpha_1(s+t)}x_1+e^{-\alpha_1 s}\sqrt{1-e^{-2\alpha_1 t}}z_1+\sqrt{1-e^{-2\alpha_1 s}}y_1,\dots,$$

$$e^{-\alpha_N(s+t)}x_N+e^{-\alpha_N s}\sqrt{1-e^{-2\alpha_N t}}z_N+\sqrt{1-e^{-2\alpha_N s}}y_N)+\sum_{i=1}^N\alpha_i(y_i^2+z_i^2)\big\}$$

We consider the change of variables

$$\begin{cases} e^{-\alpha_j s}\sqrt{1-e^{-2\alpha_j t}}z_j+\sqrt{1-e^{-2\alpha_j s}}y_j=\sqrt{1-e^{-2\alpha_j(s+t)}}u_j\\ -\sqrt{1-e^{-2\alpha_j s}}z_j+e^{-\alpha_j s}\sqrt{1-e^{-2\alpha_j t}}y_j=v_j\end{cases}$$

for $j=1,\dots,N$.

$$u_0(e^{-\alpha_1(s+t)}x_1+e^{-\alpha_1 s}\sqrt{1-e^{-2\alpha_1 t}}z_1+\sqrt{1-e^{-2\alpha_1 s}}y_1,\dots,$$

$$e^{-\alpha_N(s+t)}x_N+e^{-\alpha_N s}\sqrt{1-e^{-2\alpha_N t}}z_N+\sqrt{1-e^{-2\alpha_N t}}y_N)+\sum_{i=1}^N\alpha_i(y_i^2+z_i^2)=$$

$$u_0(e^{-\alpha_1(s+t)}x_1+\sqrt{1-e^{-2\alpha_1(t+s)}}u_1,\dots,$$

$$e^{-\alpha_N(s+t)}x_N+\sqrt{1-e^{-2\alpha_N(s+t)}}u_N)+\sum_{i=1}^N\alpha_i u_i^2+\sum_{i=1}^N\frac{\alpha_i}{1-e^{-2\alpha_1(t+s)}}v_j^2$$

We call $F(x,t,s,y,z)$ the function which appears on the left hand side and $Q(x,t,s,u,v)$ in the right one, the above equality reads

$$F(x,t,s,y,z)=Q(x,t,s,u,v)$$

The proof of the formula

$$\min_{(y,z)\in\mathbb{R}^{2N}}F(x,t,s,y,z)=\min_{(u,v)\in\mathbb{R}^{2N}}Q(x,t,s,u,v)$$

and

$$\min_{(y,z)\in\mathbb{R}^{2N}}F(x,t,s,y,z)=\min_{u\in\mathbb{R}^N}\big\{\min_{v\in\mathbb{R}^N}Q(x,t,s,u,v)\big\},$$

is trivial. From this observation, we see that the minimum is attained for $v=0$. Substituting this value inside the formula, we see that

$$Q_t^\alpha(Q_s^\alpha u_0)(x_1,\dots,x_N)=$$

$$=\min_{u\in\mathbb{R}^N}[u_0(e^{-\alpha_1(s+t)}x_1+\sqrt{1-e^{-2\alpha_1(t+s)}}u_1,\dots,$$

$$e^{-\alpha_N(s+t)}x_N+e^{-\alpha_N(s+t)}\sqrt{1-e^{-2\alpha_N s}}u_N)+\sum_{i=1}^N\alpha_i u_i^2]=$$

$$(Q_{s+t}^\alpha u_0)(x_1,\dots,x_N)$$

We fix $N=n+m$ and we represent the N-ple of $\mathbb{R}^N$ as $(x,x')\in\mathbb{R}^n\times\mathbb{R}^m$, $x=(x_1,\dots,x_n)$; $x'=(x_1',\dots,x_m')$, and the function f defined in $\mathbb{R}^N=\mathbb{R}^n\times\mathbb{R}^m$, which we represent with the notation $f(x,x')=f(x_1,\dots,x_n,x_1',\dots,x_m')$. In a similar way we will use $u(x,x',t)=u(x_1,\dots,x_n,x_1',\dots,x_m',t)$, and we will denote the gradient (D,D') with respect to the variables in $\mathbb{R}^n\times\mathbb{R}^m$, with the position $D=(\partial_{x_1},\dots,\partial_{x_n})$ and $D'=(\partial_{x'_1},\dots,\partial_{x'_n})$ We will deal with the Cauchy problem

$$\begin{cases} u_t(x,x',t)+\frac{1}{2}|Du(x,x',t)|^2+ \\ \frac{1}{2}|D'u(x,x',t)|^2+\sum_{i=1}^n \alpha_i x_i u_{x_i}(x,x',t)=0 \qquad \text{in} \quad \mathbb{R}^{\mathrm{N}}\times(0,+\infty) \\ u(0,x,x')=u_0, \qquad \text{in}\, \mathbb{R}^{\mathrm{N}} \end{cases}$$

A candidate function to be a solution is

$$u(x,x',t)=\min_{(y,y')\in\mathbb{R}^N}\big\{u_0(y,y')+\sum_{j=1}^n \frac{\alpha_j}{1-e^{-2\alpha_j t}}(y_j-e^{-\alpha_j t}x_j)^2+\frac{1}{2t}|x'-y'|^2\big\}.$$

We need to define together with the semigroup $Q_t^{\alpha_1},\dots,Q_t^{\alpha_n}$ we introduced above (from now denoted by $Q_{t,x_1}^{\alpha_1},\dots,Q_{t,x_n}^{\alpha_n}$) the semigroup

$$Q_{t,x_1}^{\alpha_1},\dots,Q_{t,x_n}^{\alpha_n},Q_{t,x'_1}^{0},\dots,Q_{t,x'_m}^{0}$$

where Q_{t,x_1}^0 is the usual Hopf-Lax semigroup applied to the variable $x_j, \quad j=1,\dots,m$

$$Q_{t,x'_j}^0(f)(x'_j)=\min_{y'_j}\big\{f(y'_j)+\frac{1}{2t}(y'_j-x'_j)^2\big\}.$$

We observe that the one dimensional semigroups

$$Q_{t,x_1}^{\alpha_1},\dots,Q_{t,x_n}^{\alpha_n},Q_{t,x'_1}^{0},\dots,Q_{t,x'_m}^{0}$$

applied to functions of $n+m$ variables $x_1,\dots,x_n,x'_1,\dots,x'_m$ are pairwise permutable. Then the function will be

$$u(x,x',t)=Q_{t,x_1}^{\alpha_1},\dots,Q_{t,x_n}^{\alpha_n},Q_{t,x'_1}^{0},\dots,Q_{t,x'_m}^{0}(u_0)(x,x'),$$

and also

$$u(x,x',t)=\min_{(z,z')}\big\{u_0(e^{-\alpha_1 t}x_1+\sqrt{1-e^{-2\alpha_1 t}}z_1,\dots,$$

$$e^{-\alpha_n t}x_n+\sqrt{1-e^{-2\alpha_n t}}z_n,x'_1+\sqrt{t}z'_1,\dots,x'_m+\sqrt{t}z'_m)+\sum_{i=1}^N \alpha_i z_i^2+\frac{1}{2}|z'|^2\big\},$$

obtained by the change of variables

$$\begin{cases} y_j=e^{-\alpha_j t}x_j+\sqrt{1-e^{-2\alpha_j t}}z_j \qquad & j=1,\dots,n \\ y'_l=x'_l+\sqrt{t}z'_l, & l=1,\dots,m \end{cases}$$

In the following we shall use the notation

$$Q_t^{(\alpha,0)}(u_0)(x,x')=u(x,x',t)$$

It is not difficult to show that

$$Q_t^{(\alpha,0)}Q_s^{(\alpha,0)}=Q_{t+s}^{(\alpha,0)} \qquad \forall s,t\in\mathbb{R}^+.$$

Indeed we can use the pairwise permutability of the one dimensional semigroups. In a similar way from the Lipschitzianity of $u_0(x,x')$ we deduce the same property (with a different constant) for the function $u(x,x',t)$, and,

also, the uniform convergence on compact subset to the initial datum as $t \to 0^+$.
The semigroup properties allow us to show tha u is viscosity solution of the Cauchy problem. Moreover, denoting by

$$Q_t^{(\alpha,\alpha')} = Q_{t,x_1}^{\alpha_1}, \dots, Q_{t,x_n}^{\alpha_n}, Q_{t,x'_1}^{\alpha'_1}, \dots, Q_{t,x'_m}^{\alpha'_n}$$

the following holds

Theorem 2.6. If $u_0 \in \mathrm{Lip}(\mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{m}})$, then for any compact subset K of $\mathbb{R}^n \times \mathbb{R}^m$ we have

$$\lim_{\alpha' \to 0} Q_t^{(\alpha,\alpha')}(u_0)(x,x') = Q_t^{(\alpha,0)}(u_0)(x,x'),$$

uniformly on K.

2.2. **General case α is real positive number.** $H^\star$ the conjugate of H, i.e. the Legendre trasform of H, defined by

$$H^\star(\zeta) = \sup_{x\in\mathbb{R}^N} \{x\zeta - H(x)\} \qquad \zeta \in \mathbb{R}^N$$

As well known $H^\star$ is non negative, convex function and positively homogeneous of degree q, where $\frac{1}{p} + \frac{1}{q} = 1$.

We introduce

$$Q_t u_0(x,t) = u(x,t) = \min_{y\in\mathbb{R}^n}\left\{ u_0(y) + \left(\frac{\alpha p}{1-e^{-\alpha p t}}\right)^{q-1} H^\star(y - e^{-\alpha t}x)\right\} =$$

$$\min_{z\in\mathbb{R}^n}\left\{ u_0\left(e^{-\alpha t}x + \left(\frac{1-e^{-\alpha p t}}{\alpha p}\right)^{\frac{1}{p}} z\right) + H^\star(z)\right\}$$

Then we have the following

Theorem 2.7. Under the assumptions

- (a1) $H : \mathbb{R}^N \to \mathbb{R}$ is an even, non negative, convex function and positively homogeneous of degree p, with $p > 1$.
- (a2) u_0 (the initial data) are Lipschitz continuous, with Lipschitz constant L_{u_0}; α is a real positive number.

the application $t \to Q_t$ has the semigroup property

$$Q_t^\alpha(Q_s^\alpha u_0)(x) = (Q_{s+t}^\alpha u_0)(x)$$
$$\lim_{t\to 0^+} Q_t^\alpha u_0(x) = u_0(x)$$

uniformly on the compact sets of $\mathbb{R}^N$.

Let us show that it is a viscosity solution

$$u(x,t) \le u(y,s) + \left(\frac{\alpha p}{1-e^{-p\alpha(t-s)}}\right)^{q-1} H^\star(y - e^{-\alpha p(t-s)}x),$$

for any $y \in \mathbb{R}^N$ and $s \in (0,t)$.

We fix $\phi \in C^1(\mathbb{R}^N \times R_+)$ and we assume that (x_0,t_0) is a relative maximum point to $u - \phi$, so we assume that there exists a neighborhood I_0 of (x_0,t_0) such that

$$u(x_0,t_0) - \phi(x_0,t_0) \ge u(x,t) - \phi(x,t)$$

We have that for $(y,s) \in I_0$, and $s \in (0,t)$:

$$\phi(x_0,t_0) - \phi(y,s) \le u(x_0,t_0) - u(y,s) \le$$
$$\left(\frac{\alpha p}{1-e^{-p\alpha(t_0-s)}}\right)^{q-1} H^\star(y - e^{-\alpha p(t_0-s)}x)$$

We set

$$h = 1 - e^{-\alpha(t_0-s)} \qquad y - e^{-p\alpha(t_0-s)}x_0 = -h\kappa$$

from which

$$s = t_0 - \frac{1}{\alpha}\log\frac{1}{1-h} \qquad y = x_0 - h(x_0+\kappa)$$

We have

$$\phi(x_0,t_0)-\phi(x_0-h(x_0+\kappa),t_0-\frac{1}{\alpha}\log\frac{1}{1-h})\le h\left(\frac{\alpha p}{1-(1-h)^p}\right)^{q-1}H^\star(h\kappa)$$

then

$$\frac{\phi(x_0,t_0)-\phi(x_0-h(x_0+\kappa),t_0-\frac{1}{\alpha}\log\frac{1}{1-h})}{h}\le\left(\frac{\alpha ph}{1-(1-h)^p}\right)^{q-1}H^\star(\kappa)$$

Taking into account

$$\lim_{h\to 0^+}\frac{h}{\big(1-(1-h)^p\big)}=\frac{1}{p},$$

sending $h\to 0$, we obtain that

$$D\phi(x_0,t_0)(x_0+\kappa)+\frac{1}{\alpha}\phi_t(x_0,t_0)\le\alpha^{q-1}H^\star(\kappa),$$

for any $\kappa\in\mathbb{R}^N$.

$$\kappa D\phi(x_0,t_0)-\alpha^{q-1}H^\star(\kappa)=\frac{1}{\alpha}\Big\{(\kappa\alpha D\phi(x_0,t_0)-H^\star(\alpha\kappa)\Big\}$$

Hence, using the Legendre trasform we finally get

$$\phi_t(x_0,t_0)+\alpha x_0 D\phi(x_0,t_0)+H(D\phi(x_0,t_0))\le 0,$$

and u is a viscosity subsolution.

Next we show the u is a supersolution.

Now assume that $u-\chi$ has a local minimum point in (x_0,t_0).

By assumption there exists a neighbourhood I_0 such that

$$u(x_0,t_0)-\chi(x_0,t_0)\le u(x,t)-\chi(x,t),\qquad\forall(x,t)\in I_0$$

or

$$\chi(x_0,t_0)-\chi(x,t)\ge u(x_0,t_0)-u(x,t),\qquad\forall(x,t)\in I_0$$

We have to prove

$$\chi_t(x_0,t_0)+H(D\chi(x_0,t_0))+\alpha x_0 D\chi_(x_0,t_0)\ge 0,$$

We argue by contradiction, and we assume that for all (x,t) in a neighbourhood J of (x_0,t_0) and for some positive θ

$$\chi_t(x,t)+H(D\chi(x_0,t_0))+\alpha x_0 D\chi(x,t)\le-\theta<0\qquad\forall(x,t)\in J_{(x_0,t_0)}\cap I_0,$$

We use the Legendre trasform, and we observe that $(p(q-1)=q)$

$$\frac{1}{\alpha}H(D\chi(x,t))=\alpha^{q-1}H\left(\frac{D\chi(x,t)}{a^{q-1}}\right)\ge\alpha^{q-1}\left\{\kappa\left(\frac{D\chi(x,t)}{\alpha^{q-1}}-H^\star(\kappa)\right\}=\right.$$

$$\kappa D\chi(x,t)-\alpha^{q-1}H^\star(\kappa)$$

Then

$$\frac{1}{\alpha}\chi_t(x,t)+(x_0+\kappa)D\chi(x,t)\le -\frac{\theta}{\alpha}+\alpha^{q-1}H^\star(\kappa)$$

We have

$$\chi(x_0,t_0)-\chi(x,t)\ge u(x_0,t_0)-u(x,t),\qquad \forall(x,t)\in I_0$$

We take a neighbourhood $J=I\cap I_0$, For h positive and small enough we fix y the point where the minimum is realized such that $(y,s)\in J$. Then we set

$$\begin{cases} y=x_1\\ s=t_0-\frac{1}{\alpha}\log\frac{1}{1-h}, \end{cases}$$

and

$$u(x_0,t_0)-u\big(x_1,t_0-\frac{1}{\alpha}\log\frac{1}{1-h}\big)=$$
$$\left(\frac{\alpha p}{1-e^{-p\alpha(t_0-s)}}\right)^{q-1}H^\star(x_1-e^{-\alpha p(t_0-s)}x_0)=$$
$$h\alpha^{q-1}H^\star(\kappa)+ho(1)$$

where

$$\kappa=\frac{-x_1+(1-h)x_0}{h},\quad \text{i.e.}\quad h(\kappa+x_0)=x_0-x_1$$

On the other hand

$$\chi(x_0,t_0)-\chi\big(x_1,t_0-\frac{1}{\alpha}\log\frac{1}{(1-h)}\big)=$$
$$\int_0^1\frac{d}{ds}\chi\big(x_1+s(x_0-x_1),t_0+(s-1)\frac{1}{\alpha}\log\frac{1}{(1-h)}\big)ds=$$
$$\int_0^1 D\chi\big(x_1+s(x_0-x_1),t_0+(s-1)\frac{1}{\alpha}\log\frac{1}{(1-h)}\big)(x_0-x_1)ds+$$
$$\int_0^s\frac{1}{\alpha}\log\frac{1}{(1-h)}\chi_t\big(x_1+s(x_0-x_1),t_0+(s-1)\frac{1}{\alpha}\log\frac{1}{(1-h)}\big)ds$$

We set

$$\begin{cases} x(s)=x_1+s(x_0-x_1)\\ t(s)=t_0+(s-1)\frac{1}{\alpha}\log\frac{1}{(1-h)}, \end{cases}$$

Taking $\omega(h):=\frac{-\log(1-h)}{h}$, we have

$$\chi(x_0,t_0)-\chi\big(x_1,t_0-\frac{1}{\alpha}\log\frac{1}{(1-h)}\big)=h\int_0^1 D\chi(x(s),t(s))(q+x_0)+\omega(h)\frac{1}{\alpha}\chi_t(x(s),t(s))ds$$

From which

$$\chi(x_0,t_0)-\chi\big(x_1,t_0-\frac{1}{\alpha}\log\frac{1}{(1-h)}\big)=$$
$$h\int_0^1\left[D\chi(x(s),t(s))(q+x_0)+\frac{1}{\alpha}\chi(x(s),t(s))\right]ds+$$

$$h(\omega(h)-1)\int_0^1 \frac{1}{\alpha}\chi_t(x(s),t(s))ds$$

Hence,

$$\chi(x_0,t_0)-\chi\big(x_1,t_0-\frac{1}{\alpha}\log\frac{1}{(1-h)}\big)\leq$$
$$h\int_0^1\left(\alpha^{q-1}H^\star(\kappa)-\frac{1}{\alpha}\theta\right)ds+h(\omega(h)-1)\frac{1}{\alpha}\int_0^1\chi_t(x(s),t(s))ds=$$
$$h\alpha^{q-1}H^\star(\kappa)-h\frac{\theta}{\alpha}+h(\omega(h)-1)\frac{1}{\alpha}\int_0^1\chi_t(x(s),t(s))ds$$

Finally

$$\chi(x_0,t_0)-\chi\big(x_0-h(x_0+\kappa),t_0-\frac{1}{\alpha}\log\frac{1}{(1-h)}\big)\leq$$
$$u(x_0,t_0)-u\big(x_0-h(x_0+\kappa,t_0-\frac{1}{\alpha}\log\frac{1}{(1-h)}\big)-\sigma(h),$$

for h small enough

$$\sigma(h)=h\frac{\theta}{\alpha}-ho(1)-h\frac{(\omega(h)-1))}{\alpha}\int_0^1\chi_t(x(s),t(s))ds>0$$

which contradicts the assumption that (x_0,t_0) is a relative mimimum point to $u-\chi$.

2.3. **Hypercontractivity.**

$$Q_t u_0(x) = u(x,t) = \min_{y\in\mathbb{R}^n}\left\{u_0(y) + \left(\frac{\alpha p}{1-e^{-\alpha p t}}\right)^{q-1} H^\star(y - e^{-\alpha t}x)\right\}$$

We fix the numbers η and ω such that

$$0 < \eta \le \omega,$$

and we set

$$a = \frac{\eta}{\omega} e^{-\alpha t}$$

and we introduce the functions

$$u(x) = \exp[\omega e^{\alpha t}(Q_t u_0)(x)]; \qquad v(x) = \exp[-\gamma H^\star(x)];$$

$$w(x) = \exp[\eta u_0(\frac{\omega}{\eta}x)]$$

where γ has to be fixed later. u_0 is an admissible function, this means that u_0 belong to a suitable functional space to justify the computation we are going to do.

$$(u(x))^a = \exp\left[\eta(Q_t u_0)(x)\right] \le \exp\left\{\eta[u_0(e^{-\alpha t}x + \left(\frac{1-e^{-\alpha p t}}{\alpha p}\right)^{\frac{1}{p}} z) + H^\star(z)]\right\} =$$

$$\exp\left\{\eta\left(u_0(\frac{\omega}{\eta}(\frac{\eta}{\omega}e^{-\alpha t}x + \frac{\eta}{\omega}\left(\frac{1-e^{-\alpha p t}}{\alpha p}\right)^{\frac{1}{p}} z)\right) + \eta H^\star(z)\right\}$$

Now we set

$$\frac{\eta}{\omega}\left(\frac{1-e^{-\alpha p t}}{\alpha p}\right)^{\frac{1}{p}} z = \left(1 - \frac{\eta}{\omega}e^{-\alpha t}\right) y,$$

which means

$$z = \frac{1 - \frac{\eta}{\omega}e^{-\alpha t}}{\frac{\eta}{\omega}\left(\frac{1-e^{-\alpha p t}}{a p}\right)^{\frac{1}{p}}} y$$

We have

$$(u(x))^a (v(y))^{1-a} \le \exp\left\{\eta u_0\left[\frac{\omega}{\eta}(ax + (1-a)y\right] +\right.$$

$$\left.\eta H^\star\left(\frac{\omega}{\eta}\frac{(\alpha p)^{\frac{1}{p}}(1-a)}{(1-e^{-\alpha p t})^{\frac{1}{p}}} y\right) - \gamma(1-a)H^\star(y)\right\}$$

Then we select γ

$$\gamma = \frac{\omega^q}{\eta^{q-1}}\left(\frac{(\alpha p)(1-a)}{(1-e^{-\alpha p t})}\right)^{q-1}$$

Since $\frac{q}{p} = q-1$ and $H^\star$ is q-homogeneous we have

$$(u(x))^a (v(y))^{1-a} \le \exp\left\{\eta u_0\left[\frac{\omega}{\eta}(ax + (1-a)y)\right]\right\} = w(ax + (1-a)y)$$

We apply the Brunn-Minkowski inequality, and we get

$$\Big(\int_{\mathbb{R}^N} u(x)dx\Big)^a\Big(\int_{\mathbb{R}^N} v(x)dx\Big)^{1-a} \le \int_{\mathbb{R}^N} w(x)dx.$$

Now we compute

$$\int_{\mathbb{R}^N} w(x)dx = \int_{\mathbb{R}^N} \exp[\eta u_0(\frac{\omega}{\eta})x]dx = (\frac{\eta}{\omega})^N \int_{\mathbb{R}^N} e^{\eta u_0(x)}dx$$

$$\int_{\mathbb{R}^N} v(x)dx = \int_{\mathbb{R}^N} \exp[-\gamma H^\star(x)]dx = \int_{\mathbb{R}^N} \exp[-H^\star(\gamma^{\frac{1}{q}}x)]dx = \frac{1}{\gamma^{\frac{N}{q}}}\int_{\mathbb{R}^N} e^{-H^\star(x)}dx$$

Finally we get

$$\|e^{Q_t u_0}\|_{L^{\omega e^{\alpha t}}(\mathbb{R}^N)} \le c_\alpha \|e^{u_0}\|_{L^\eta(\mathbb{R}^N)}$$

$$c_\alpha = \frac{\big(\frac{\eta}{\omega}\big)^{\frac{N}{\eta}}}{\Big(\int_{\mathbb{R}^N} v(x)dx\Big)^{\frac{1-a}{\eta}}} = \frac{\big(\frac{\eta}{\omega}\big)^{\frac{N}{\eta}}\gamma^{\frac{N}{q}(\frac{1}{\eta}-\frac{1}{\omega e^{\alpha t}})}}{\Big(\int_{\mathbb{R}^N} e^{-H^\star(x)}dx\Big)^{(\frac{1}{\eta}-\frac{1}{\omega e^{\alpha t}})}} =$$

$$\Big(\frac{\alpha p(\omega-\eta e^{-\alpha t})}{1-e^{-\alpha p t}}\Big)^{\frac{N}{p}(\frac{1}{\eta}-\frac{1}{\omega e^{\alpha t}})} \frac{\eta^{N(\frac{1}{\eta q}+\frac{1}{p e^{\alpha \omega t}})}}{\omega^{N(\frac{1}{\eta p}+\frac{1}{\omega q e^{\alpha t}})}\Big(\int_{\mathbb{R}^N} e^{-H^\star(x)}dx\Big)^{(\frac{1}{\eta}-\frac{1}{\omega e^{\alpha t}})}}$$

As α goes to 0^+ the constant goes to c_0 where

$$c_0 = \Big(\frac{\omega-\eta}{t}\Big)^{\frac{N}{p}(\frac{1}{\eta}-\frac{1}{\omega})} \frac{\eta^{N(\frac{1}{\eta q}+\frac{1}{p})}}{\omega^{N(\frac{1}{\eta p}+\frac{1}{\omega q})}\Big(\int_{\mathbb{R}^N} e^{-H^\star(x)}dx\Big)^{(\frac{1}{\eta}-\frac{1}{\omega})}}$$

and it is also the constant found by I. Gentil in the case of the Hopf-Lax formula. To get strict hyperconctractivity, i.e. $c_\alpha \le 1$, we rewrite c_α in the following form

$$c_\alpha = \Big(\frac{\eta}{\omega}\Big)^{\frac{1}{\omega e^{\alpha t}}} \eta^{\frac{N}{q}(\frac{1}{\eta}-\frac{1}{\omega e^{\alpha t}})} C$$

$$C = \Big(\frac{\alpha p(1-a)}{1-e^{-\alpha p t}}\Big)^{\frac{N}{p}(\frac{1}{\eta}-\frac{1}{\omega e^{\alpha t}})} \frac{1}{\Big(\int_{\mathbb{R}^N} e^{-H^\star(x)}dx\Big)^{(\frac{1}{\eta}-\frac{1}{\omega e^{\alpha t}})}} =$$

$$\left(\frac{\eta}{\omega}\right)^{\frac{1}{\omega e^{\alpha t}}}\left[\left(\frac{\alpha p(1-a)}{1-e^{-\alpha pt}}\right)^{\frac{1}{p}}\eta^{\frac{1}{q}}\frac{1}{\left(\int_{\mathbb{R}^N}e^{-H^\star(x)}dx\right)^{\frac{1}{N}}}\right]^{N(\frac{1}{\eta}-\frac{1}{\omega e^{\alpha t}})}$$

Then, since $\eta\le\omega$, to satisfy $c_\alpha\le 1$ we impose

$$\left(\frac{\alpha p(1-\frac{\eta}{\omega}e^{-\alpha pt})}{1-e^{-\alpha pt}}\right)^{\frac{1}{p}}\eta^{\frac{1}{q}}\le\left(\int_{\mathbb{R}^N}e^{-H^\star(x)}dx\right)^{\frac{1}{N}}$$

Then we can state the following theorem

Theorem 2.8. The semigroup is hyperconctractive from $L^\eta(\mathbb{R}^N)$ to $L^{\omega e^{\alpha t}}(\mathbb{R}^N)$, i.e.

$$\|e^{Q_t u_0}\|_{L^{\omega e^{\alpha t}}(\mathbb{R}^N)}\le\|e^{u_0}\|_{L^\eta(\mathbb{R}^N)}$$

for all the triple of real positive numbers (η,ω,t) for which $\eta\le\omega$ and $t\in\mathbb{R}^+$ such the above condition is verified.

Corollary 2.9. If

$$(\alpha p)^{\frac{1}{p}}<\left(\int_{\mathbb{R}^N}e^{-H^\star(x)}dx\right)^{\frac{1}{N}},$$

then for t large enough we have

$$\|e^{Q_t u_0}\|_{L^{\omega e^{\alpha t}}(\mathbb{R}^N)}\le\|e^{u_0}\|_{L^1(\mathbb{R}^N)}$$

Au	Hyper	Ultra
IG(02)	$\|e^{Q_t(f)}\|_{L^q}\le C(p,q,t)\|e^f\|_{L^p}$ $\left[\left(\frac{q-p}{2\pi t}\right)^{\frac{1}{p}-\frac{1}{q}}\left(\frac{p}{q}\right)^{\frac{1}{p}+\frac{1}{q}}\right]^{\frac{N}{2}}$	q $\to\infty$ $p=1$ $\left(\frac{1}{2\pi t}\right)^{\frac{N}{2}}$
AA-PL	$\|e^{Q_t(f)}\|_{L^q}\le C(\alpha,p,q,t)\|e^f\|_{L^p}$ $\left(\frac{(q-pe^{-\alpha t})\alpha}{\pi(1-e^{-2\alpha t})}\right)^{\frac{N}{2}(\frac{1}{p}-\frac{1}{qe^{-\alpha t}})}\left(\frac{p}{q}\right)^{\frac{N}{2}(\frac{1}{p}+\frac{1}{qe^{-\alpha t}})}$	q $\to\infty$ $\left(\frac{p\alpha}{\pi(1-e^{-2\alpha t})}\right)^{\frac{N}{2p}}$

In the ultracontractive case we take $p=1$ and $N=1$. Then since

$$\frac{1}{2\pi t}<\frac{\alpha}{\pi(1-e^{-2\alpha t})}$$

the constant found by Gentil is smaller than ours.
This does not means that our constant is not good!
Both the estimates are optimal

Ge (03)	$\frac{1}{\left(\int_{\mathbb{R}^N}e^{-H^\star(x)}dx\right)^{(\frac{1}{\eta}-\frac{1}{\omega})}}\left(\frac{\omega-\eta}{t}\right)^{\frac{N}{p}(\frac{1}{\eta}-\frac{1}{\omega})}\left(\frac{\eta^{\frac{\eta}{p}+\frac{\omega}{q}}}{\omega^{\frac{\omega}{p}+\frac{\eta}{q}}}\right)^{\frac{N}{\omega\eta}}$
AL	$\frac{1}{\left(\int_{\mathbb{R}^N}e^{-H^\star(x)}dx\right)^{(\frac{1}{\eta}-\frac{1}{\omega e^{\alpha t}})}}\left(\frac{(\omega-\eta e^{-\alpha t})\alpha p}{(1-e^{-\alpha pt})}\right)^{\frac{N}{p}(\frac{1}{\eta}-\frac{1}{\omega e^{\alpha t}})}\left(\frac{\eta^{\frac{\eta}{p}+\frac{\omega}{q}}}{\omega^{\frac{\omega}{p}+\frac{\eta}{q}}}\right)^{\frac{N}{\omega\eta}}$

Here the table of functions giving the optimality.

Au	Optimal
IG(03)	-b $H^\star(x-\overline{x})$
AA-PL	-$H^\star\Big(\big(\frac{Ap\alpha}{\alpha-A(1-e^{-\alpha pt})}\big)^{\frac{1}{p}}(e^{-\alpha t}x-\overline{x})\Big)$

3. LSI

We show in the simpler case how to get logarithmic Sobolev inequality by hypercontractivity. For the general case see [5].

We introduce the function

$$F(t)=\|e^{Q_tu_0}\|_{L^{q(t)}}, \tag{9}$$

then

$$\big(F(t)\big)^{q(t)}=e^{q(t)\log F(t)}=\int_{\mathbb{R}}e^{q(t)Q_tu_0(x)}dx, \tag{10}$$

and

$$\log F(t)=\frac{1}{q(t)}\log\int_{\mathbb{R}}e^{q(t)Q_tu_0(x)}dx,$$

We differentiate (10)

$$\big(F(t)\big)^{q(t)}\Big(q'(t)\log F(t)+q(t)\frac{F'(t)}{F(t)}\Big)=$$

$$\int_{\mathbb{R}}e^{q(t)Q_tu_0(x)}\Big(q'(t)Q_tu_0(x)+q(t)\frac{\partial}{\partial t}Q_tu_0(x)\Big)dx, \tag{11}$$

and, using (11)

$$q(t)\big(F(t)\big)^{q(t)-1}F'(t)=-q'(t)\big(F(t)\big)^{q(t)}\log F(t)+$$

$$\int_{\mathbb{R}}e^{q(t)Q_tu_0(x)}\Big(q'(t)Q_tu_0(x)+q(t)\frac{\partial}{\partial t}Q_tu_0(x)\Big)dx=$$

$$-\frac{q'(t)}{q(t)}\big(F(t)\big)^{q(t)}\log\int_{\mathbb{R}}e^{q(t)Q_tu_0(x)}dx+$$

$$\int_{\mathbb{R}}e^{q(t)Q_tu_0(x)}q'(t)Q_tu_0(x)dx+\int_{\mathbb{R}}q(t)e^{q(t)Q_tu_0(x)}\frac{\partial}{\partial t}Q_tu_0(x)dx$$

Using (10) we get

$$q(t)\big(F(t)\big)^{q(t)-1}F'(t)=-\frac{q'(t)}{q(t)}\int_{\mathbb{R}}e^{q(t)Q_tu_0(x)}dx\log\int_{\mathbb{R}}e^{q(t)Q_tu_0(x)}dx+$$

$$q'(t)\int_{\mathbb{R}}e^{q(t)Q_tu_0(x)}Q_tu_0(x)dx+q(t)\int_{\mathbb{R}}e^{q(t)Q_tu_0(x)}\frac{\partial}{\partial t}Q_tu_0(x)dx$$

From which, recalling that $Q_tu_0(x)$ is a solution,

$$\int_{\mathbb{R}}e^{q(t)Q_tu_0(x)}\frac{\partial}{\partial t}Q_tu_0(x)dx=$$

$$-\int_{\mathbb{R}} e^{q(t)Q_tu_0(x)}\left(\frac{1}{2}\frac{\partial}{\partial x}|Q_tu_0(x)|^2+\alpha x\frac{\partial}{\partial x}Q_tu_0(x)\right)dx,$$

we have

$$q^2(t)\big(F(t)\big)^{q(t)-1}F'(t)=-q'(t)\bigg(\int_{\mathbb{R}} e^{q(t)Q_tu_0(x)}dx\log\int_{\mathbb{R}} e^{q(t)Q_tu_0(x)}dx+$$

$$q'(t)q(t)\int_{\mathbb{R}} e^{q(t)Q_tu_0(x)}Q_tu_0(x)dx-$$

$$q^2(t)\int_{\mathbb{R}} e^{q(t)Q_tu_0(x)}\big[\frac{1}{2}\frac{\partial}{\partial x}|Q_tu_0(x)|^2+\alpha x\frac{\partial}{\partial x}Q_tu_0(x)\big]dx$$

We set

$$h(x,t)=e^{q(t)Q_tu_0(x)},\qquad Q_tu_0(x)=\frac{1}{q(t)}\log h(x,t)$$

and we recall the definition of entropy of a function h

$$E(h):=\int_{\mathbb{R}} h\log h dx-\int_{\mathbb{R}} hdx\log\int_{\mathbb{R}} hdx \tag{12}$$

Then, we have

$$q^2(t)\big(F(t)\big)^{q(t)-1}F'(t)=$$

(13)

$$q'(t)E(e^{q(t)Q_tu_0(x)})-q^2(t)\int_{\mathbb{R}} e^{q(t)Q_tu_0(x)}\big[\frac{1}{2}\frac{\partial}{\partial x}|Q_tu_0(x)|^2+\alpha x\frac{\partial}{\partial x}Q_tu_0(x)\big]dx$$

We select the function F as

$$F^\star(t)=\|e^{Q_{t_1}u_0}\|_{L^{pe^{\alpha t}}}$$

Hypercontracitity gives

Lemma 3.1. For every $p\in(0,\frac{\pi}{\alpha}]$ the function $F^\star(t)$ is non increasing for $t\in(0,\frac{1}{\alpha}\log\frac{\pi}{\alpha p})$.

Now we pass to prove a logarithmic Sobolev inequality. We consider (13). From the lemma (3.1) we have

$$F^\star(t)\leq 0\qquad\forall t\in(0,\frac{1}{\alpha}\log\frac{\pi}{\alpha p})$$

$$\alpha pe^{\alpha t}E(e^{pe^{\alpha t}Q_tu_0(x)})\leq$$

$$\frac{p^2e^{2\alpha t}}{2}\int_{\mathbb{R}} e^{pe^{\alpha t}Q_tu_0(x)}\big[\frac{\partial}{\partial x}|Q_tu_0(x)|^2+2\alpha x\frac{\partial}{\partial x}Q_tu_0(x)\big]dx$$

Taking the limit as $t\to 0$, for any admissible u_0, by the continuity of Q_t, we obtain

$$\alpha E(e^{pu_0(x)})\leq\frac{p}{2}\int_{\mathbb{R}} e^{pu_0(x)}\big[\frac{\partial}{\partial x}|u_0(x)|^2+2\alpha x\frac{\partial}{\partial x}u_0(x)\big]dx\qquad\forall\alpha<\pi \tag{14}$$

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