# SYMMETRIC DEGENERATIONS ARE NOT IN GENERAL INDUCED BY TYPE A DEGENERATIONS

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Abstract. We consider a symmetric quiver with relations. Its (symmetric) representations of a fixed symmetric dimension vector are encoded in the (symmetric) representation varieties. The orbits by a (symmetric) base change group action are the isomorphism classes of (symmetric) representations. The symmetric orbits are induced by simply restricting the non-symmetric orbits. However, when it comes to orbit closure relations, it is so far an open question under which assumptions they are induced. We describe an explicit example of a quiver of finite representation type for which orbit closure relations are induced in types B and C, but not in type D.

## Contents



## 1. Introduction

Let $\mathcal{A} = \mathrm{k}\mathcal{Q}/I$ be a symmetric quiver algebra. We fix a $\mathcal{Q}_0$-graded k-vector space $V$ and denote the representation variety of representations with underlying vector space $V$ by $R(\mathcal{A}, V)$. Inside of $R(\mathcal{A}, V)$ there is a subvariety $R(\mathcal{A}, V)^{\langle -,- \rangle,\varepsilon}$ of so-called $\varepsilon$-representations; here $\varepsilon$ is a sign and $\langle -,- \rangle$ is a non-degenerate bilinear form on $V$. An $\varepsilon$-representation is a symmetric representation which has an orthogonal or a symplectic structure. There are natural (symmetric) base change actions on these varieties; their orbits correspond to isomorphism classes of (symmetric) representations. It is natural to ask, whether the orbits and their closures can be translated easily between the two group actions. Concerning the orbits, Derksen and Weyman proved in 2002 that by restricting an orbit $\mathcal{O} \subseteq R(\mathcal{A}, V)$ to $R(\mathcal{A}, V)^{\langle -,- \rangle,\varepsilon}$, we get an orbit of $\varepsilon$-representations; and more substantial that in this way every symmetric orbit is obtained. The question under which assumptions the orbit closure relations are induced by restricting in the same way, is still open. In this article, we present a counterexample of a quiver of finite representation type, that is, an example of such algebra for which orbit closure relations are not induced by type A.

This work is heavily based on our preceding article [5] where many details on the symmetric representation theory of a symmetric quiver algebra can be found. In particular, in said article we prove that for $\mathcal{Q}$ a Dynkin quiver with symmetry, orbit closure relations are induced by type A.

We structure this article as follows: The setup of our Main Question 2.3 is explained in Section 2 where aforesaid question is posed. At the same time we recall some general knowledge on algebras with self-dualities. In Section 3, we define several partial orders which embed our Main Question into a representation-theoretical and homological frame. Our counterexample is described in Section 4 where we look at the so-called seesaw algebra and define particular symmetric representations which degenerate in type A, but not in type D. We end the paper by posing conjectures which are likely to hold true from our current perspective on the topic.

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## 2. Setup

Let $\mathrm{k} = \mathbf{C}$ be the field of complex numbers and let $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, s, t)$ be a finite *quiver*, that is, an oriented graph with a finite set of *vertices* $\mathcal{Q}_0$, a finite set of *edges* $\mathcal{Q}_1$ and two maps $s, t : \mathcal{Q}_1 \to \mathcal{Q}_0$ which provide the orientation $\alpha : s(\alpha) \to t(\alpha)$ of the edges. Let us consider the elements of $\mathcal{Q}_1$ as arrows. A sequence of arrows $\omega = \alpha_s \cdots \alpha_1$ is called a *path* in $\mathcal{Q}$ whenever $t(\alpha_i) = s(\alpha_{i+1})$ for all $i$; we formally include a path $\varepsilon_i : i \to i$ of length zero for each $i \in \mathcal{Q}_0$. The *path algebra* $\mathrm{k}\mathcal{Q}$ of $\mathcal{Q}$ is the k-algebra spanned as a k-vector space by the set of all paths in $\mathcal{Q}$ together with the concatenation of paths as multiplication. Let $R \subseteq \mathrm{k}\mathcal{Q}$ be the 2-sided ideal generated by all arrows in $\mathcal{Q}_1$; it is called the *arrow ideal.* Then every ideal $I \subseteq \mathrm{k}\mathcal{Q}$ which determines an integer $s$ with $R^s \subseteq I \subseteq R^2$ is called *admissible.* If $I$ is admissible, then the quotient algebra $\mathcal{A} := \mathrm{k}\mathcal{Q}/I$ is a finite-dimensional and associative *quiver algebra* [3].

Now assume that $\mathcal{Q}$ comes with a symmetry as defined in [9], that is, we consider a tuple $(\mathcal{Q}, \sigma)$ where $\sigma : Q \to Q^{op}$ is an involutive bijection of $\mathcal{Q}_0$ and an arrow-reversing involution of $\mathcal{Q}_1$. Then $(\mathcal{Q}, \sigma)$ is called *symmetric quiver.* Assume that an admissible ideal $I \subset \mathrm{k}\mathcal{Q}$ fulfills $\sigma(I) = I$, then $\mathcal{A} = \mathrm{k}\mathcal{Q}/I$ is isomorphic (via $\sigma$) to its opposite $\mathcal{A}^{\mathrm{op}} = \mathrm{k}\mathcal{Q}^{\mathrm{op}}/\sigma(I)$ and the pair $(\mathcal{A}, \sigma)$ is called a *symmetric quiver algebra.*

### 2.1. Quiver representations.

Our main question in this article arises in the context of (symmetric) representations. Let $V = \oplus_{i \in \mathcal{Q}_0} V_i$ be a finite dimensional $\mathcal{Q}_0$-graded vector space of graded dimension $\mathbf{d} = \mathbf{dim}\, V = (\dim V_i)_{i \in \mathcal{Q}_0}$. We denote by $\mathrm{Rep}(\mathcal{A})$ the representation category and by $R(\mathcal{A}, V)$ the variety of $\mathcal{A}$-representations having $V$ as underlying vector space, that is, its elements are collections $f = (f_\alpha : V_{s(\alpha)} \to V_{t(\alpha)})_{\alpha \in \mathcal{Q}_1}$ of linear maps such that $f_\pi = 0$ for every $\pi \in I$:

$$R := R(\mathcal{A}, V) \subseteq R(\mathrm{k}\mathcal{Q}, V) := \bigoplus_{\alpha : i \to j \in \mathcal{Q}_1} \mathrm{Hom}_{\mathrm{k}}(V_i, V_j).$$

The vector $\mathbf{d}$ is called the *dimension vector* of these representations. Let $\mathrm{GL}^\bullet(V) := \prod_{i \in \mathcal{Q}_0} \mathrm{GL}(V_i)$ be the group of graded automorphisms of $V$, then $\mathrm{GL}^\bullet(V)$ acts on $R(\mathcal{A}, V)$ by change of basis: given $g = (g_i)_{i \in \mathcal{Q}_0} \in \mathrm{GL}^\bullet(V)$ and $M = (M_\alpha)_{\alpha \in \mathcal{Q}_1} \in R(\mathcal{A}, V)$ the representation $g \cdot M$ is defined by $(g \cdot M)_\alpha = g_{t(\alpha)} \circ M_\alpha \circ g_{s(\alpha)}^{-1}$. The $\mathrm{GL}^\bullet(V)$-orbits are the isomorphism classes of $\mathcal{A}$-representations with underlying vector space $V$ in the representation category of $\mathcal{A}$.

Let $M \in R(\mathcal{A}, V)$, let $B_i$ be a k-basis of $V_i$ for every $i \in \mathcal{Q}_0$ and let $B$ be the disjoint union of these sets $B_i$. We define the *coefficient quiver* $\Gamma(M) := \Gamma(M, B)$ of $M$ with respect to the basis $B$ to be the quiver with exactly one vertex for each element of $B$, such that for each arrow $\alpha \in \mathcal{Q}_1$ and every element $b \in B_{s(\alpha)}$ we have

$$M_\alpha(b) = \sum_{c \in B_{t(\alpha)}} \lambda^\alpha_{b,c} c$$

with $\lambda^\alpha_{b,c} \in \mathrm{k}$. For each $\lambda^\alpha_{b,c} \neq 0$ we draw an arrow $b \to c$ with label $\alpha$ [18]. Thus, the quiver reflects the coefficients corresponding to the representation $M$ with respect to the chosen basis $B$ and will help us to depict representations in a nice way in the remainder of the article. In case there are no multiple arrows between two vertices, we label the arrows of the coefficient quiver with the actual value of $\lambda^\alpha_{b,c}$.

We include a basic example in order to display the ideas behind our setup. We will come back to this example throughout this section.

*Example* 2.1. Let $\mathcal{Q}$ be the one-loop quiver, that is, $\mathcal{Q}_0 = \{x\}$ and $\mathcal{Q}_1 = \{\alpha : x \to x\}$, let $V = \mathrm{k}^n$, and consider the admissible ideal $I = (\alpha^n) \subseteq \mathrm{k}\mathcal{Q}$. Then $R(\mathcal{A}, V) = \mathcal{N} = \{N \in \mathrm{k}^{n \times n} \mid N^n = 0\}$ equals the nilpotent cone and $\mathrm{GL}^\bullet(V) = \mathrm{GL}_n(\mathrm{k})$. Thus, the $\mathrm{GL}^\bullet(V)$-action on $R(\mathcal{A}, V)$ is the usual conjugation action, its orbits are described by the Jordan canonical form [14]; or by partitions, that is, combinatorial objects named Young diagrams. The closure relations are known by Gerstenhaber [11] and are given by box dropping of Young diagrams.

2.2. **Symmetric quiver representations.** We come back to the symmetric quiver algebra $\mathcal{A} = \mathrm{k}\mathcal{Q}/I$ now. This algebra comes along with a self-duality on its representations which we will define briefly; more details can be found in [5].

The anti-involution $\sigma$ can be extended to an isomorphism $\sigma : \mathcal{A} \to \mathcal{A}^{op}$ of $\mathcal{A}$ to its opposite algebra. This isomorphism induces an equivalence $\sigma : \mathrm{Rep}(\mathcal{A}) \to \mathrm{Rep}(\mathcal{A}^{op})$ of the representation categories; and a self-duality $^* : \mathrm{Rep}(\mathcal{A}) \to \mathrm{Rep}(\mathcal{A})$ on $\mathrm{Rep}(\mathcal{A})$ by composing with the standard k-duality $D = \mathrm{Hom}(-, \mathrm{k})$ . Let $V^* = \mathrm{Hom}(V, \mathrm{k})$ be the linear dual of a k-vector space $V$ and $f^* : V^* \to U^*$ be the linear dual of a linear map $f : U \to V$ defined by $f^*(h)(u) = h(f(u))$ for every $h \in V^*$ and $u \in U$.

For a $\mathcal{Q}_0$-graded vector space $V = \oplus_{i \in \mathcal{Q}_0} V_i$, we define its *twisted dual* $\nabla V$ as the $\mathcal{Q}_0$-graded vector space whose $i$-th component is $(\nabla V)_i = V^*_{\sigma(i)}$.

*Definition* 2.2. Let $\nabla : \mathrm{Rep}(\mathcal{A}) \to \mathrm{Rep}(\mathcal{A})$ be the functor defined by

- $\nabla(M)_\alpha = -M^*_{\sigma(\alpha)}$ for every arrow $\alpha$ on the objects $M$
- $(\nabla h)_i = h^*_{\sigma(i)}$, for every vertex $i \in \mathcal{Q}_0$ on the morphisms $h : M \to N$

Notice that $\nabla V = V^*$ for the semi-simple representation $V = \oplus_{i \in \mathcal{Q}_0} V_i$.

Let us fix $\varepsilon \in \{\pm 1\}$ and let $\langle -, - \rangle : V \times V \to \mathrm{k}$ be a non-degenerate bilinear form which fulfills two conditions:

(1) the form $\langle -, - \rangle$ is *compatible with* $\sigma$, i.e. $\langle -, - \rangle|_{V_i \times V_j} = 0$ if $j \neq \sigma(i)$;
(2) the form $\langle -, - \rangle$ is an *$\varepsilon$-form*: i.e. $\langle v, w \rangle = \varepsilon \langle w, v \rangle$ for every $v, w \in V$.

Then the dimension vector of $V$ is *$\sigma$-symmetric*, i.e. $\mathbf{d}_{\sigma(i)} = \mathbf{d}_i$ for every $i \in \mathcal{Q}_0$. Every endomorphism $f$ of $V$ has a unique *adjoint* $f^\star$ with respect to $\langle -, - \rangle$ defined by the condition $\langle f(v), w \rangle = \langle v, f^\star(w) \rangle$, for all $v, w, \in V$. We denote by $\mathrm{G}(V, \langle -, - \rangle) = \{g \in \mathrm{GL}(V) | g = (g^\star)^{-1}\}$ the group of isometries of $(V, \langle -, - \rangle)$, by $\mathrm{O}_n$ the orthogonal group and by $\mathrm{SP}_n$ the symplectic group of $n \times n$-matrices.

$M \in R(\mathcal{A}, V)$ is an *$\varepsilon$-representation* of $(\mathcal{A}, \sigma)$ with respect to $(V, \langle -, - \rangle)$ if [9]

(iii) $M^\star + M = 0$.

Condition (iii) means that $M$, interpreted as an endomorphism of $V$, lies in the Lie algebra of $\mathrm{G}(V, \langle -,- \rangle)$. A $+1$-representation is called *orthogonal* and a $-1$-representation is called *symplectic*. We collect all $\varepsilon$-representations in a variety $R(\mathcal{A}, V)^{\langle -,- \rangle, \varepsilon} = \{M \in R(\mathcal{A}, V) | \, M^\star + M = 0\}$ and denote by $\mathrm{G}^\bullet(V, \langle -,- \rangle) := \mathrm{G}(V, \langle -,- \rangle) \cap \mathrm{GL}^\bullet(V)$ the group of graded isometries of $(V, \langle -,- \rangle)$.

Then the action of $\mathrm{GL}^\bullet(V)$ on $R$ induces an action of $\mathrm{G}^\bullet(V, \langle -,- \rangle)$ on $R(\mathcal{A}, V)^{\langle -,- \rangle, \varepsilon}$ by change of basis ([9, 5]). One first question which suggests itself is whether or not

$$\mathrm{G}^\bullet(V, \langle -,- \rangle) \cdot M = \mathrm{GL}^\bullet(V) \cdot M \cap R(\mathcal{A}, V)^{\langle -,- \rangle, \varepsilon}$$

holds true for every $M \in R(\mathcal{A}, V)^{\langle -,- \rangle, \varepsilon}$, that is, whether the orbits of the smaller group are induced by the orbits of the bigger group. This question is answered positively by Derksen and Weyman in [9] and with different techniques in [5]. The main question which we address in this article follows immediately:

**Main Question 2.3.** *Is it true that*

$$\overline{\mathrm{G}^\bullet(V, \langle -,- \rangle) \cdot M} = \overline{\mathrm{GL}^\bullet(V) \cdot M} \cap R(\mathcal{A}, V)^{\langle -,- \rangle, \varepsilon}$$

*for every* $M \in R(\mathcal{A}, V)^{\langle -,- \rangle, \varepsilon}$*?*

Main Question 2.3 is answered positively in [5] for Dynkin quivers. Its answer is particularly interesting when the algebra $\mathcal{A}$ is of finite representation type, that is, in case there is only a finite number of $\mathrm{GL}^\bullet(V)$-orbits in $R(\mathcal{A}, V)$, as in Example 2.4. Our aim in this article is to give a counterexample of a representation-finite algebra for which the answer to Main Question 2.3 is negative. This is indeed unexpected, since our example is closely related to the fundamental Example 2.1 which does not give a counterexample.

*Example* 2.4. In case of Example 2.1, we fix $\varepsilon$ to be $+1$ or $-1$. Let $J_k$ be the $k \times k$-anti-diagonal matrix with every entry on the anti-diagonal being one and every other entry being zero. The non-degenerate bilinear form $\langle -,- \rangle : V \times V \to \mathrm{k}$ given by the matrix

$$F_\varepsilon = \begin{bmatrix} 0 & J_l \\ \varepsilon J_l & 0 \end{bmatrix}$$

if $\varepsilon = -1$ and by $J_n$ if $\varepsilon = 1$ fulfills conditions (1) and (2). Then $\mathrm{G}^\bullet(V, \langle -,- \rangle) = \mathrm{O}_n$ if $\varepsilon = 1$ and $\mathrm{G}^\bullet(V, \langle -,- \rangle) = \mathrm{SP}_n$ if $\varepsilon = -1$ and the $\mathrm{G}^\bullet(V, \langle -,- \rangle)$-action on $R(\mathcal{A}, V)^{\langle -,- \rangle, \varepsilon} = \mathcal{N} \cap \mathrm{Lie}\, G$ is given by orthogonal/symplectic conjugation. The orbits of the latter are classified by Springer and Steinberg by so-called $\varepsilon$-partitions and their closures are known by Hesselink (these results are e.g. described by Kraft and Procesi in [15]). Main Question 2.3 is answered positively. Indeed, assume that there is an orbit closure relation between two orbits, in $R(\mathcal{A}, V)$, i.e. partitions. If both partitions are $\varepsilon$-partitions, then there also is an orbit closure relation between them in $R(\mathcal{A}, V)^{\langle -,- \rangle, \varepsilon}$.

2.3. **Motivation.** Example 2.1 shows that - in addition to being interesting from a quiver representation-theoretic point of view - the answer of Main Question 2.3 has further applications to algebraic Lie Theory. This will be worked out in more detail in Remark 4.3 in Section 4.

*Remark* 2.5. Our setup fits into a more general context described by Magyar, Weyman and Zelevinsky in [16]. In fact, given a complex algebraic variety $X$ together with an action of a group $\mathrm{G}$ and two involutions $\rho : \mathrm{G} \to \mathrm{G}$ and $\Delta : X \to X$ such that $\Delta(g \cdot \Delta x) = g^\rho \cdot x$, we denote the fixed point sets by $\mathrm{G}^\rho \subset \mathrm{G}$ and $X^\Delta \subset X$. Assume that

(1) the group G is a subgroup of the group of invertible elements $E^\times$ of a finite-dimensional associative algebra $E$ over k;
(2) the anti-involution of G given by $g \mapsto g^* := (g^\rho)^{-1}$ extends to a k–linear anti-involution $f \mapsto f^*$ on the algebra $E$;
(3) for every fixed point $x \in X^\Delta$, its stabilizer $H = \mathrm{Stab}_\mathrm{G}(x)$ is the group of invertible elements of its linear span $\mathrm{Span}_\mathrm{k}(H) \subset E$.

Then $\mathrm{G}\,x \cap X^\Delta = \mathrm{G}^\rho\, x$ holds true for all $x \in X^\Delta$ by [16, Section 2.1]. The natural subsequent (and open) question is

$$\text{"Is it true that } \overline{\mathrm{G}\,x} \cap X^\Delta = \overline{\mathrm{G}^\rho\, x} \text{ for every } x \in X^\Delta\text{?"} \tag{2.1}$$

As described in [5, Subsection 2.4], $R(\mathcal{A}, V)$ and $R(\mathcal{A}, V)^{\langle -,-\rangle,\varepsilon}$ can be realized as $X$ and $X^\Delta$; and $\mathrm{GL}^\bullet(V)$ and $\mathrm{G}^\bullet(V, \langle -,-\rangle)$ can be realized as $G$ and $G^\rho$. Thus, our counterexample on Main Question 2.3 can be thought of as a counterexample for (2.1).

## 3. Ext-, deg- and hom-order

Let $\mathcal{A}$ be a quiver algebra, let $\mathbf{d} \in \mathbf{Z}_{\geq 0}^{\mathcal{Q}_0}$ be a dimension vector and let $V$ be a $\mathcal{Q}_0$-graded complex vector space of dimension vector $\mathbf{d}$.

Let $M, N \in R(\mathcal{A}, V)$. We denote $[M, N] := \dim \mathrm{Hom}_\mathcal{A}(M, N)$ and $[M, N]^1 := \dim \mathrm{Ext}^1_\mathcal{A}(M, N)$ and define three partial orders on $R(\mathcal{A}, V)$ which were first described by Abeasis-Del Fra for quivers of Dynkin type $A$ [1, 2], before being generalized to quiver algebras by Riedtmann [17], Bongartz [4] and Zwara [19].

- The *degeneration order* $\leq_{\deg}$ is defined by
$$M \leq_{\deg} N :\Longleftrightarrow N \in \overline{\mathrm{GL}^\bullet(V)M}$$
- The *Hom-order* $\leq_{\mathrm{Hom}}$ is defined by
$$M \leq_{\mathrm{Hom}} N :\Longleftrightarrow [M, E] \leq [N, E] \text{ for every indecomposable } E.$$
- The *Ext-order* $\leq_{\mathrm{Ext}}$ is defined by
$$M \leq_{\mathrm{Ext}} N \quad :\Longleftrightarrow \quad \begin{array}{c} \exists M_1, \cdots, M_k \in R(\mathcal{A}, V) \text{ and short exact} \\ \text{sequences } 0 \to U_i \to M_{i-1} \to V_i \to 0 \ (\forall i) \\ \text{such that } M_1 = M, M_k = N,\ M_i \simeq U_i \oplus V_i. \end{array}$$

It is known by [4, Lemma 1.1] (first implication) and [17, Proposition 2.1] (second implication) that

$$M \leq_{\mathrm{Ext}} N \Longrightarrow M \leq_{\deg} N \Longrightarrow M \leq_{\mathrm{Hom}} N.$$

If $\mathcal{A}$ is an algebra of finite representation type, then Zwara [19, Corollary of Theorem 1] shows

$$M \leq_{\deg} N \Longleftrightarrow M \leq_{\mathrm{Hom}} N.$$

If furthermore all indecomposables are rigid, i.e. $[E, E]^1 = 0$ for all indecomposables $E$, then all three orders coincide [19, Theorem 2]. In particular, they are equivalent for Dynkin quivers. Note that the result on Dynkin quivers also follows from work of Bongartz where he shows that all three partial orders coincide for representation-directed algebras [4, Proposition 3.2,Corollary 4.2].

Following [5], we introduce symmetric versions of $\leq_{\deg}$ and $\leq_{\mathrm{Ext}}$ now. Thus, we assume $\mathcal{A}$ to be a symmetric quiver algebra, let $\varepsilon$ be $+1$ or $-1$ and let $\langle -,-\rangle$ be a bilinear form as in the Section 2. Then we consider the following partial orders on $R(\mathcal{A}, V)^{\langle -,-\rangle,\varepsilon}$.

- The *symmetric degeneration order* $\leq^\varepsilon_{\deg}$ is defined by
$$M \leq^\varepsilon_{\deg} N :\Longleftrightarrow N \in \overline{\mathrm{GL}^\bullet(V, \langle -,-\rangle)M}.$$

- The *symmetric Ext-order* $\leq^{\varepsilon}_{\mathrm{Ext}}$ is defined by

$$M \leq_{\mathrm{Ext}} N \quad :\Longleftrightarrow \quad \begin{array}{l} \exists M_1, \cdots, M_k \in R(\mathcal{A}, V)^{\langle -,- \rangle, \varepsilon} \text{ and sh. ex. seqs} \\ 0 \to U_i \to M_{i-1} \to V_i \to 0 \ (\forall i) \text{ s.th. } M_1 = M, M_k = N, \\ U_i \text{ is isotropic in } M_{i-1}, \text{ and } M_i \simeq U_i \oplus \nabla U_i \oplus U_i^{\perp}/U_i. \end{array}$$

It is known by [5, Corollary 3.3] and since $\mathrm{GL}^{\bullet}(V, \langle -,- \rangle) \subseteq \mathrm{GL}^{\bullet}(V)$ is a subgroup that

$$M \leq^{\varepsilon}_{\mathrm{Ext}} N \Longrightarrow M \leq^{\varepsilon}_{\mathrm{deg}} N \Longrightarrow M \leq_{\mathrm{deg}} N \ (\Longrightarrow M \leq_{\mathrm{Hom}} N).$$

**Main Question 3.1.** *Does*

$$\leq^{\varepsilon}_{\mathrm{deg}} \Longleftrightarrow \leq_{\mathrm{deg}}$$

*hold true on* $R(\mathcal{A}, V)^{\langle -,- \rangle, \varepsilon}$*?*

From our considerations before, it is clear that Main Question 2.3 and Main Question 3.1 coincide; we can thus answer either of them.

## 4. The Seesaw algebra

Let $n \in \{2l, 2l+1\}$ be an integer and let $\mathcal{A} = \mathrm{k}\mathcal{Q}/I$ be the symmetric quiver algebra given by the symmetric quiver

$$\begin{tikzcd} \mathcal{Q}: \ 1 \arrow[r, "a_1"] & 2 \arrow[r, "a_2"] & \cdots \arrow[r, "a_{l-1}"] & l \arrow[r, "a_l"] & \omega \arrow[r, "a_l^*"] \arrow[loop below, "\alpha = \alpha^*"] & l^* \arrow[r, "a_{l-1}^*"] & \cdots \arrow[r, "a_2^*"] & 2^* \arrow[r, "a_1^*"] & 1^* \end{tikzcd}$$

where $\sigma(i) = i^*$ for $i \in \mathcal{Q}_0 \cup \mathcal{Q}_1$ and by the admissible ideal $I = (\alpha^2, a_l^* a_l)$. We call it the *Seesaw algebra* due to the appearance of $\mathcal{Q}$. We consider the symmetric dimension vector

$$\mathbf{d} := (\mathbf{d}_i)_i = (1, 2, \cdots, l-1, l, n, l, l-1, \cdots, 2, 1)$$

Let us fix $n = 2l$ for now, let $V = \oplus_{i \in \mathcal{Q}_0} V_i = \oplus_{i \in \mathcal{Q}_0} \mathrm{k}^{\mathbf{d}_i}$ and fix $\varepsilon = 1$ (that is, we work in orthogonal type D). Let $\langle -,- \rangle$ be a bilinear $\varepsilon$-form on $V$ as in Section 2. In order to be able to work in coordinates (and to depict our representations nicely), let us fix a basis $\mathcal{B}_s = \{v_k^{(s)} \mid 1 \leq k \leq i\}$ of each $V_s$ where $s \in \{i, i^*\}$ and $\mathcal{B}_\omega := \{v_k^{(\omega)}, v_k^{(\omega^*)} \mid 1 \leq k \leq l\}$ of $V_\omega$, such that on the basis elements the form is zero unless

$$\langle v_k^{(i)}, v_k^{(i^*)} \rangle = 1, \ \langle v_k^{(i^*)}, v_k^{(i)} \rangle = \varepsilon, \ \langle v_k^{(\omega)}, v_k^{(\omega^*)} \rangle = 1 \text{ or } \langle v_k^{(\omega^*)}, v_k^{(\omega)} \rangle = \varepsilon.$$

Let $M = (M_\beta)_{\beta \in \mathcal{Q}_1}$ and $N = (N_\beta)_{\beta \in \mathcal{Q}_1}$ be two representations. Here, $M_{\alpha_i} = N_{\alpha_i}$, are the standard embeddings into the first $i$ copies of k, $M_{\alpha_{i^*}} = N_{\alpha_{i^*}}$ equals minus the standard projection of the last $i$ copies of k onto $\mathrm{k}^i$. Furthermore, $M_\alpha$ sends $v_1^{(\omega)}$ to $v_l^{(\omega^*)}$, $v_l^{(\omega)}$ to $v_1^{(\omega^*)}$ and $N_\alpha$ sends $v_1^{(\omega)}$ to $v_l^{(\omega)}$, $v_l^{(\omega^*)}$ to $v_1^{(\omega^*)}$ and every other basis element is mapped to zero by $M_\alpha$ and $N_\alpha$. We depict them by their coefficient quivers for $n = 4$.

$$M = \quad \begin{tikzcd} v_1^{(1)} \arrow[r, "1"] & v_1^{(2)} \arrow[r, "1"] & v_1^{(\omega)} \arrow[dd, bend left, "1"] & & \\ & v_2^{(2)} \arrow[r, "1"] & v_l^{(\omega)} \arrow[dd, bend right, "-1"'] & & \\ & & v_l^{(\omega^*)} \arrow[r, "-1"'] & v_2^{(2^*)} & \\ & & v_1^{(\omega^*)} \arrow[r, "-1"'] & v_1^{(2^*)} \arrow[r, "-1"'] & v_1^{(1^*)} \end{tikzcd}$$

$$N = \begin{tikzcd}[row sep=small]
v_1^{(1)} \arrow[r, "1"] & v_1^{(2)} \arrow[r, "1"] & v_1^{(\omega)} \arrow[d, bend left, "1"] & & \\
& v_2^{(2)} \arrow[r, "1"] & v_2^{(\omega)} & & \\
& & v_2^{(\omega^*)} \arrow[r, "-1"'] \arrow[d, bend right, "-1"'] & v_2^{(2^*)} & \\
& & v_1^{(\omega^*)} \arrow[r, "-1"'] & v_1^{(2^*)} \arrow[r, "-1"'] & v_1^{(1^*)}
\end{tikzcd}$$

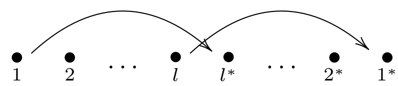
Then the relations $M_\alpha^2 = N_\alpha^2 = 0 = \pi_l \circ \iota_l$ are fulfilled and $M, N \in R(\mathcal{A}, V)^{\langle -,- \rangle, \varepsilon}$. The following proposition gives our counterexample.

**Proposition 4.1.** *In type D,,*

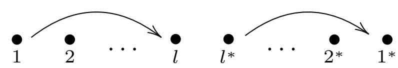
*(1)* $N \in \overline{\mathrm{GL}^\bullet(V) \cdot M} \subseteq R(\mathcal{A}, V)$*, i.e.* $M \leq_{\deg} N$

*(2)* $N \notin \overline{\mathrm{G}^\bullet(V, \langle -,- \rangle) \cdot M} \subseteq R(\mathcal{A}, V)^{\langle -,- \rangle, \varepsilon}$*, i.e.* $M \not\leq^{\varepsilon}_{\deg} N$

*Proof.* In a similar way as in [7], the representation $M$ corresponds to the so-called oriented link pattern (representing the part of the coefficient quiver which describes the loop $\alpha$ at vertex $\omega$)



and the representation $N$ corresponds to



In order to prove (1), we make use of the description of degenerations given in [7]:

$$M \leq_{\deg} N \Longleftrightarrow (p_i^M \leq p_i^{M'}) \wedge (q_{i,j}^M \leq q_{i,j}^{M'}) \ \forall i, j \in \{1, \ldots, l, 1^*, \ldots, l^*\},$$

where $p_i^X$ and $q_{i,j}^X$ are data which depend on the oriented link pattern of a representation $X$ as follows:

- $p_i^X$ equals the number of vertices to the left of $i$ which are not incident with an arrow, plus the number of arrows whose target vertex is to the left of $i$.
- $q_{i,j}^X$ equals $p_j^X$ plus the number of arrows whose source vertex lies to the left of $j$ and whose target vertex lies to the left of $i$.

By simple counting, we see that $p_i^M \leq p_i^N$ and $q_{i,j}^M \leq q_{i,j}^N$ for all $i, j$, and, thus, that $M \leq_{\deg} N$.

In order to prove (2), we calculate the dimensions of $\overline{\mathrm{G}^\bullet(V, \langle -,- \rangle) \cdot M} = \mathrm{G}^\bullet(V, \langle -,- \rangle) \cdot M$ and $\mathrm{G}^\bullet(V, \langle -,- \rangle) \cdot N$. Note that

$$\dim \mathrm{G}^\bullet(V, \langle -,- \rangle) = \dim(\prod_{i=1}^{l} \mathrm{GL}_i(\mathrm{k}) \times \mathrm{O}_n) = \sum_{i=1}^{l} i^2 + 2l^2 - l.$$

The stabilizer dimension can e.g. be calculated by basic methods of linear algebra when going over to Borel-orbits of 2-nilpotent matrices as explained in Remark 4.3. Another option is a calculation in terms symmetric endomorphism spaces [6] or of certain Crawley Boevey triples [8] (since $\mathcal{A}$ is a string algebra). It follows that

$$\dim \mathrm{Stab}_{\mathrm{G}^\bullet(V, \langle -,- \rangle)} M = \dim \mathrm{Stab}_{\mathrm{G}^\bullet(V, \langle -,- \rangle)} N = l^2 - 3l + 4$$

In particular, the dimensions of the orbits of $M$ and $N$ coincide:

$$\begin{array}{rcl}
\mathrm{G}^\bullet(V, \langle -,- \rangle) \cdot M & = & \sum_{i=1}^{l} i^2 + 2l^2 - l - (l^2 - 3l + 4) \\
& = & \frac{1}{3}l^3 + \frac{3}{2}l^2 + \frac{13}{6}l - 4 \\
& = & \mathrm{G}^\bullet(V, \langle -,- \rangle) \cdot N
\end{array}$$

Since the orbit closure $\overline{\mathrm{G}^\bullet(V,\langle -,-\rangle)\cdot M}$ equals the union of $\mathrm{G}^\bullet(V,\langle -,-\rangle)\cdot M$ and orbits of smaller dimension, we have therefore shown $N\notin\overline{\mathrm{G}^\bullet(V,\langle -,-\rangle)\cdot M}$ and, thus, $M\not\leq^{\varepsilon}_{\mathrm{deg}} N$ $\square$

Proposition 4.1 leads to negative answers for Main Question 2.3 and Main Question 3.1 for the seesaw algebra and thus we have the following corollary.

**Corollary 4.2.** *Given a symmetric quiver algebra of finite representation type, the equivalences*

$$\leq_{\mathrm{deg}}\Longleftrightarrow\leq^{\varepsilon}_{\mathrm{deg}}$$

$$\leq^{\varepsilon}_{\mathrm{deg}}\Longleftrightarrow\leq_{\mathrm{Hom}}$$

*are not in general true. This particularly means that orbit closure relations between symmetric representations are not in general induced by type A degenerations.*

*Remark* 4.3. The (symmetric) representation theory of the seesaw algebra can be translated to a particular Lie-theoretic setup [6] in a related, but more involved way as in Example 2.4: Let $B\subset\mathrm{GL}_n(\mathrm{k})$ be the standard Borel subgroup of upper-triangular matrices. Let $G$ be a classical Lie group, then $B_G=B\cap G\subset G$ is the standard Borel subgroup of $G$. If $G$ is orthogonal, we fix $\varepsilon=1$ and if $G$ is symplectic, we fix $\varepsilon=-1$. We consider the dimension vector $\mathbf{d}$, the k-vector space $V$ and the form $\langle -,-\rangle$ on $V$ as described in the beginning of this section. Let $\mathcal{N}^{(2)}\subseteq\mathcal{N}$ be the subvariety of 2-nilpotent matrices and notice that $B$ acts on $\mathcal{N}^{(2)}$ and $B_G$ acts on $\mathcal{N}^{(2)}\cap\mathrm{Lie}\,G$ via conjugation. Let furthermore $R^0(\mathcal{A},V)\subseteq R(\mathcal{A},V)$ and $R^0(\mathcal{A},V)^{\langle -,-\rangle,\varepsilon}\subseteq R(\mathcal{A},V)^{\langle -,-\rangle,\varepsilon}$ denote the subvarieties of representations of which the linear maps at $\alpha_i$ are injective and at $\alpha_{i^*}$ are surjective for all $i,i^*$. Following [6], there are associated fibre bundle constructions and isomorphisms of complex varieties

$$R^0(\mathcal{A},V)\cong\mathrm{GL}^\bullet(V)\times^B\mathcal{N}^{(2)}$$
$$R^0(\mathcal{A},V)^{\langle -,-\rangle,\varepsilon}\cong\mathrm{G}^\bullet(V,\langle -,-\rangle)\times^{B_G}\mathcal{N}^{(2)}\cap\mathrm{Lie}\,G$$

This means that orbits and orbit closure relations between the Borel-actions on 2-nilpotent matrices and the isomorphism classes of representations can be translated. In particular, our counterexample of Proposition 4.1 shows that orthogonal Borel-orbit closure relations in $\mathcal{N}^{(2)}$ are not induced by Type A. Note that further Lie-theoretic actions can be translated to quiver-theoretic settings via associated fibre bundles, for example certain parabolic group actions on different subvarieties of the nilpotent cone.

Let us consider an explicit example now.

*Example* 4.4. Let $(n,\varepsilon)=(5,1)$ in Type B, $(n,\varepsilon)=(4,-1)$ in Type C and $(n,\varepsilon)=(4,1)$ in Type D. Let us fix a basis as in the proof of Proposition 4.1 and add a basis vector $v$ for odd type B; this vector fulfills $\langle v,v\rangle=1$ and $\langle v,v'\rangle=0$ for every other basis vector $v'$. Up to isomorphism, every representation $M$ can be assumed to have the following coefficient quiver, together with a matrix $M_\alpha$ at the loop:

$$v_1^{(1)}\xrightarrow{1}v_1^{(2)}\xrightarrow{1}v_1^{(\omega)}$$
$$v_2^{(2)}\xrightarrow{1}v_2^{(\omega)}$$
$$(v)$$
$$v_2^{(\omega^*)}\xrightarrow[-1]{}v_2^{(2^*)}$$
$$v_1^{(\omega^*)}\xrightarrow[-1]{}v_1^{(2^*)}\xrightarrow[-1]{}v_1^{(1^*)}$$

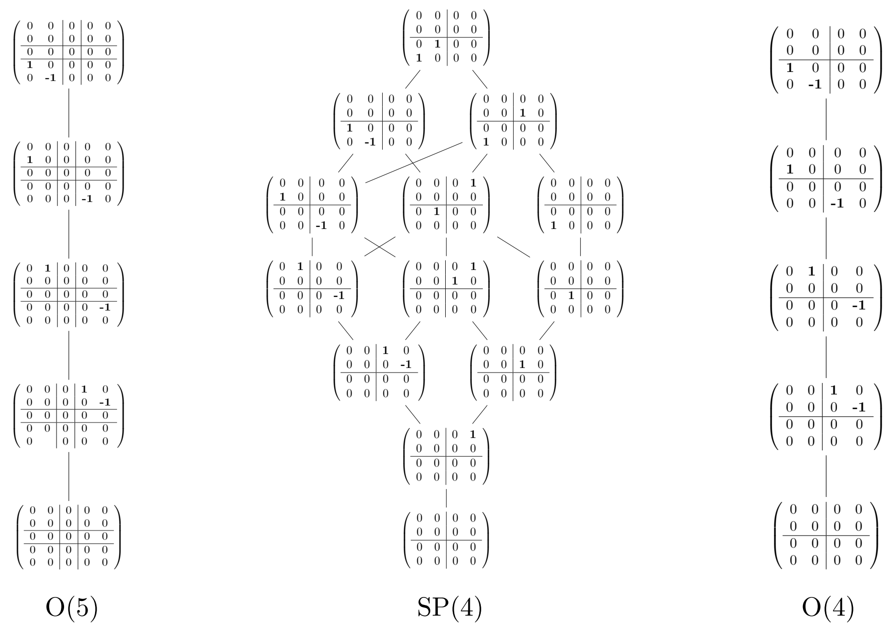



Figure 1. Diagrams of (Type A-)degenerations

We depict an isomorphism class of $\varepsilon$-representations by such representative matrix $M_\alpha$ in the following. Figure 1 shows complete representative systems of orbits of $\varepsilon$-representations in the different Lie types. Furthermore, all Type A-degenerations $\leq_{\text{deg}}$ between these $\varepsilon$-representations are depicted. In Types B and C, every such Type-A-degeneration is indeed an $\varepsilon$-degeneration which can be seen by calculation of $\varepsilon$-extensions $\leq_{\text{Ext}}^{\varepsilon}$ or by describing explicit curves which go over from one orbit into another. In Type D, however, the picture of $\varepsilon$-degenerations is different, since $\varepsilon$-degenerations are not induced (see Proposition 4.1). Figure 4.4 shows the actual orbit closure relations in $R(\mathcal{A}, V)^{\langle -,- \rangle, \varepsilon}$; every connecting line stands for a relation $\leq_{\text{deg}}^{\varepsilon}$.

## 5. Conjectures

Following Example 4.4, we state the following (quite educated) conjectures about degenerations for the seesaw algebra. In type C, this conjecture is built on work of Gandini, Möseneder Frajria and Papi [10] and we aim to present a proof of Conjecture 5.1 in a subsequent article.

*Conjecture* 5.1. Let $\mathcal{A}$ be the seesaw algebra.

- Type B ($\varepsilon = 1$ and $n$ odd): Main Question 3.1 is true when restricted to single $\mathrm{GL}^\bullet(V)$-orbits in $R(\mathcal{A}, V)$. We expect it to be true in general.
- Type C ($\varepsilon = -1$ and $n$ even): Main Question 3.1 is true.

In Type D, by Proposition 4.1 we know that Main Question 2.3 is answered negatively for every $n$. We note that the seesaw algebra is not representation-directed [6].

*Conjecture* 5.2. Let $\mathcal{A}$ be representation-directed. Then Main Question 3.1 is true.

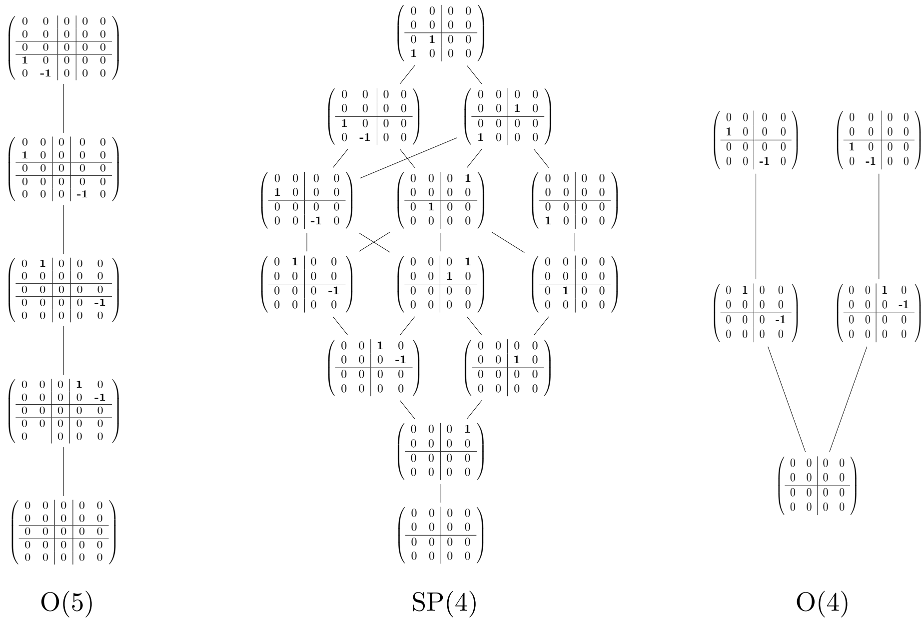



FIGURE 2. Diagrams of $\varepsilon$-degenerations

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