1. Geometric Mean generalization in ^{n}

Given n positive numbers $x_1, x_2, \quad x_n$ we de ne the geometric mean as follows

$$f(x_1, x_2, \quad, x_n) = \left(\prod_{i=1}^{n} x_i \right)^{-}$$

1.1. **Gradient of f**

$$Df(x_1, x_2, \quad, x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \quad, \frac{\partial f}{\partial x_n} \right)$$

where

$$\frac{\partial f}{\partial x_i}(x) = \frac{1}{n} (x_1 x_2 \quad x_i \quad x_n)^{- \quad 1} \; x_1 x_2 \quad x_{i \;\; 1} x_{i \;\; 1} \quad x_{n \;\; 1} x_n \big) \qquad i = 1, \qquad, n$$

1.2. **Partial Second Derivative wrt to the variable x_i**

$$\frac{\partial^2 f}{\partial x_i^2}(x) = \frac{1}{n} \left(\frac{1}{n} \quad 1 \right) (x_1 \quad x_n)^{- \quad 2} (x_1 \quad x_{i \;\; 1} x_{i \;\; 1} \quad x_n)^2 =$$

$$(x_1 \quad x_n)^{-} \left[\left(\frac{1}{n^2} \quad \frac{1}{n} \right) \frac{x_1^2 x_2^2 \quad x_{i \;\; 1}^2 x_{i \;\; 1}^2 \quad x_n^2}{x_1^2 x_2^2 \quad x_n^2} \right] =$$

$$(x_1 x_2 \quad x_n)^{-} \left[\frac{1}{n^2} \frac{1}{x_i^2} \quad \frac{1}{n} \frac{1}{x_i^2} \right]$$

1.3. **Partial Second Derivative wrt to the variable x_i and x_j with $j \neq i$**

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(x) = \frac{1}{n} \left(\frac{1}{n} \quad 1 \right) (x_1 \quad x_n)^{- \quad 2} (x_1 \quad x_{i \;\; 1} x_{i \;\; 1} \quad x_n)(x_1 \quad x_{j \;\; 1} x_{j \;\; 1} \quad x_n) +$$

$$\frac{1}{n} (x_1 \quad x_n)^{- \quad 1} (x_1 \quad x_{i \;\; 1} x_{i \;\; 1} \quad x_{j \;\; 1} x_{j \;\; 1} \quad x_n) =$$

$$(x_1 \quad x_n)^{-} \left[\left(\frac{1}{n^2} \quad \frac{1}{n} \right) \frac{x_1^2 \quad x_i \quad x_j \quad x_n^2}{(x_1 \quad x_n)^2} + \frac{1}{n} \frac{x_1 \quad x_{i \;\; 1} x_{i \;\; 1} \quad x_{j \;\; 1} x_{j \;\; 1} \quad x_n}{x_1 \quad x_n} \right] =$$

$$(x_1 \quad x_n)^{-} \left[\left(\frac{1}{n^2} \quad \frac{1}{n} \right) \frac{1}{x_i x_j} + \frac{1}{n} \frac{1}{x_i x_j} \right] = (x_1 \quad x_n)^{-} \frac{1}{n^2} \frac{1}{x_i x_j}$$

2. Concavity and negative non positive) quadratic form

We consider the quadratic form associated to the Hessian matrix

$$\sum_{j \; k=1}^{n} \frac{\partial^2 f}{\partial x_j x_k} v_j v_k = \underbrace{\frac{1}{n^2} \left(\prod_{i=1}^{n} x_i \right)^{-}}_{\geq 0} \underbrace{\left[\sum_{j=1}^{n} \frac{v_j}{x_j} \sum_{k=1}^{n} \frac{v_k}{x_k} \quad n \sum_{j=1}^{n} \frac{v_j^2}{x_j^2} \right]} \tag{1}$$

Since

$$\left(\sum_{j=1}^{n} \frac{v_j}{x_j} \right)^2 = \sum_{j=1}^{n} \frac{v_j}{x_j} \sum_{j=1}^{n} \frac{v_k}{x_k}$$

we have only to show

$$\left(\sum_{j=1}^{n} \frac{v_j}{x_j} \right)^2 \leq n \sum_{j=1}^{n} \frac{v_j^2}{x_j^2}$$

(see inequality on linear regression).