A FUJITA TYPE RESULT FOR A DEGENERATE NEUMANN PROBLEM IN DOMAINS WITH NON COMPACT BOUNDARY

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1. Introduction

Let $\Omega \subset \boldsymbol{R}^N$, $N \geq 2$, be an unbounded domain with non compact boundary $\partial\Omega$. We study the behaviour of nonnegative solutions in $Q_T = \Omega \times (0, T)$, $T \leq \infty$, of the following Neumann problem

$$(w^\beta)_t - \operatorname{div}(|Dw|^{m-1}Dw) = w^\nu \,, \qquad \text{in } Q_T \,, \tag{1.1}$$

$$|Dw|^{m-1}Dw \cdot \mathbf{n} = 0 \,, \qquad \text{on } \partial\Omega \times (0, T) \,, \tag{1.2}$$

$$w^\beta(x, 0) = w_0^\beta(x) \,, \qquad \text{in } \Omega \,, \tag{1.3}$$

where $0 < \beta \leq 1$, $m > 1$, $\nu > 1$, $w_0(x) \geq 0$ $x \in \Omega$, with $w_0^\beta \in L^1_{loc}(\overline{\Omega})$; finally $\mathbf{n}$ denotes the outer normal to $\partial\Omega$. We are primarily concerned with existence or non existence of global in time solutions, and with estimates of the finite speed of propagation of the support of w. We show how these properties are connected with the geometry of Ω.

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In fact, we work mostly with the following euivalent formulation, obtained by changing variables. Setting $u = w^{\beta}$, we have from (1.1)–(1.3)

$$u_t - c\,\mathrm{div}(u^{\alpha}|Du|^{m-1}Du) = u^{\mu}\,, \qquad \text{in } Q_T\,, \tag{1.4}$$

$$u^{\alpha}|Du|^{m-1}Du \cdot \mathbf{n} = 0\,, \qquad \text{on } \partial\Omega \times (0,T)\,, \tag{1.5}$$

$$u(x,0) = u_0(x)\,, \qquad \text{in } \Omega\,, \tag{1.6}$$

where $\alpha = (1-\beta)m/\beta \geq 0$, $\mu = \nu/\beta > 1$; c is a positive constant depending on m, β.

Our approach relies on sharp integral estimates of solutions to the problems above, which in turn are a consequence of embedding results involving geometrical properties of the domain Ω. It is therefore natural to introduce the following quantity $\ell(v)$, obviously related to isoperimetrical inequalities in Ω

$$\ell(v) = \inf\{|\partial G \cap \Omega|_{N-1} : G \subset \Omega\,, |G| = v\,, \partial G \text{ Lipschitz}\}\,, \quad \text{for all } v > 0\,,$$

(we use the symbol $|\cdot|$ to denote the N dimensional Lebesgue measure, while the $N-1$ dimensional Hausdorff measure is denoted by $|\cdot|_{N-1}$). We are going to assume that $\ell(v) > 0$ for $v > 0$; in fact, all our arguments are given in terms of a continuous function g satisfying

$$0 < g(v) \leq \ell(v)\,, \quad v > 0\,, \tag{1.7}$$

$$\omega(v) := \frac{v^{\frac{N-1}{N}}}{g(v)} \quad \text{is non decreasing for } v > 0\,. \tag{1.8}$$

Let us also introduce the volume function V and its inverse R

$$V(\rho) = |\Omega_\rho|\,, \quad \rho > 0\,, \quad \Omega_\rho = \Omega \cap \{|x| < \rho\}\,, \qquad R = V^{(-1)}\,. \tag{1.9}$$

Definition 1.1. Let $\Omega \subset \boldsymbol{R}^N$, $N \geq 2$ be an unbounded connected open set, satisfying $|\Omega| = \infty$, with a Lipschitz continuous boundary $\partial\Omega$, $0 \in \partial\Omega$. Assume moreover that a function $g \in C((0,\infty))$ is given, as in (1.7)–(1.8). Then we say that Ω belongs to the class $\mathcal{B}_1(g)$.

Definition 1.2. The class $\mathcal{B}_2(g)$ comprises all the domains $\Omega \in \mathcal{B}_1(g)$ which also satisfy

$$c_0 \frac{v}{g(v)} \le R(v) \le c_1 \frac{v}{g(v)} , \qquad \text{for all } v > 0, \tag{1.10}$$

for two suitable constants c_0, $c_1 > 0$.

Assuming (1.10) essentially amounts to requiring that the volume $V(\rho)$ is equivalent to $\rho g(V(\rho))$.

It is easy to see that the requirement $\ell(v) > 0$ rules out the case of Ω shaped like an "infinite cusp" (with infinite volume). Then domains in the classes defined above are sometimes referred to as "expanding" or "non-contracting" domains. Domains in classes similar to $\mathcal{B}_1(g)$, $\mathcal{B}_2(g)$ were considered by Gushchin, see [7] and subsequent papers.

Example (paraboloid-like domains). Let $0 \le h \le 1$ be fixed, and define

$$\Omega = \{x \in \boldsymbol{R}^N \mid |x'| < x_N^h\} , \qquad x' = (x_1, \dots, x_{N-1}) . \tag{1.11}$$

It follows from the results of [14], Chapter 4, that $\Omega \in \mathcal{B}_1(g)$, with

$$g(v) = \gamma \min(v^{\frac{N-1}{N}}, v^{\eta}) , \quad v > 0 ; \qquad \eta = \frac{h(N-1)}{1 + h(N-1)} \le \frac{N-1}{N} .$$

Thus in this case

$$\omega(v) = \gamma \max(1, v^{\frac{N-1}{N} - \eta}) , \quad v > 0 .$$

In fact, it is clear that $\Omega \in \mathcal{B}_2(g)$.

Note that Ω is a cone when $h = 1$, while it is a cylinder when $h = 0$. See also remarks 1.1, 3.1 for further comments on this class of examples.

Definition 1.3. We say that u is a weak solution of (1.4)–(1.6) if $u \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \times (0, T))$, $u \in C((0, T); L^2_{\mathrm{loc}}(\overline{\Omega}))$, $u^\alpha |Du|^{m+1} \in L^1_{\mathrm{loc}}(\overline{\Omega} \times (0, T))$, and for all $\zeta \in C^1(\boldsymbol{R}^N \times [0, T])$, such that $\operatorname{supp}\zeta$ is contained in a cylinder $\{|x| \le \rho < \infty\,, 0 \le t < T\}$, we have

$$\int_0^T\int_{\boldsymbol{R}^N} [-u\zeta_t + u^\alpha |Du|^{m-1} Du\cdot D\zeta - u^\mu\zeta]\,\mathrm{d}x\,\mathrm{d}t = -\int_{\boldsymbol{R}^N} u_0(x)\zeta(x,0)\,\mathrm{d}x\,.$$

The notion of weak solution to (1.1)–(1.3), can be derived immediately from the Definition above.

Let us also define for all $q > 0$

$$\begin{aligned} &\psi_q := \text{ the inverse function over } [0,+\infty) \text{ of} \\ &\Psi_q(z) = z^{\frac{\alpha+m-1}{q}+\frac{m+1}{N}}\omega(z)^{m+1} = z^{\frac{\alpha+m-1}{q}+m+1} g(z)^{-(m+1)}\,, \quad z \ge 0\,. \end{aligned} \tag{1.12}$$

We denote in the following $\psi = \psi_1$. We also use the notation $\|u\|_{p,G} = \|u\|_{L^p(G)}$, and denote by γ, γ_i $i = 0, 1, \dots,$ generic positive constants, whose dependence on N, ν, β, m, and c_0, c_1 in (1.10) is implicitly understood.

Let us state first our results on global solvability and blow up of solutions to (1.1)–(1.3).

Theorem 1.1. *Let us assume that $\Omega \in \mathcal{B}_1(g)$, and that $\psi = \psi_1$ as in* (1.12) *satisfies*

$$\int^{+\infty} \psi(z)^{-(\frac{\nu}{\beta}-1)}\,\mathrm{d}z < +\infty\,. \tag{1.13}$$

Then problem (1.1)–(1.3) *has a solution w defined for all positive times, provided the initial datum fulfils*

$$\|w_0^\beta\|_{1,\Omega} + \|w_0^\beta\|_{q,\Omega} \le \delta\,, \tag{1.14}$$

where $q > 1$ is such that $N(\nu - m) < q\beta(m+1)$, and $\delta = \delta(N, m, \beta, \nu, q, g)$ is chosen suitably small. Moreover w satisfies

$$\|w(\cdot,t)\|^\beta_{\infty,\Omega} \le \gamma \frac{\|w_0^\beta\|_{1,\Omega}}{\psi(t\|w_0^\beta\|_{1,\Omega}^{\frac{m-\beta}{\beta}})}\,, \qquad \textit{for all } t > 0. \tag{1.15}$$

Theorem 1.2. *Assume that $\Omega \in \mathcal{B}_1(g)$, $\nu > m$, and that*

$$\int_0 V\big(\tau^{-\frac{\nu-m}{m+1}}\big)^{1/\beta}\,\mathrm{d}\tau < \infty\,. \tag{1.16}$$

Then all non negative solutions w to (1.1)–(1.3) *become unbounded in a finite time (in some bounded subset of Ω), excepting of course the trivial solution $w \equiv 0$, provided we assume also that there exist a number $0 < \lambda < \beta(m+1)/(\nu - m)$, and a non increasing function $\chi : (0, \infty) \to (0, \infty)$ such that*

$$C_1 \chi(\rho) \leq V(\rho)\rho^{-\lambda} \leq C_2 \chi(\rho) , \qquad \textit{for large } \rho, \tag{1.17}$$

for two suitable positive constants C_1, C_2.

Remark 1.1. If Ω is a "paraboloid" as in (1.11), we can check that

$$(1.13) \Leftrightarrow \frac{b}{\Lambda(b)}(\mu - 1) > 1 , \quad \text{and} \quad (1.16) \Leftrightarrow \frac{b}{\Lambda(b)}(\mu - 1) < 1 , \tag{1.18}$$

where $\Lambda(b) = b(\alpha + m - 1) + m + 1$, $b = 1 + h(N - 1)$ (we select $\lambda = b$ and $\chi \equiv 1$ in Theorem 1.2). Therefore the conditions we give for existence or non existence of global solutions are sharp, at least for this class of examples. Let us note here that we do not deal with the threshold case $b(\mu - 1) = \Lambda(b)$. The parameter ranges in (1.18) should be compared with their counterparts in the case of the Cauchy problem $\Omega = \boldsymbol{R}^N$, where it is known (at least if $\alpha = 0$ or $m = 1$, see [2], [6]) that global existence is possible if $1 < N(\mu - 1)/\Lambda(N)$, while every non trivial solution blows up in a finite time if $1 \geq N(\mu - 1)/\Lambda(N)$. We see that the homogeneous Neumann problem in a cone ($h = 1$), respectively in a cylinder ($h = 0$), behaves like the Cauchy problem in spatial dimension N, respectively in spatial dimension 1, at least from this point of view. In fact, the Cauchy problem is covered by our methods, when one takes $g(v) = \gamma_0 v^{\frac{N-1}{N}}$. Note, by contrast, that the Cauchy problem and the Dirichlet problem in cones (with homogeneous boundary data) are different, see [12].

Remark 1.2. Provided we assume g to be smooth enough, we may change variable $t = \psi(z)$ in (1.13), and prove by integrating by parts that $N(\nu - m)/\beta(m + 1) \geq 1$ if (1.13) is fulfilled.

It follows from (1.16) that $V(\rho)\rho^{-\beta\frac{m+1}{\nu - m}} \to 0$ as $\rho \to \infty$ (see (5.5)). Then our technical assumption (1.17) does not seem to be too restrictive, also in view of the examples in Remark 1.1.

Due to the degeneracy of the equation, one expects solutions with compactly supported initial datum to exhibit the property of finite speed of propagation. In our next result we give a sharp estimate for the support of $u(\cdot,t)$, for large t. Let us define

$$Z(t) = \inf\{\rho > 0 \mid \operatorname{supp} u(\cdot,t) \subset \Omega_\rho\}\,, \quad t > 0\,.$$

Theorem 1.3. *Assume $\Omega \in \mathcal{B}_2(g)$, and let the assumptions of Theorem 1.1 be satisfied, for a suitable $\delta > 0$ in* (1.14)*. Moreover, we require that*

$$\operatorname{supp} w_0 \subset \Omega_{\rho_0}\,, \qquad \text{for a given } 0 < \rho_0 < \infty, \tag{1.19}$$

and that g is non decreasing. Then, for all $t > \bar{t} > 0$, we have

$$\gamma_0 R(\psi(t\|w_0^\beta\|_{1,\Omega}^{\frac{m-\beta}{\beta}})) \le Z(t) \le \gamma R(\psi(t\|w_0^\beta\|_{1,\Omega}^{\frac{m-\beta}{\beta}}))\,, \tag{1.20}$$

and

$$\|w(\cdot,t)\|_{\infty,\Omega}^\beta \ge \gamma_0 \frac{\|w_0^\beta\|_{1,\Omega}}{\psi(t\|w_0^\beta\|_{1,\Omega}^{\frac{m-\beta}{\beta}})}\,, \tag{1.21}$$

where $\bar{t}$ depends on $\|u_0^\beta\|_{1,\Omega}$, ρ_0 and N, m, β, $g(1)$.

Note that (1.21) implies that the sup estimate (1.15) is sharp.

Remark 1.3. 1) If $\Omega \in \mathcal{B}_1(g)$ in Theorem 1.3, our proof still implies that

$$Z(t) \le \gamma G_0(\psi(t\|u_0^\beta\|_{1,\Omega}^{\frac{m-\beta}{\beta}}))\,, \qquad t > \bar{t}\,, \qquad \text{where } G_0(s) = s/g(s)\,, \quad s > 0\,.$$

2) In fact, the assumption that g be monotonic can be relaxed somehow. We refer to the proof in Section 6 below.

Remark 1.4. It is well known that equations containing nonlinear sources of the type of (1.4), require local regularity conditions on the initial datum in order to be solvable (see [2] and the literature quoted therein; note that q is suitably large in Theorem 1.1). The optimal regularity condition for initial data measures could be found in the present setting by means of the methods of [1], but we do not dwell on this point, as the main interest here is on the behaviour of solutions for large times.

Pioneering work on the subject of Neumann problem for parabolic equations in unbounded domains is due to Gushchin [7], [8], and following papers, see [9], where the author considered a class (close to our $\mathcal{B}_1(g)$, $\mathcal{B}_2(g)$), of domains with non compact boundary satisfying isoperimetrical inequalities essentially similar to (1.7), (1.8). However, those papers were only concerned with the study of linear parabolic equations. For such equations, [8] gave the optimal stabilization rate as $t \to \infty$, for $u \in L^1(\Omega) \cap L^\infty(\Omega)$.

For the degenerate case we treat in this paper, but without the nonlinear source term, the optimal stabilization rate was given in [17], and finite speed of propagation was proven in [16], for solutions satisfying the global integrability requirement $|Du| \in L^2(Q_T)$, in the more general setting of higher order equations. Moreover, in [16], Bernis' approach [4] was employed, relying on a weighted interpolation technique, requiring in turn additional assumptions on Ω. Here we estimate the finite speed of propagation by means of the method introduced in [3].

In Section 2, we establish some preliminary technical facts needed in the following. In Section 3 we prove the essential a priori L^∞ estimates for u. In Section 4 we prove Theorem 1.1, and in Section 5 we give the proof of Theorem 1.2. Section 6 is devoted to the proof of Theorem 1.3. Please, note that in sections 3, 4 and 6, we work with formulation (1.4)–(1.6), while the setting (1.1)–(1.3) is used in Section 5.

2. Preliminaries

Lemma 2.1. *Let $\Omega \in \mathcal{B}_1(g)$, $v \in L^\infty((0,T); L^r(\Omega))$, $Dv \in (L^p(\Omega))^N$, with $p > 1$, $r \geq 1$, and assume that $\sup_{(0,T)} |\operatorname{supp} v(\cdot,t)| < \infty$. Then*

$$\int_0^T\!\!\int_\Omega |v|^{p+\frac{pr}{N}} \,\mathrm{d}x\,\mathrm{d}t \leq \gamma \sup_{0<t<T} \left[\omega(|\operatorname{supp} v(\cdot,t)|)^p \Big(\int_\Omega |v(x,t)|^r \,\mathrm{d}x \Big)^{\frac{p}{N}} \right] \int_0^T\!\!\int_\Omega |Dv|^p \,\mathrm{d}x\,\mathrm{d}t \,, \tag{2.1}$$

where the non decreasing function $\omega : [0,\infty) \to [0,\infty)$ is given by $\omega(\tau) = \tau^{1-1/N}/g(\tau)$, and $\gamma = \gamma(p,r,N)$.

Proof. We start from the "elliptic" embedding proven in [16], which we state in the notation used here: Let $v \in L^h(\Omega) \cap L^s(\Omega)$, $Dv \in (L^p(\Omega))^N$. Then

$$\|v\|_{s,\Omega} \leq \gamma \omega(V) V^{\frac{1}{s}-\frac{1}{p}+\frac{1}{N}} \|Dv\|_{p,\Omega} \,, \tag{2.2}$$

where $s \geq 1$, $s > h > 0$, $p > 1$, $s(N-p) \leq Np$, $V = (\|v\|_{h,\Omega} \|v\|_{s,\Omega}^{-1})^{\frac{hs}{s-h}}$, and $\gamma = \gamma(s,h,p,N)$. If we assume that $|\operatorname{supp} v| < \infty$, we infer from an application of Hölder's inequality that $V \leq |\operatorname{supp} v|$. By using this estimate in (2.2) we get

$$\|v\|_{s,\Omega}^{1+\frac{hs}{s-h}\left(\frac{1}{s}-\frac{1}{p}+\frac{1}{N}\right)} \leq \gamma \omega(|\operatorname{supp} v|) \|v\|_{h,\Omega}^{\frac{hs}{s-h}\left(\frac{1}{s}-\frac{1}{p}+\frac{1}{N}\right)} \|Dv\|_{p,\Omega} \,. \tag{2.3}$$

To prove (2.1) we discriminate between the cases $p < N$ and $p \geq N$. If $p < N$, we choose $s = Np/(N-p)$, and use the corresponding specialisation of (2.3) as follows

$$\begin{aligned} \int_0^T\!\!\int_\Omega |v(t)|^{p+\frac{pr}{N}} \,\mathrm{d}x\,\mathrm{d}t &\leq \int_0^T \Big(\int_\Omega |v(t)|^s \,\mathrm{d}x \Big)^{\frac{N-p}{N}} \Big(\int_\Omega |v(t)|^r \,\mathrm{d}x \Big)^{\frac{p}{N}} \mathrm{d}t \\ &\leq \gamma \int_0^T \omega(|\operatorname{supp} v(t)|)^p \Big(\int_\Omega |v(t)|^r \,\mathrm{d}x \Big)^{\frac{p}{N}} \Big(\int_\Omega |Dv(t)|^p \,\mathrm{d}x \Big) \,\mathrm{d}t \,, \end{aligned}$$

whence (2.1).

If $p \geq N$, we may choose $s = p + pr/N > r$, $h = r$, and check that (2.3) reduces to

$$\int_\Omega |v|^s \,\mathrm{d}x \leq \gamma \omega(|\operatorname{supp} v|)^p \Big(\int_\Omega |v|^r \,\mathrm{d}x \Big)^{\frac{p}{N}} \int_\Omega |Dv|^p \,\mathrm{d}x \,;$$

(2.1) follows easily on integrating the above inequality, written for $v = v(\cdot, t)$, over $(0, T)$. $\square$

Remark 2.1. Estimates (2.1), (2.3), should be compared with the embeddings of Chapter I of [11] and Chapter II of [5]. The embeddings given here reduce the ones there when Ω is a cone, or $\boldsymbol{R}^N$ (so that ω is constant). See also [8] for the case $p = 2$.

Remark 2.2. In [16] the proof of (2.2) contains a formal mistake in the case $p < h$, which can be easily fixed (we refer the reader to the proof of the embedding result in the forthcoming paper [3]).

Lemma 2.2. *Let* $\{Y_n\}$*,* $n \geq 0$*, be a sequence of non negative real numbers satisfying*

$$Y_{n+1} \leq C b^n Y_n^{1+\varepsilon} f(C_1 b_1^n Y_{n-1}) \,, \; n \geq 1 \,, \tag{2.4}$$

where $b > 1$*, and* ε*,* C*,* C_1*,* $b_1 > 0$ *are real numbers. We also assume* $Y_1 \leq Y_0$*, and that the function* $f : [0, \infty) \to [0, \infty)$ *is non decreasing. Then* $Y_n \to 0$ *as* $n \to \infty$*, provided*

$$C \beta^{1+1/\varepsilon} Y_0^{\varepsilon} f(C_1 \beta^{2/\varepsilon} Y_0) \leq 1 \,, \qquad \beta = \max(b, b_1^{\varepsilon}) \,. \tag{2.5}$$

Proof. The claim made in the statement can be proven as Lemma 5.6 in Chapter II of [11]. Anyway, for the reader's convenience, we give here a short proof. We are in fact going to show that

$$Y_n \leq \beta^{-\frac{n-1}{\varepsilon}} Y_0 \,, \text{ for all } n \geq 0 \,, \tag{2.6}$$

whence the result, keeping in mind that $\beta > 1$ according to its definition in (2.5). Of course (2.6) holds when $n = 0$, $n = 1$, because $Y_1 \leq Y_0$ by assumption. Next we proceed by induction, assuming it holds also for all $0 \leq i \leq n$. We have

$$\begin{aligned} Y_{n+1} &\leq C b^n (\beta^{-\frac{n-1}{\varepsilon}} Y_0)^{1+\varepsilon} f(C_1 b_1^n \beta^{-\frac{n-2}{\varepsilon}} Y_0) \\ &\leq (\beta^{-\frac{n}{\varepsilon}} Y_0) [C \beta^{1+1/\varepsilon} Y_0^{\varepsilon} f(C_1 \beta^{2/\varepsilon} Y_0)] \leq \beta^{-\frac{n}{\varepsilon}} Y_0 \,, \end{aligned} \tag{2.7}$$

owing to (2.5). Therefore (2.6) is in force for $i = n+1$ too, and the proof is concluded. □

We conclude this section with the following technical lemma, whose results are employed without further mention throughout the paper.

Lemma 2.3. *For g as in* (1.7) *and ψ_q as in* (1.12)*, we have for all $z > 0$*

$$g(\gamma z) \le \gamma^{\frac{N-1}{N}} g(z)\,, \quad \gamma > 1\,; \qquad g(\delta z) \ge \delta^{\frac{N-1}{N}} g(z)\,, \quad 0 < \delta < 1\,; \tag{2.8}$$

$$\psi_q(\gamma z) \le \gamma^{\frac{Nq}{\mathcal{K}_q}} \psi_q(z)\,, \quad \gamma > 1\,; \qquad \psi_q(\delta z) \ge \delta^{\frac{Nq}{\mathcal{K}_q}} \psi_q(z)\,, \quad 0 < \delta < 1\,, \tag{2.9}$$

$\mathcal{K}_q = N(\alpha + m - 1) + q(m+1)$, $q > 0$.

Proof. The first inequality in (2.8) follows from

$$\frac{(\gamma z)^{\frac{N-1}{N}}}{g(\gamma z)} \ge \frac{z^{\frac{N-1}{N}}}{g(z)}\,,$$

which holds true because $\gamma > 1$ and ω is nondecreasing by assumption. From the definition of Ψ_q and from (2.8), we infer at once

$$\Psi_q(\gamma z) \ge \gamma^{\frac{N(\alpha+m-1)+q(m+1)}{Nq}} \Psi_q(z)\,, \qquad \gamma > 1\,,$$

whence we get the bound below for $\psi_q(\gamma z)$ in (2.9) (after redefining z, γ). The other estimates are proven in the same way. □

3. The main a priori estimate

Lemma 3.1. *Let u be a bounded solution to problem* (1.4)–(1.6) *in $\Omega_{2\rho} \times (0,t)$. Let $Q_0 = \Omega_{(1+\sigma)\rho} \times (t(1-\sigma)/2, t)$, $Q = \Omega_\rho \times (t/2, t)$, for a given $0 < \sigma < 1$. Then, if*

$$\frac{t}{\rho^{m+1}} \|u\|_{\infty,Q_0}^{\alpha+m-1} + t\|u\|_{\infty,Q_0}^{\mu-1} \le 1\,, \tag{3.1}$$

we have for any $q > 0$

$$\|u\|_{\infty,Q} \le \gamma \frac{\left(t^{-1} \iint_{Q_0} u^q \,\mathrm{d}x\,\mathrm{d}\tau\right)^{\frac{1}{q}}}{\psi_q\left(t\left(t^{-1} \iint_{Q_0} u^q \,\mathrm{d}x\,\mathrm{d}\tau\right)^{\frac{\alpha+m-1}{q}}\right)^{\frac{1}{q}}}\,, \tag{3.2}$$

where $\gamma = \gamma(\sigma, q, N, m, \alpha)$, *and* $\psi_q : [0, \infty) \to [0, \infty)$ *has been defined in* (1.12).

Proof. Let $k > 0$, $q > 0$ be constants, and let $\zeta \in C^1(\boldsymbol{R}^N \times \boldsymbol{R})$ be a standard cutoff function satisfying

$$\zeta \equiv 1 \text{ in } B_{r'} \times (t', t)\,, \ \operatorname{supp} \zeta \subset B_{r''} \times (t'', t)\,,$$
$$0 \le \zeta \le 1\,, \ |D\zeta| \le \gamma(r'' - r')^{-1}\,, \ 0 \le \zeta_t \le \gamma(t' - t'')^{-1}\,,$$

where $2\rho > r'' > r' > 0$ and $t > t' > t'' > 0$ are given. If we choose $(u-k)_+^q \zeta^{m+1}$ as a testing function in the weak formulation of the problem, we get by standard calculations (in fact also exploiting a Steklov averaging procedure)

$$\sup_{0<\tau<t} \int\limits_{\Omega(\tau)} (u-k)_+^{q+1} \zeta^{m+1} \,\mathrm{d}x + \int\limits_0^t \int\limits_\Omega u^\alpha |D(u-k)_+^{\frac{q+m}{m+1}} \zeta|^{m+1} \,\mathrm{d}x \,\mathrm{d}\tau$$
$$\le \gamma_1 \int\limits_0^t \int\limits_\Omega \left\{ (u-k)_+^{q+1} \zeta^m \zeta_\tau + u^\alpha |D\zeta|^{m+1} (u-k)_+^{q+m} + u^\mu (u-k)_+^q \zeta^{m+1} \right\} \,\mathrm{d}x \,\mathrm{d}\tau\,. \tag{3.3}$$

In what follows we let for all $n \ge 0$, $B_n = \{|x| < \rho_n\}$, $Q_n = (\Omega \cap B_n) \times (t_n, t)$, where

$$\rho_n = \rho\left(1 + \frac{\sigma}{2^n}\right)\,, \ t_n = \frac{t}{2}\left(1 - \frac{\sigma}{2^n}\right)\,, \ k_n = k\left(1 - \frac{1}{2^{n+1}}\right)\,,$$

so that $\{Q_n\}$ is a sequence of cylindrical domains interpolating between Q_0 and Q. We also denote by ζ_n a cut off function as above, with $r' = \rho_n$, $r'' = \rho_{n-1}$, $t' = t_n$, $t'' = t_{n-1}$, $n \ge 1$. We obtain from (3.3), choosing $k = k_{n+1}$, $\zeta = \zeta_{n+1}$ there,

$$\sup_{0<\tau<t} \int\limits_{\Omega(\tau)} (u-k_{n+1})_+^{q+1} \zeta_{n+1}^{m+1} \,\mathrm{d}x + k^\alpha \int\limits_0^t \int\limits_\Omega |D(u-k_{n+1})_+^{\frac{q+m}{m+1}} \zeta_{n+1}|^{m+1} \,\mathrm{d}x \,\mathrm{d}\tau$$
$$\le \gamma_2 \frac{2^{nm}}{t\sigma^{m+1}} \left\{ 1 + \frac{t}{\rho^{m+1}} \|u\|_{\infty,Q_0}^{\alpha+m-1} + t\|u\|_{\infty,Q_0}^{\mu-1} \right\} \iint\limits_{Q_n} (u-k_n)_+^{q+1} \,\mathrm{d}x \,\mathrm{d}\tau\,. \tag{3.4}$$

Next, as a consequence of the embedding Lemma 2.1, we establish the recursive inequality which is the core of the proof. Define

$$A_{n+1}=\{(x,\tau)\in Q_n \mid u(x,\tau)>k_{n+1}\}\subset \boldsymbol{R}^{N+1}\,,$$
$$A_{n+1}(\tau)=\{x\in \Omega\cap B_n \mid u(x,\tau)>k_{n+1}\}\subset \boldsymbol{R}^{N}\,,\; Y_n=\iint\limits_{Q_n}(u-k_n)_+^{q+1}\,\mathrm{d}x\,\mathrm{d}\tau\,.$$

We find, on applying firstly Hölder's inequality and then the embedding result just quoted,

$$\begin{aligned}
Y_{n+1} &\le \iint\limits_{Q_n}(u-k_{n+1})_+^{q+1}\zeta_{n+1}^{\theta}\le |A_{n+1}|^{1-\frac{q+1}{\lambda}}\Big(\iint\limits_{Q_n}(u-k_{n+1})_+^{\lambda}\zeta_{n+1}^{\frac{\lambda\theta}{q+1}}\,\mathrm{d}x\,\mathrm{d}\tau\Big)^{\frac{q+1}{\lambda}}\\
&\le \gamma_3|A_{n+1}|^{1-\frac{q+1}{\lambda}}\Bigg(\Big[\sup_{t_n<\tau<t}\omega(|A_{n+1}(\tau)|)^{m+1}\Big(\int\limits_{\Omega(\tau)}(u-k_{n+1})_+^{q+1}\zeta_{n+1}^{m+1}\,\mathrm{d}x\Big)^{\frac{m+1}{N}}\Big]\times\\
&\Big[\int_0^t\!\!\int\limits_{\Omega}|D(u-k_{n+1})_+^{\frac{q+m}{m+1}}\zeta_{n+1}|^{m+1}\,\mathrm{d}x\,\mathrm{d}\tau+\frac{2^{nm}}{\rho^{m+1}\sigma^{m+1}}\|u\|_{\infty,Q_0}^{\alpha+m-1}k^{-\alpha}Y_n\Big]\Bigg)^{\frac{q+1}{\lambda}}\,,
\end{aligned}\tag{3.5}$$

where $\theta>1$ is chosen large enough, and, as required by the embedding result,

$$\lambda=q+m+\frac{(m+1)(q+1)}{N}\,.$$

Note that $\lambda>q+1$ due to the assumptions on m. We are going to exploit estimate (3.4) in (3.5); we also need the following consequences of Chebichev's inequality (and of the definition of k_n)

$$|A_{n+1}|\le \gamma_4 2^{n(q+1)}k^{-q-1}Y_n\,,$$
$$|A_{n+1}(\tau)|\le \gamma_5 2^{n(q+1)}k^{-q-1}\int\limits_{B_n(\tau)}(u-k_n)_+^{q+1}\,\mathrm{d}x\le \gamma_6\frac{2^{n(m+q)}k^{-q-1}}{t\sigma^{m+1}}Y_{n-1}\,,\; n\ge 1\,.$$

In the last inequality we have made use again of (3.4), as well as of assumption (3.1). Collecting the estimates above we have for $n\ge 1$, after elementary algebraic calculations,

$$Y_{n+1} \le \gamma_7 b^n k^{-\frac{q+1}{\lambda}(\alpha+\lambda-q-1)} \frac{Y_n^{1+\frac{(q+1)(m+1)}{\lambda N}}}{(t\sigma^{m+1})^{\frac{q+1}{\lambda}\left(1+\frac{m+1}{N}\right)}} \omega\Big(\gamma_8 b_1^n \frac{k^{-q-1}}{t\sigma^{m+1}} Y_{n-1}\Big)^{(m+1)\frac{q+1}{\lambda}},$$

with b, b_1 suitable constants. According to Lemma 2.2, we have $Y_n \to 0$ as $n \to \infty$, provided k is chosen in such a way that the formula corresponding to (2.5) is fulfilled. More specifically, we choose k so as to have

$$\gamma_9 k^{-\frac{q+1}{\lambda}(\alpha+\lambda-q-1)} \frac{\|u\|_{q+1,Q_0}^{(q+1)\frac{(q+1)(m+1)}{\lambda N}}}{(t\sigma^{m+1})^{\frac{q+1}{\lambda}\left(1+\frac{m+1}{N}\right)}} \omega\Big(\gamma_{10} \frac{k^{-q-1}}{t\sigma^{m+1}} \|u\|_{q+1,Q_0}^{q+1}\Big)^{(m+1)\frac{q+1}{\lambda}} = 1\,. \tag{3.6}$$

Note that the left-hand side of (3.6) is decreasing in k, and that $\|u\|_{\infty,Q} \le k$, because $Y_n \to 0$, so that

$$\gamma_9 \|u\|_{\infty,Q}^{-(\alpha+\lambda-q-1)} \frac{\|u\|_{q+1,Q_0}^{(q+1)\frac{m+1}{N}}}{(t\sigma^{m+1})^{1+\frac{m+1}{N}}} \omega\Big(\gamma_{10} \frac{\|u\|_{\infty,Q}^{-q-1}}{t\sigma^{m+1}} \|u\|_{q+1,Q_0}^{q+1}\Big)^{m+1} \ge 1\,. \tag{3.7}$$

In order to conclude the proof let us define

$$r_0 = \rho\,,\ r_{i+1} = r_i + 2^{-i-1}\sigma\rho\,, \qquad t_0 = t/2\,,\ t_{i+1} = t_i - 2^{-i-2}\sigma t\,, \quad i \ge 0\,,$$
$$Q^i = (\Omega \cap \{|x| < r_i\}) \times (t_i, t) \subset Q^{i+1} \subset Q_0\,,\ U_i = \|u\|_{\infty,Q^i}\,,$$

so that $Q^0 = Q$, and $r_i \to (1+\sigma)\rho$, $t_i \to (1-\sigma)t/2$ as $i \to \infty$. In fact, estimate (3.7) has been proven for the pair of cylinders Q, Q_0 for the sake of notational simplicity, but we are going to apply it for a suitable pair Q^j, Q^{j+1}. Indeed, we define the integer j as follows, for a $\delta \in (0,1)$ to be chosen:

$$\begin{aligned} &j = 0\,, && \text{if } U_0 \ge \delta U_1\,,\\ &j = \sup\{k \ge 1 \mid U_{i-1} < \delta U_i \text{ for all } 1 \le i \le k\}\,, && \text{if } U_0 < \delta U_1\,. \end{aligned}$$

We may assume that j is finite, as we would have otherwise $\|u\|_{\infty,Q} \le \delta^i \|u\|_{\infty,Q_0}$ for all $i > 0$, implying $u \equiv 0$ in Q, and therefore, trivially, (3.2). Then we have

$$\text{a) } U_{j+1} \le \delta^{-1} U_j\,, \quad \text{b) } U_0 \le \delta^j U_j\,, \tag{3.8}$$

and also

$$\|u\|_{q+1,Q^{j+1}}^{q+1} \le U_{j+1}\|u\|_{q,Q^{j+1}}^{q} \le \delta^{-1}U_j\|u\|_{q,Q_0}^{q}\,, \tag{3.9}$$

by virtue of part a) of (3.8). (Here we use the notation $\|u\|_{q,Q}$ even if $q<1$.) It is clear that we may formally replace in (3.7) Q with Q^j, and Q_0 with Q^{j+1}, of course also replacing σ with $2^{-(j+1)}\sigma$ because of the change in the geometry. If we also use (3.9) in the resulting inequality, we find

$$\gamma_9 U_j^{-(\alpha+m-1+\frac{m+1}{N}q)}\delta^{-\frac{m+1}{N}}\frac{\|u\|_{q,Q_0}^{q\frac{m+1}{N}}}{(t2^{-(j+1)(m+1)}\sigma^{m+1})^{1+\frac{m+1}{N}}}\times \\ \omega\Big(\gamma_{10}\frac{U_j^{-q}}{t2^{-(j+1)(m+1)}\sigma^{m+1}}\delta^{-1}\|u\|_{q,Q_0}^{q}\Big)^{m+1}\ge 1\,. \tag{3.10}$$

Finally we make use of part b) of (3.8), obtaining

$$\tilde\gamma\bar\gamma^j U_0^{-(\alpha+m-1+\frac{m+1}{N}q)}\frac{\|u\|_{q,Q_0}^{q\frac{m+1}{N}}}{(t\sigma^{m+1})^{1+\frac{m+1}{N}}}\omega\Big(\tilde\gamma_1\bar\gamma_1^j\frac{U_0^{-q}}{t\sigma^{m+1}}\|u\|_{q,Q_0}^{q}\Big)^{m+1}\ge 1\,, \tag{3.11}$$

where now $\tilde\gamma$, $\tilde\gamma_1$ depend on δ too, and

$$\bar\gamma = \delta^{\alpha+m-1+\frac{m+1}{N}q}2^{(m+1)(1+\frac{m+1}{N})}\,,\qquad \bar\gamma_1=\delta^q 2^{m+1}\,.$$

If we choose δ so that $\bar\gamma\le 1$, $\bar\gamma_1\le 1$, the left-hand side of (3.11) does not depend essentially on j. In other words, the constants $\tilde\gamma$, $\bar\gamma^j$, $\tilde\gamma_1$ and $\bar\gamma_1^j$ can be estimated a priori in terms of the parameters m, q, N, α and μ only. Estimate (3.2) follows now from (3.11), upon a simple step of functional inversion. □

Remark 3.1. In the example of paraboloid-like domains pointed out in the Introduction, we have $g(\tau)=\gamma\min(\tau^{\frac{N-1}{N}},\tau^{\eta})$; then we see that (3.2) takes the form

$$\|u\|_{\infty,Q}\le\gamma\max(W_{1/N},W_{1-\eta})\,, \tag{3.12}$$

where for $s>0$ we set

$$W_s = t^{-\frac{1}{\alpha+m-1+qs(m+1)}}\Big(t^{-1}\iint\limits_{Q_0} u^q\,\mathrm{d}x\,\mathrm{d}\tau\Big)^{\frac{s(m+1)}{\alpha+m-1+qs(m+1)}}\,,$$

under the assumptions stated in Lemma 3.1.

4. Proof of Theorem 1.1

We have gathered some essential technical facts in the following

Lemma 4.1. *Let ψ_q be as in (1.12), $q > 0$; let U, s, θ, t denote positive numbers; $\mathcal{K}_q$ is the constant defined in Lemma 2.3. Then*

$$\begin{aligned} &\mathcal{D}(U) := U^s/\psi_q(U^{\alpha+m-1}) \text{ is non decreasing in } U > 0 \text{ for } s \geq s_0, \\ &s_0 := Nq(\alpha+m-1)/\mathcal{K}_q . \quad \text{Also, } \mathcal{D}(0+) = 0, \text{ if } s > s_0. \end{aligned} \tag{4.1}$$

$$J := \int_0^t \psi_q(\tau U^{\alpha+m-1})^{-\theta}\,\mathrm{d}\tau \leq \gamma t \psi_q(tU^{\alpha+m-1})^{-\theta} < \infty\,, \text{ if } Nq\theta < \mathcal{K}_q. \tag{4.2}$$

$$\begin{aligned} &\text{Assume } I(U) := \int_a^b U^\theta \psi_q(\tau U^{\alpha+m-1})^{-\theta/q}\,\mathrm{d}\tau < \infty, \text{ for all } U > 0, \\ &\text{for some fixed } 0 \leq a < b \leq \infty. \text{ Then } I(U) \to 0 \text{ as } U \to 0. \end{aligned} \tag{4.3}$$

Proof. (4.1): Define $z = \psi_q(U^{\alpha+m-1})$; then we have to show that $\Psi_q(z)^{s/(\alpha+m-1)}/z$ is non decreasing in z. But this is an obvious consequence of the choice $s \geq s_0$, and of (1.8), once we write explicitly

$$\mathcal{D}(U) = z^{\frac{s}{q} + \frac{s(m+1)}{N(\alpha+m-1)} - 1} \omega(z)^{\frac{s(m+1)}{\alpha+m-1}} . \tag{4.4}$$

By the same token, $\mathcal{D}(0+) = 0$ if $s > s_0$, keeping in mind that $z \to 0$ if $U \to 0$.
(4.2): We may use (4.1) to calculate

$$J = \int_0^t \left[\frac{(\tau^{\frac{1}{\alpha+m-1}} U)^{s_0}}{\psi_q(\tau U^{\alpha+m-1})} \right]^\theta \frac{\mathrm{d}\tau}{(\tau^{\frac{1}{\alpha+m-1}} U)^{s_0\theta}} \leq \left[\frac{(t^{\frac{1}{\alpha+m-1}} U)^{s_0}}{\psi_q(tU^{\alpha+m-1})} \right]^\theta \frac{t^{1-\frac{\theta s_0}{\alpha+m-1}}}{U^{s_0\theta}\big(1 - \frac{\theta s_0}{\alpha+m-1}\big)} .$$

(4.3): This follows obviously from (4.1), and from $I(U) < \infty$, if we take into account that $q > s_0$ and that

$$U^\theta \psi_q(\tau U^{\alpha+m-1})^{-\theta/q} = [U^q \psi_q(\tau U^{\alpha+m-1})^{-1}]^{\theta/q} .$$

□

Let us introduce the sequence of approximating solutions $u_n \geq 0$, $n \geq 1$, where u_n solves

$$u_t - c\,\mathrm{div}(u^\alpha |Du|^{m-1} Du) = \min(u^\mu, n)\,, \qquad \text{in } \Omega_n \times (0,\infty)\,,$$
$$u^\alpha |Du|^{m-1} Du \cdot \mathbf{n} = 0\,, \quad \text{on } \partial\Omega \cap \partial\Omega_n\,, \qquad u = 0\,, \quad \text{on } \Omega \cap \partial\Omega_n\,,$$
$$u(x,0) = u_{0n}(x)\,, \qquad \text{in } \Omega_n\,,$$

and $u_{0n} \in C^\infty(\overline{\Omega_n})$, with $u_{0n} \to u_0$ in $L^1(\Omega) \cap L^q(\Omega)$; note that we always understand u_n to be defined over Ω, by setting $u_n \equiv 0$ out of Ω_n.

It follows from the results of [18] that the problem above is globally solvable. In the following, for the sake of simplicity, we denote $u_n = u$. Note that Lemma 3.1 can be applied to u; more precisely, no localization in space is necessary in the present case, due to the global integrability information stated in (1.14). Therefore, assumption (3.1) can be replaced with

$$t \| u(\cdot,t) \|_{\infty,\Omega}^{\mu-1} \leq 1\,, \tag{4.5}$$

and the sup and integral norms in (3.2) are global norms in Ω. Let T be the supremum of all times $\bar{t}$ such that (4.5) holds for all $t \leq \bar{t}$. Our goal is to show that $T = \infty$. Then we may apply Lemma 3.1 and the global estimates of the integral norms of u proven below to infer, with the help of the results of [10], [15], that $\{u_n\}$ is equicontinuous in every compact subset of $\overline{\Omega} \times (0,T)$. Moreover integral gradient estimates for u follows as in [2], [3]. With the help of these estimates, and exploiting the monotonicity of the operator, we can pass to the limit in the weak formulation of the problem written for u_n (perhaps extracting a suitable subsequence), thereby obtaining a global solution u.

Multiplying the differential equation satisfied by u against u^{s-1}, $s \geq 1$ (here $u^0 \equiv 1$ by convention), and integrating over $\Omega_n \times (0,t)$, we find for $0 < t < T$

$$\int_{\Omega(t)} u^s \,\mathrm{d}x \leq \int_{\Omega} u_{0n}^s \,\mathrm{d}x + \gamma \int_0^t \int_{\Omega} u^{\mu+s-1} \,\mathrm{d}x \,\mathrm{d}\tau \leq \int_{\Omega} u_{0n}^s \,\mathrm{d}x \,\mathrm{d} + \gamma \Phi_s(t) \Xi(t) \,, \tag{4.6}$$

$$\Xi(t) := \int_0^t \frac{\Phi_q(\tau)^{\frac{\mu-1}{q}}}{\psi_q(\tau \Phi_q(\tau)^{\frac{\alpha+m-1}{q}})^{\frac{\mu-1}{q}}} \,\mathrm{d}\tau \,,$$

where we have used the sup bound for u of Lemma 3.1, and we defined

$$\Phi_s(t) = \sup_{0<\tau<t} \int_{\Omega(\tau)} u^s(x,\tau) \,\mathrm{d}x \,, \qquad s \geq 1 \,.$$

Also note that $\Xi(t) < \infty$ as a consequence of (4.2), and of the choice of q. In fact the assumption in (4.2), i.e., $Nq\theta < \mathcal{K}_q$, with $\theta = (\mu-1)/q$, is equivalent to the restriction placed on q in the statement of Theorem 1.1 (recall the definition of $\mathcal{K}_q$ in Lemma 2.3). Take $s = q$ in (4.6); of course we may assume $\|u_{0n}\|_{q,\Omega} \leq \gamma \|u_0\|_{q,\Omega}$. Then, defining

$$T_0 = \sup\{t > 0 \mid \Xi(t) \leq \varepsilon\} \,,$$

for a sufficiently small $\varepsilon > 0$, we have from (4.6)

$$\Phi_q(t) \leq \gamma_1 \|u_0\|_{q,\Omega} \,, \qquad 0 < t < \min(T, T_0) \,. \tag{4.7}$$

Thus $T_0 \geq \min(T, T_0')$, where T_0' is defined by

$$\gamma_2(\gamma_1) \int_0^{T_0'} \frac{\|u_0\|_{q,\Omega}^{\mu-1}}{\psi_q(\tau \|u_0\|_{q,\Omega}^{\alpha+m-1})^{\frac{\mu-1}{q}}} \,\mathrm{d}\tau = \varepsilon$$

(we are using (4.1) with $s = q$). Moreover, if $t < \min(T, T_0')$, we also have (recalling that ψ_q is non decreasing)

$$t \|u(\cdot,t)\|_{\infty,\Omega}^{\mu-1} \leq \gamma \frac{t \|u_0\|_{q,\Omega}^{\mu-1}}{\psi_q(t \|u_0\|_{q,\Omega}^{\alpha+m-1})^{\frac{\mu-1}{q}}} \leq \gamma \int_0^t \frac{\|u_0\|_{q,\Omega}^{\mu-1}}{\psi_q(\tau \|u_0\|_{q,\Omega}^{\alpha+m-1})^{\frac{\mu-1}{q}}} \,\mathrm{d}\tau \leq \frac{1}{2} \,, \tag{4.8}$$

provided ε is redefined if necessary. Thus $T_0' \le T$; in turn, we may assume $T_0' > 1$, invoking (4.2), (4.3) and perhaps changing δ in the statement. Incidentally, note that T_0' does not depend on n: in its definition u_0 is the original datum.

We can now proceed to estimate the $L^1(\Omega)$ norm of u, uniformly in t; for small t we use (4.6) again, but choosing now $s = 1$ there. From the arguments above we infer

$$\int\limits_{\Omega(t)} u \,\mathrm{d}x \le \gamma_5 \int\limits_{\Omega} u_0 \,\mathrm{d}x\,, \qquad \text{for } 0 < t < T_0'. \tag{4.9}$$

Moreover, we have as in (4.6)

$$\int\limits_{\Omega(t)} u \,\mathrm{d}x \le \int\limits_{\Omega(1)} u \,\mathrm{d}x + \gamma_6 \int\limits_1^t \frac{\Phi_1(\tau)^\mu}{\psi(\tau\Phi_1(\tau)^{\alpha+m-1})^{\mu-1}} \,\mathrm{d}\tau\,, \qquad 1 < t < T\,. \tag{4.10}$$

From (4.9), (4.10) we get

$$\Phi_1(t) \le \gamma_5 \|u_0\|_{1,\Omega} + \gamma_6 \int\limits_1^t \frac{\Phi_1(\tau)^\mu}{\psi(\tau\|u_0\|_{1,\Omega}^{\alpha+m-1})^{\mu-1}} \,\mathrm{d}\tau\,, \qquad 1 < t < T\,.$$

Then $\Phi_1(t)$ is majorised for $T > t > 1$ by the increasing function $y(t)$, where y solves

$$\begin{cases} y' = \gamma_6 y^\mu \psi(t\|u_0\|_{1,\Omega}^{\alpha+m-1})^{-\mu+1}\,, & t > 1\,, \\ y(1) = \gamma_5 \|u_0\|_{1,\Omega}\,. \end{cases}$$

It follows from the explicit representation

$$y(t) = y(1)\Big[1 - \gamma_7 y(1)^{\mu-1} \int\limits_1^t \psi(\tau\|u_0\|_{1,\Omega}^{\alpha+m-1})^{-\mu+1} \,\mathrm{d}\tau\Big]^{-1/(\mu-1)}\,, \qquad t \ge 1\,. \tag{4.11}$$

that y is defined and bounded over $(1, \infty)$, provided δ in (1.14) is chosen small enough, which we are going to assume. Indeed, (4.3) and (1.13) imply that the quantity in brackets in (4.11) is bounded away from zero, if δ is small enough. It is therefore clear that we have $\Phi_1(t) \le \gamma_8 \|u_0\|_{1,\Omega}$ for all $1 < t < T$; then for all such t, invoking

(4.3),

$$t\|u(\cdot,t)\|_{\infty,\Omega}^{\mu-1} \le \gamma_9 t \frac{\|u_0\|_{1,\Omega}^{\mu-1}}{\psi(t\|u_0\|_{1,\Omega}^{\alpha+m-1})^{\mu-1}} = \gamma_9(t-1+1)\frac{\|u_0\|_{1,\Omega}^{\mu-1}}{\psi(t\|u_0\|_{1,\Omega}^{\alpha+m-1})^{\mu-1}}$$

$$\le \gamma_9 \int_1^\infty \frac{\|u_0\|_{1,\Omega}^{\mu-1}}{\psi(\tau\|u_0\|_{1,\Omega}^{\alpha+m-1})^{\mu-1}}\,\mathrm{d}\tau + \gamma_9 \frac{\|u_0\|_{1,\Omega}^{\mu-1}}{\psi(\|u_0\|_{1,\Omega}^{\alpha+m-1})^{\mu-1}} \le \frac{1}{2}\,, \quad (4.12)$$

by possibly redefining δ. The definition of T and (4.12) would yield a contradiction if $T < \infty$. Therefore $T = \infty$ and the proof is concluded.

5. Proof of Theorem 1.2

We use here the formulation (1.1)–(1.3). We define $V_*(\rho) = \max(V(\rho), 1)$. We may assume without loss of generality that $V(1) = 1$, and that (1.17) holds true for $\rho \ge 1$. Recalling (1.16), we also set

$$f(w) = h(w)^{-\varepsilon} := \left[\int_0^w V_*(\tau^{-\frac{\nu-m}{m+1}})^{1/\beta}\,\mathrm{d}\tau\right]^{-\varepsilon}, \quad w > 0\,; \qquad f(w) = 0\,, \quad w = 0\,,$$

with $0 < \varepsilon < 1$ to be chosen. Let $\zeta \in C_0^\infty(\boldsymbol{R}^N)$, $\zeta \equiv 0$ for $|x| \ge \rho$, $\zeta \equiv 1$ for $|x| \le \rho/2$, $0 \le \zeta \le 1$, $|D\zeta| \le \gamma/\rho$. We may take $f(w)\zeta^s$, $s > m+1$, as a testing function in (1.1) (in fact an easy approximation argument should be employed here). Setting

$$\varphi(w) = \int_0^w \tau^{\beta-1} f(\tau)\,\mathrm{d}\tau\,,$$

we find

$$\beta\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \varphi(w)\zeta^s\,\mathrm{d}x = -\int_{\Omega(t)} |Dw|^{m+1} f'(w)\zeta^s\,\mathrm{d}x - s\int_{\Omega(t)} |Dw|^{m-1}Dw\cdot D\zeta\zeta^{s-1}f(w)\,\mathrm{d}x$$
$$+\int_{\Omega(t)} w^\nu f(w)\zeta^s\,\mathrm{d}x =: J_1 + J_2 + J_3\,.$$

We bound $|J_2|$ by means of Young's inequality, and of

Fact 5.1. $\frac{w}{\varepsilon} \le \frac{f(w)}{|f'(w)|} \le \gamma w$, for all $w > 0$, (the proofs of Fact 5.1, and of other technical Facts, are gathered at the end of this section). We find

$$\beta\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega(t)}\varphi(w)\zeta^s\,\mathrm{d}x \geq -\frac{\gamma_1}{\rho^{m+1}}\int\limits_{\Omega(t)} f(w)w^m\zeta^{s-m-1}\,\mathrm{d}x + \int\limits_{\Omega(t)} w^\nu f(w)\zeta^s\,\mathrm{d}x\,. \tag{5.1}$$

Note that, if $0 < s_1 m < s - m - 1$, we can use in (5.1)

$$w^m f(w)\zeta^{s-m-1} \leq [w\zeta^{s_1}]^m f(w\zeta^{s_1})\,,$$

because f is decreasing and $0 \leq \zeta \leq 1$. Next we may apply the inequality in p. 331 of [13] to the function $z\Phi(z)$, where

Fact 5.2. Φ is defined by $\Phi(z^{\nu-m}) = z^m f(z)$, so that if ε is small

$$a\Phi(b) \leq \sigma a\Phi(a) + \gamma(\sigma)b\Phi(b)\,, \quad \text{for all } \sigma\in(0,1),\ a,\ b>0.$$

Then we find for all $1 > \sigma > 0$

$$\begin{aligned}\rho^{-m-1}\Phi((w\zeta^{s_1})^{\nu-m}) &\leq \sigma(w\zeta^{s_1})^{\nu-m}\Phi((w\zeta^{s_1})^{\nu-m}) + \gamma(\sigma)\rho^{-m-1}\Phi(\rho^{-m-1})\\ &\leq \gamma\sigma w^\nu\zeta^{s_1\nu-s_1\varepsilon}f(w) + \gamma(\sigma)\rho^{-\nu\frac{m+1}{\nu-m}}f(\rho^{-\frac{m+1}{\nu-m}})\,,\end{aligned}$$

where we have also used

Fact 5.3. $f(w\zeta^{s_1}) \leq \gamma_2\zeta^{-\varepsilon s_1}f(w)$.

By choosing $s_1(\nu-\varepsilon) = s$, which is consistent with the condition $s_1 m < s - m - 1$ if s is taken large enough and ε is small enough so as to have $\nu - \varepsilon > m$, we get from (5.1), also selecting a suitable σ,

$$\beta\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega(t)}\varphi(w)\zeta^s\,\mathrm{d}x \geq -\gamma_3\frac{V(\rho)}{\rho^{\nu\frac{m+1}{\nu-m}}}f\big(\rho^{-\frac{m+1}{\nu-m}}\big) + \frac{1}{2}\int\limits_{\Omega(t)} w^\nu f(w)\zeta^s\,\mathrm{d}x\,. \tag{5.2}$$

Next, we note that

Fact 5.4. $(\beta-\varepsilon)^{-1}z^\beta f(z) \geq \varphi(z) \geq \beta^{-1}z^\beta f(z)$, for all $z > 0$ provided $\varepsilon < \beta$.

Then for $\varepsilon < \beta/2$, we have $w^\nu f(w) \geq (\beta/2)\Theta(\varphi(w))$, where

Fact 5.5. the function Θ defined by $\Theta(\varphi(z)) = z^{\nu-\beta}\varphi(z)$, $z > 0$, is convex, if $2\varepsilon < \nu$.

Then, by Jensen inequality,

$$\Theta(\lambda(t)) \le \int\limits_{\Omega(t)} \zeta^s \Theta(\varphi(w))\,\mathrm{d}x \Big(\int\limits_{\Omega} \zeta^s\,\mathrm{d}x \Big)^{-1}, \quad \lambda(t) :== \int\limits_{\Omega(t)} \zeta^s \varphi(w)\,\mathrm{d}x \Big(\int\limits_{\Omega} \zeta^s\,\mathrm{d}x \Big)^{-1}.$$

Using the inequalities above in (5.2), we infer that

$$\lambda'(t) \ge \frac{1}{4}\Theta(\lambda(t)) - P(\rho) \Big(\int\limits_{\Omega} \zeta^s\,\mathrm{d}x \Big)^{-1}, \tag{5.3}$$

where $-P(\rho)$ is the first term on the right hand side of (5.2). It follows from (5.3), and from

Fact 5.6. $\int^{+\infty} \Theta(z)^{-1}\,\mathrm{d}z < \infty$,

that we have blow up of $\lambda(t)$ in a finite time, unless

$$\Theta(\lambda(t)) \le 4P(\rho) \Big(\int\limits_{\Omega} \zeta^s\,\mathrm{d}x \Big)^{-1}, \tag{5.4}$$

for all $t > 0$, $\rho > 0$. Then, we take into account that the inverse function H of Θ satisfies

Fact 5.7. $H(\gamma z) \le \gamma H(z)$, for all $z > 0$, $\gamma > 1$.

Set $Z_0 = \int_\Omega \zeta^s\,\mathrm{d}x$, and note that $V(\rho)/Z_0 > 1$. We conclude from (5.4) and Fact 5.4, that for $\rho > 1$,

$$\begin{aligned}\int\limits_{\Omega(t)} \zeta^s \varphi(w)\,\mathrm{d}x &\le Z_0 H(4P(\rho)Z_0^{-1}) \le 4\gamma_3 V(\rho) H\Bigg(f\big(\rho^{-\frac{m+1}{\nu-m}}\big)\rho^{-\nu\frac{m+1}{\nu-m}} \Bigg) \\ &\le \gamma_4 V(\rho) H\Bigg(\varphi\big(\rho^{-\frac{m+1}{\nu-m}}\big)\rho^{-(\nu-\beta)\frac{m+1}{\nu-m}} \Bigg) = \gamma_4 V(\rho)\varphi(\rho^{-\frac{m+1}{\nu-m}}) \\ &\le \gamma_5 V(\rho)\rho^{-\beta\frac{m+1}{\nu-m}} \big[\rho^{-\frac{m+1}{\nu-m}} V(\rho)^{1/\beta}\big]^{-\varepsilon} = \gamma_5 \big[\rho^{-\frac{m+1}{\nu-m}} V(\rho)^{1/\beta}\big]^{\beta-\varepsilon} =: \xi(\rho),\end{aligned}$$

where we have used also the definitions of f, V_*. We conclude the proof by noting that $\xi(\rho) \to 0$ as $\rho \to \infty$, if $\beta > \varepsilon$, owing to (1.16) and to

$$\rho^{-\frac{m+1}{\nu-m}} V(\rho)^{1/\beta} = tV(t^{-\frac{\nu-m}{m+1}})^{1/\beta} \leq \int_0^t V(\tau^{-\frac{\nu-m}{m+1}})^{1/\beta} \, \mathrm{d}\tau \,, \tag{5.5}$$

where $t = \rho^{-\frac{m+1}{\nu-m}} \to 0$ as $\rho \to \infty$.

Proof of Fact 5.1 We have immediately

$$\frac{f(w)}{|f'(w)|} = \frac{1}{\varepsilon} \frac{\int_0^w V_*(\tau^{-\delta})^{1/\beta} \, \mathrm{d}\tau}{V_*(w^{-\delta})^{1/\beta}} \geq \frac{w}{\varepsilon} \,, \tag{5.6}$$

because V_* is obviously increasing; here $\delta = (\nu - m)/(m+1)$. If $w \leq 1$, then $w^{-\delta} \geq 1$ and we get from assumption (1.17)

$$\begin{aligned} \int_0^w V_*(\tau^{-\delta})^{1/\beta} \, \mathrm{d}\tau &= \int_0^w V(\tau^{-\delta})^{1/\beta} \, \mathrm{d}\tau = w \int_0^1 \left(\frac{V(s^{-\delta} w^{-\delta})}{(s^{-\delta} w^{-\delta})^{\lambda}} \right)^{1/\beta} (s^{-\delta} w^{-\delta})^{\lambda/\beta} \, \mathrm{d}s \\ &\leq C_2^{1/\beta} w \int_0^1 \chi(s^{-\delta} w^{-\delta})^{1/\beta} (s^{-\delta} w^{-\delta})^{\lambda/\beta} \, \mathrm{d}s \\ &\leq \frac{C_2^{1/\beta}}{C_1^{1/\beta}} w \left(\frac{V(w^{-\delta})}{w^{-\delta\lambda}} \right)^{1/\beta} \frac{w^{-\delta\lambda/\beta}}{1 - \delta\lambda/\beta} \leq \gamma w V(w^{-\delta})^{1/\beta} \,. \end{aligned}$$

If $w > 1$, we have however

$$\int_0^w V_*(\tau^{-\delta})^{1/\beta} \, \mathrm{d}\tau = \int_0^1 V(\tau^{-\delta})^{1/\beta} \, \mathrm{d}\tau + w - 1 \leq \tilde{\gamma} + w \leq (\tilde{\gamma} + 1) w V_*(w^{-\delta})^{1/\beta} \,.$$

We conclude by substituting the estimates above in (5.6).

Proof of Fact 5.2 As remarked in [13], the sought after inequality is an elementary consequence of $z\Phi'(z) \geq \delta\Phi(z)$, for a suitable $\delta > 0$. In turn, the last bound is proven immediately, for all $z > 0$, by differentiating Φ, and using the already proven fact $wf'(w) \geq -\varepsilon f(w)$, with small ε.

Proof of Fact 5.3 We have, with $\delta = (\nu - m)/(m+1)$,

$$h(w\zeta^{s_1}) \geq w\zeta^{s_1} V_*((w\zeta^{s_1})^{-\delta})^{1/\beta} \geq w\zeta^{s_1} V_*(w^{-\delta})^{1/\beta} \geq \gamma_0 \zeta^{s_1} h(w) \,,$$

whence the claim follows.

Proof of Fact 5.4 From Fact 5.1 and integrating by parts one gets

$$\varphi(z) \geq -\frac{1}{\varepsilon}\int\limits_0^z \tau^\beta f'(\tau)\,\mathrm{d}\tau = -\frac{z^\beta}{\varepsilon} f(z) + \frac{\beta}{\varepsilon}\varphi(z)\,,$$

proving the upper bound for φ. The lower bound follows trivially from the definition, recalling that f is decreasing.

Proof of Fact 5.5 Obviously we have $\Theta(z) = z\eta(z)^{\nu-\beta}$, where η is the inverse function of φ. After lengthy but trivial calculations we find

$$\Theta''(z) = (\nu-\beta)\frac{\eta(z)^{\nu-2\beta}}{f(\eta(z))^2}\big\{2f(\eta(z)) + (\nu-2\beta)z\eta(z)^{-\beta} - z\eta(z)^{1-\beta}f'(\eta(z))\big\}\,.$$

Clearly, $\Theta'' \geq 0$ if $\nu \geq 2\beta$, because $f' < 0$. If $\nu < 2\beta$, we recall from Fact 5.4 that $(\beta-\varepsilon)z\eta(z)^{-\beta} \leq f(\eta(z))$; therefore certainly $\Theta'' \geq 0$ provided $2 \geq (2\beta-\nu)/(\beta-\varepsilon)$, which amounts to $2\varepsilon \leq \nu$.

Proof of Fact 5.6 If we define η as in the proof of Fact 5.5, we have

$$\int\limits_1^\infty \frac{\mathrm{d}z}{\Theta(z)} = \int\limits_1^\infty \frac{\mathrm{d}z}{z\eta(z)^{\nu-\beta}} = \int\limits_{\eta(1)}^\infty \frac{r^{\beta-1}f(r)}{\varphi(r)r^{\nu-\beta}}\,\mathrm{d}r \leq \beta\int\limits_{\eta(1)}^\infty \frac{\mathrm{d}r}{r^{\nu-\beta+1}} < \infty\,,$$

because $\nu-\beta+1 \geq \nu > 1$.

Proof of Fact 5.7 We may calculate directly

$$t\frac{H'(t)}{H(t)} = \frac{t}{t + (\nu-\beta)tH(t)\eta(H(t))^{-\beta}f(\eta(H(t)))^{-1}} \leq 1\,,$$

whence the claim follows upon integrating $H'(t)/H(t) \leq 1/t$ over $(z, \gamma z)$.

6. Proof of Theorem 1.3

We define a sequence of cutoff functions ζ_n, $n \geq 0$, so that

$$\zeta_n(x) = 1\,,\quad x \in \Omega_{\rho_n}\setminus\Omega_{\bar\rho_n}\,,\quad \zeta_n(x) = 0\,,\quad x \notin \Omega_{\rho_{n-1}}\setminus\Omega_{\bar\rho_{n-1}}\,,\quad |D\zeta_n| \leq \gamma 2^n/(\sigma\rho)\,,$$

where $0 < \sigma < 1/2$ is given and, for ρ_0 as in the statement,

$$\rho_n = \rho + \sigma 2^{-n}\rho\,,\quad \bar\rho_n = (\rho - \sigma 2^{-n}\rho)/2\,,\qquad n \geq 0\,,\ \rho > 4\rho_0\,.$$

Note that $\rho_n \geq \rho_{n+1} \geq \rho$, $\bar{\rho}_n \leq \bar{\rho}_{n+1} \leq \rho/2$, $n \geq 0$. Also, the support of each ζ_n is bounded away from the support of u_0. If we use the testing function $\zeta_n^{m+1} u^\theta$ in the weak formulation of problem (1.4)–(1.6), where $0 < \theta < (m+1)/N$, we find by means of straightforward calculations

$$\sup_{0<\tau<t} \int\limits_{\Omega(\tau)} u^{1+\theta} \zeta_n^{m+1} \,\mathrm{d}x + \int\limits_0^t \int\limits_\Omega u^{\alpha+\theta-1} |Du|^{m+1} \zeta_n^{m+1} \,\mathrm{d}x\,\mathrm{d}\tau$$

$$\leq \gamma \frac{2^{n(m+1)}}{(\rho\sigma)^{m+1}} \int\limits_0^t \int\limits_{\Omega_{\rho_{n-1}} \setminus \Omega_{\bar{\rho}_{n-1}}} u^{m+\alpha+\theta} \,\mathrm{d}x\,\mathrm{d}\tau + \gamma \int\limits_0^t \int\limits_\Omega u^{\mu+\theta} \zeta_n^{m+1} \,\mathrm{d}x\,\mathrm{d}\tau =: A_1 + A_2 \,. \quad (6.1)$$

As a first remark, we note that

$$A_2 \leq \gamma \int\limits_0^t \|u(\cdot,\tau)\|_{\infty,\Omega}^{\mu-1} \,\mathrm{d}\tau \sup_{0<\tau<t} \int\limits_{\Omega(\tau)} \zeta_n^{m+1} u^{1+\theta} \,\mathrm{d}x \,.$$

Thus, reasoning as in the proof of Theorem 1.1 (see (4.8) and (4.12)), we may guarantee that

$$A_2 \leq \frac{1}{2} \sup_{0<\tau<t} \int\limits_{\Omega(\tau)} \zeta_n^{m+1} u^{1+\theta} \,\mathrm{d}x \,, \quad (6.2)$$

for all $t > 0$, provided δ in (1.14) is small enough. Therefore, in (6.1) we may absorb the term A_2 into the left hand side. Then, on setting $v_n = u^{\frac{\alpha+m+\theta}{m+1}} \zeta_n^s$, for a sufficiently large $s > 1$, we have for $n \geq 1$

$$Y_n := \sup_{0<\tau<t} \int\limits_{\Omega(\tau)} v_n^\varepsilon \,\mathrm{d}x + \int\limits_0^t \int\limits_\Omega |Dv_n|^{m+1} \,\mathrm{d}x\,\mathrm{d}\tau \leq \frac{\gamma_1 2^{n(m+1)}}{(\rho\sigma)^{m+1}} \int\limits_0^t \int\limits_\Omega v_{n-1}^{m+1} \,\mathrm{d}x\,\mathrm{d}\tau \,, \quad (6.3)$$

where $\varepsilon = (m+1)(1+\theta)/(\alpha+m+\theta)$. We introduce the increasing functions

$$F_1(s) = s^{m+\frac{m+1}{\varepsilon}} g\big(s^{-1}\big)^{m+1} \,, \quad F_2(s) = \big[F_1^{(-1)}(s)\big]^{\frac{m+1-\varepsilon}{\varepsilon}} \,.$$

The monotonic characyer of F_1 follows from (1.8), and from $\varepsilon < m+1$. Next we apply the embedding (2.2) with $h = \varepsilon$, $s = m+1$, $p = m+1$, $v = v_{n-1}(\cdot,\tau)$. After integrating in time the resulting inequality, and defining

$$\mathcal{E}(t) = \int_0^t \Big(\int_{\Omega(\tau)} v_{n-1}^{\varepsilon} \,\mathrm{d}x \Big)^{\frac{m+1}{\varepsilon}} \mathrm{d}\tau \,, \qquad \mathcal{I}(t) = \int_0^t\int_{\Omega} |Dv_{n-1}|^{m+1} \,\mathrm{d}x \,\mathrm{d}\tau \,,$$

we find, making use of Jensen's inequality,

$$\int_0^t\int_{\Omega} v_{n-1}^{m+1} \,\mathrm{d}x \,\mathrm{d}\tau \le \gamma_2 \int_0^t \Big(\int_{\Omega(\tau)} v_{n-1}^{\varepsilon} \,\mathrm{d}x \Big)^{\frac{m+1}{\varepsilon}} F_2\left(\frac{\int_{\Omega(\tau)} |Dv_{n-1}|^{m+1} \,\mathrm{d}x}{\big(\int_{\Omega(\tau)} v_{n-1}^{\varepsilon} \,\mathrm{d}x \big)^{\frac{m+1}{\varepsilon}}} \right) \mathrm{d}\tau$$

$$\le \gamma_3 \mathcal{E}(t) F_2\left(\frac{\mathcal{I}(t)}{\mathcal{E}(t)} \right) . \tag{6.4}$$

Note that we assume provisionally here that

$$F_2^{(-1)} \text{ is convex,}$$

We prove at the end of this section that this extra assumption can be removed. We have also used

$$F_2(\gamma s) \le \gamma F_2(s) \,, \text{ for all } s > 0, \, \gamma > 1.$$

which is a consequence of the definition of F_2, and of the second estimate in (2.8). Again by the definitions of F_1, F_2 and from (1.8), we infer that for any fixed $A > 0$

$$s \mapsto s F_2\big(\frac{A}{s}\big) \quad \text{is non decreasing for } s > 0. \tag{6.5}$$

Next we apply (6.4) in (6.3), using (6.5) and the obvious inequalities $\mathcal{E}(t) \le t Y_{n-1}^{(m+1)/\varepsilon}$, $\mathcal{I}(t) \le Y_{n-1}$ to bound the last term in (6.4). We find

$$Y_n \le \gamma_1 \gamma_3 \frac{2^{n(m+1)}}{(\rho\sigma)^{m+1}} t Y_{n-1}^{\frac{m+1}{\varepsilon}} F_2\left(\frac{1}{t Y_{n-1}^{\frac{m+1-\varepsilon}{\varepsilon}}} \right)$$

$$\le \gamma_4 2^{n(m+1)} \rho^{-m-1} t^{\frac{(1+\theta)(m+1)}{\mathcal{K}_{1+\theta}}} Y_{n-1}^{1+\frac{(\alpha+m-1)(m+1)}{\mathcal{K}_{1+\theta}}} f_0(t Y_{n-1}^{\frac{\alpha+m-1}{1+\theta}}) \,, \tag{6.6}$$

where (recalling that $\mathcal{K}_q$ has been introduced in Lemma 2.3), by definition,

$$f_0(s) = \left[F_1^{(-1)}\big(\frac{1}{s}\big) s^{\frac{N(1+\theta)}{\mathcal{K}_{1+\theta}}} \right]^{\frac{\alpha+m-1}{1+\theta}} , \qquad s > 0 \,.$$

It follows from (1.8) and the definition of F_1 that f_0 is non decreasing. Moreover, reasoning as in (6.1), (6.2), we have for all $n \geq 0$,

$$Y_n \leq \gamma_5 2^{n(m+1)} I_0 \,, \qquad I_0 = \frac{1}{\rho^{m+1}} \int_0^t \int_{\Omega_{2\rho}} u^{m+\alpha+\theta} \,\mathrm{d}x \,\mathrm{d}\tau \,. \tag{6.7}$$

Combining (6.6), (6.7) with the elementary inequality

$$f_0(\gamma s) \leq [F_1^{-1}(s^{-1})]^{\frac{\alpha+m-1}{1+\theta}} (\gamma s)^{\frac{N(\alpha+m-1)}{\mathcal{K}_{1+\theta}}} \leq \gamma f_0(s) \,, \qquad \gamma > 1 \,, s > 0$$

(indeed $N(\alpha + m - 1) < \mathcal{K}_{1+\theta}$), we infer

$$Y_n \leq \gamma_6 b^n \rho^{-m-1} t^{\frac{(1+\theta)(m+1)}{\mathcal{K}_{1+\theta}}} Y_{n-1}^{1+\frac{(\alpha+m-1)(m+1)}{\mathcal{K}_{1+\theta}}} f_0(tI_0^{\frac{\alpha+m-1}{1+\theta}}) \,, \quad b > 1 \,.$$

Hence, invoking Lemma 5.6 Chapter II [11], we have that $Y_n \to 0$ $n \to \infty$, i.e., $u(x, t) = 0$, $x \in \Omega_\rho \setminus \Omega_{\rho/2}$, provided

$$Y_0 \leq \gamma_0 \Big[\rho^{-m-1} t^{\frac{(1+\theta)(m+1)}{\mathcal{K}_{1+\theta}}} f_0(tI_0^{\frac{\alpha+m-1}{1+\theta}}) \Big]^{-\frac{\mathcal{K}_{1+\theta}}{(\alpha+m-1)(m+1)}} \,. \tag{6.8}$$

One can employ the inequality $Y_0 \leq \gamma I_0$, and the definitions of F_1, f_0, to show that (6.8) is implied by

$$G_0\Big(1/F_1^{(-1)} \big((tI_0^{\frac{\alpha+m-1}{1+\theta}})^{-1} \big) \Big) \leq \gamma_7 \rho \,, \quad \text{where } G_0(s) = s/g(s) \,, \quad s > 0 \,. \tag{6.9}$$

Next we estimate, using (4.2),

$$I_0 \leq \gamma_8 \frac{\|u_0\|_{1,\Omega}}{\rho^{m+1}} \int_0^t \frac{\|u_0\|_{1,\Omega}^{m+\alpha+\theta-1} \,\mathrm{d}\tau}{\psi(\tau \|u_0\|_{1,\Omega}^{\alpha+m-1})^{m+\alpha+\theta-1}} \leq \gamma_9 \frac{t \|u_0\|_{1,\Omega}^{m+\alpha+\theta}}{\rho^{m+1} \psi(t \|u_0\|_{1,\Omega}^{\alpha+m-1})^{m+\alpha+\theta-1}} \tag{6.10}$$

(here we exploit the restriction $\theta < (m+1)/N$). On substituting I_0 in (6.9) with the bound given in (6.10), one can see, by means of routine calculations, that (6.9) is in fact fulfilled for

$$\rho \geq \gamma_{10} G_0(\psi(t \|u_0\|_{1,\Omega}^{\alpha+m-1})) \,, \tag{6.11}$$

for a large enough constant γ_{10}. We remark that G_0 does not depend on θ in (6.11), and that the restriction $\rho > 4\rho_0$ is included in (6.11), provided $t > \bar{t}$, for a suitable

$\bar{t} > 0$. The proof of the bound above for Z is concluded by noting that $G_0 \simeq R$ if $\Omega \in \mathcal{B}_2(g)$.

In order to prove the bounds below, for u and for Z, we proceed as follows. We have

$$\|u_0\|_{1,\Omega} \leq \int\limits_{\Omega} u(x,t)\,\mathrm{d}x \leq V(Z(t))\|u(\cdot,t)\|_{\infty,\Omega} \leq \gamma\psi(t\|u_0\|_{1,\Omega}^{\alpha+m-1})\|u(\cdot,t)\|_{\infty,\Omega}\,, \tag{6.12}$$

from the bound above for Z, and the definitions of the functions V, R. Note that the first inequality in (6.12) can be proven by standard arguments, exploiting also the fact that $u(\,\cdot\,,t)$ is compactly supported. This shows that the bound (1.15) is optimal. Finally, by the same token,

$$V(Z(t)) \geq \frac{\|u_0\|_{1,\Omega}}{\|u(\cdot,t)\|_{\infty,\Omega}} \geq \gamma_0\psi(t\|u_0\|_{1,\Omega}^{\alpha+m-1})\,,$$

immediately implying the sought after bound below for Z.

In order to complete the proof, we need remove the assumption $F_2^{(-1)}$ convex. This is the only step where we employ the monotonicity of g. Note that

$$F_2^{(-1)}(s) = s^{1+\frac{(m+1)(1+\theta)}{\alpha+m-1}} g\big(s^{-\frac{1+\theta}{\alpha+m-1}}\big)^{m+1}\,, \qquad s > 0\,,$$

and define

$$\phi(s) = \int\limits_0^s \frac{\mathrm{d}\tau}{\tau} \int\limits_0^\tau y^{\frac{(m+1)(1+\theta)}{\alpha+m-1}} g\big(y^{-\frac{1+\theta}{\alpha+m-1}}\big)^{m+1}\,\mathrm{d}y\,.$$

We may minorize, exploiting the fact that g is non decreasing,

$$\phi(s) \geq \int\limits_0^s \frac{\mathrm{d}\tau}{\tau} \int\limits_0^\tau y^{\frac{(m+1)(1+\theta)}{\alpha+m-1}}\,\mathrm{d}y\; g\big(s^{-\frac{1+\theta}{\alpha+m-1}}\big)^{m+1} \geq \gamma_0 F_2^{(-1)}(s)\,.$$

On the other hand, $z^{m+1}g(z^{-1})^{m+1}$ is non decreasing in z, so that

$$\phi(s) \leq \int\limits_0^s \frac{1}{\tau}\tau^{1+\frac{(m+1)(1+\theta)}{\alpha+m-1}} g\big(\tau^{-\frac{1+\theta}{\alpha+m-1}}\big)^{m+1}\,\mathrm{d}\tau \leq F_2^{(-1)}(s)\,. \tag{6.13}$$

Finally, $\phi''(s) \geq 0$ follows from elementary differentiation, and reasoning as in (6.13). Therefore, we may replace $F_2^{(-1)}$ with ϕ when invoking Jensen's inequality, and then switch back to $F_2^{(-1)}$ again, exploiting the two sided estimate just proven.

References

[1] D. Andreucci. Degenerate parabolic equations with initial data measures. *Transactions American Mathematical Society*, 349:3911–3923, 1997. American Mathematical Society.

[2] D. Andreucci and E. DiBenedetto. On the Cauchy problem and initial traces for a class of evolution equations with strongly nonlinear sources. *Annali Sc. Normale Sup. Pisa*, 18:363–441, 1991.

[3] D. Andreucci and A. F. Tedeev. Sharp estimates and finite speed of propagation for a Neumann problem in domains narrowing at infinity. *Advances Diff. Eqs.*, 5:833–860, 2000. Khayyam Publ., Athens Ohio (U.S.A.).

[4] F. Bernis. Qualitative properties for some nonlinear higher order degenerate parabolic equations. *Houston Journal of Mathematics*, 14:319–352, 1988.

[5] E. DiBenedetto. *Degenerate Parabolic Equations.* Springer-Verlag, New York, NY, 1993.

[6] V. A. Galaktionov. Blow-up for quasilinear heat equations with critical Fujita's exponent. *Proc. Roy. Soc. Edinburgh Sect. A*, 124:517–525, 1994.

[7] A. K. Gushchin. The rate of stabilization of a solution of a parabolic equation in an unbounded region. *Diff. Uravn.*, 6:741–761, 1970. Engl. Transl. Diff. Eqns. 6, 1970, 567–581.

[8] A. K. Gushchin. On estimates of solutions of boundary value problems for for a second order parabolic equation. *Trudy Mat. Inst. Steklov*, 126:5–45, 1973. Engl. Transl. Proc. Steklov Inst. Math. 126, 1973.

[9] A. K. Gushchin. On the uniform stabilization of solutions of the second mixed problem for a parabolic equation. *Mat. Sbornik*, 119(161), 1982. Engl. Transl. Math. USSR Sbornik 47, 1984, 439–498.

[10] A. V. Ivanov. Holder estimates near the boundary for generalized solutions of quasilinear parabolic equations that admit double degeneration. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.*, 188:45–69, 1991.

[11] O. A. Ladyzhenskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and Quasilinear Equations of Parabolic Type*, volume 23 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 1968.

[12] H. A. Levine. The role of critical exponents in blowup theorems. *SIAM Review*, 32:262–288, 1990.

[13] G. M. Lieberman. The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations. *Communications in Partial Differential Equations*, 16:311–361, 1991.

[14] V. G. Maz'ja. *Sobolev Spaces.* Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, Germany, 1985.

[15] M. Porzio and V. Vespri. Holder estimates for local solutions of some doubly nonlinear degenerate parabolic equations. *J. Diff. Eqns.*, 103:146–178, 1993.

[16] A. F. Tedeev. Qualitative properties of solutions of Neumann problem for quasilinear higher order parabolic equations. *Ukrainian Mat. Journal*, 44:1571–1579, 1993.

[17] A. F. Tedeev. Two sided estimates as t tends to infinity of the solution of Neumann problem for quasilinear second order equations. *Ukrainian Mat. Journal*, 48:989–998, 1996.

[18] M. Tsutsumi. On solutions of some doubly nonlinear parabolic equations with absorption. *Journal of Mathematical Analysis and Applications*, 132:187–212, 1988.