On expansions in non-integer base

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Introduction

This dissertation is devoted to the study of developments in series in the form

$$(1) \qquad \sum_{i=1}^{\infty} \frac{x_i}{q^i}$$

with the coefficients x_i belonging to a finite set of positive real values named *alphabet* and the ratio q, named *base*, being either a real or a complex number. To ensure the convergence of (1), the base q is assumed to be greater than 1 in modulus. When a number x satisfies $x = \sum_{i=1}^{\infty} \frac{x_i}{q^i}$, for a sequence $(x_i)_{i \geq 1}$ with digits in the alphabet A, we say that x is *representable* in base q and alphabet A and we call $(x_i)_{i \geq 1} = x_1 x_2 \cdots$ a *representation* of x. The representability is one of the main themes of this dissertation and it can be treated starting from very familiar concepts. For example when the alphabet is $\{0, 1\}$ and the base is $q = 2$, then any value $x \in [0, 1]$ satisfies:

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$$

for an appropriate sequence $(x_i)_{i \geq 1} = x_1 x_2 \cdots$ corresponding to the classical binary expansion of x. Similarly the decimal expansion of x is nothing different from a representation $(x_i)_{i \geq 1}$ with digits in $\{0, 1, \ldots, 9\}$ and base 10. The fact that every value in $[0, 1]$ admits a decimal expansion is quite trivial, this being the usual way we represent numbers, but not general. Let us consider the *set of representable numbers* $\Lambda_{q,A} := \{\sum_{i=1}^{\infty} \frac{x_i}{q^i} \mid x_i \in A, \ i \geq 1\}$. If the alphabet A is equal to $\{0, 2\}$ and the base is $q = 3$, then

$$\Lambda_{q,A} = \Lambda_{3,\{0,2\}} = \{\sum_{i=1}^{\infty} \frac{x_i}{3^i} \mid x_i \in \{0, 2\}, i \geq 1\}$$

is well known to coincide with the middle third Cantor set. In particular $\Lambda_{3,\{0,2\}}$ is a totally disconnected set strictly included in the unitary interval. By assuming that the alphabet is $\{0, 1, 2\}$ instead of $\{0, 2\}$, namely by adding the digit 1, it can be proved that the full representability of $[0, 1]$ is restored, i.e. $\Lambda_{3,\{0,1,2\}} = [0, 1]$. To stress that the digit 1 is necessary to represent the whole unit interval, the alphabet $\{0, 2\}$ is said to be with *deleted digits*.

We then consider a generalized concept of representability. Fix an alphabet A and a base q, either real or complex, and suppose that for $x \notin \Lambda_{q,A}$ there exists a non-negative integer n such that $x/q^n \in \Lambda_{q,A}$. As

$$\frac{x}{q^n} = \sum_{i=1}^{\infty} \frac{x_i}{q^i} \quad \text{implies} \quad x = \sum_{i=0}^{n-1} x_{n-i} q^i + \sum_{i=1}^{\infty} \frac{x_{n+i}}{q^i} = x_1 q^{n-1} + \cdots + x_{n-1} q + x_n + \frac{x_{n+1}}{q} + \cdots$$

we may represent x by an expression in the form $x_1 \cdots x_n \boldsymbol{.} x_{n+1} \cdots$, with the symbol $\boldsymbol{.}$ indicating the scaling factor q^n. When we assume that the base is 10, then this general notation assumes a familiar form; e.g. $1 \boldsymbol{.} 2$ traditionally denotes the quantity $10 \times (\frac{1}{10} + \frac{2}{10^2})$.

If any value belonging to a given numerical set Λ admits a "generalized" representation in base q and alphabet A, then the couple (A, q) is said to specify a (positional) numeration system for Λ.

Let us see some examples. The non-negative real numbers can be represented by an integer base $b > 1$ and alphabet $\{0, 1, \ldots, b-1\}$ as well as by a non-integer base $q > 1$ and alphabet $\{0, 1, \ldots, \lfloor q \rfloor\}$. The choice of a negative base leads to the full representability of $\mathbb{R}$: any real number can be represented in integer base $-b$, with $b > 1$, and alphabet $\{0, 1, \ldots, b-1\}$ and any real number can be represented by the non-integer negative base $-q$ and alphabet $\{0, 1, \ldots, \lfloor q \rfloor\}$. The case of a positive non-integer base is presented in the seminal paper by Rényi [Rén57] while the negative non-integer case has been recently treated in [IS09]. The alphabets $\{0, 1, \ldots, b-1\}$ and $\{0, 1, \ldots, \lfloor q \rfloor\}$ are called *canonical*. Pedicini [Ped05] studied the representability in positive real base and non-canonical alphabets, also named *alphabets with deleted digits*, by showing that $(\{a_1, \ldots, a_J\}, q)$ is numeration system for non-negative real numbers if and only if $q \leq (a_J - a_1) / \max\{a_{j+1} - a_j\}$. The case of complex base is more complicated and representability results have been established only for some classes of complex base, e.g. the Gaussian integers in the form $-n \pm i$, with $n \in \mathbb{N}$, the integer roots $\sqrt[k]{n}$ with $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, the quadratic complex numbers.

In the case of a positive non-integer base, the set of representable values $\Lambda_{q,A}$ may have only two topological structures: either it coincides with an interval or it is a totally disconnected set. Clearly the couple (A, q) provides a full representability of $\mathbb{R}^+$ if and only if the first case occurs. Putting together these facts we may deduce that every non-negative real number can be represented in base q and alphabet A if and only if $\Lambda_{q,A}$ is a convex set. The study of $\Lambda_{q,A}$ has also been simplified by the fact that the smallest and the greatest representable number are explicitly known:

$$\min \Lambda_{q,A} = \frac{\min A}{q-1} = \sum_{i=1}^{\infty} \frac{\min A}{q^i} \qquad \text{and} \qquad \max \Lambda_{q,A} = \frac{\max A}{q-1} = \sum_{i=1}^{\infty} \frac{\max A}{q^i}.$$

When we consider a complex base, some aspects change. The convexity of $\Lambda_{q,A}$ is not a necessary condition for the representability, e.g. every complex number can be represented in base $-1+i$ and alphabet $\{0, 1\}$ but $\Lambda_{q,A}$ coincides with the space-filling twin dragon curve, a non-convex set. Moreover the boundary of $\Lambda_{q,A}$ may have a fractal nature in complex base even in the case of full representability.

Our approach to the problem of representability in complex base consists in the study of the convex hull of $\Lambda_{q,A}$ when q has a rational argument. In Chapter 2 we characterize the convex hull of $\Lambda_{q,A}$ and, by mean of such a characterization, we establish a necessary and sufficient condition for $\Lambda_{q,A}$ to be convex. As the convexity of $\Lambda_{q,A}$ is a sufficient condition for the full representability, such a result provides an inedited class of number systems with complex base.

So far we spoke about the conditions of representability: in other words we focused on the problems related to establish whether any real or complex number can be represented using a given base and a given alphabet. But when the representability is assumed, e.g. by choosing a non-integer base $\pm q$ and its canonical alphabet $A_q = \{0, 1, \ldots, \lfloor q \rfloor\}$, many other questions arise. For example one may ask how to represent the numbers, what are the combinatorial and dynamical properties of such representations, how many different representations does a real number have. In what follows we briefly recall some classical results on these topics to the end of contextualize our original results.

The proof that any non-negative real number admits a representation in base q and alphabet A_q is classically constructive and based upon the so-called *greedy algorithm*. The sequences obtained by the greedy algorithm are called *greedy expansions* or *q-expansions* and they revealed

interesting properties. For example the q-expansion of $x \in [0,1]$ is the lexicographically great-est among all the possible representations of x: this implies that if we truncate the q-expansion of a given number x, the error we do is minimal. Another interesting feature is the monotonic-ity with respect to the numerical value: if two q-expansions are one lexicographically less than the other then the corresponding values are one smaller than the other. These properties of q-expansions stimulated the research on this topic involving several fields of mathematics and theoretical computer science, like number theory, ergodic theory, symbolic dynamics, automata theory. In particular the Parry's characterization of q-expansions [Par60] allowed to study from a symbolic dynamical point of view the closure of the set of q-expansions, named *q-shift*. Fea-tures like the recognizability by a finite automaton and the entropy of the q-shift as well as the existence of finite automata performing arithmetic operations in some positive bases have been widely investigated. In Chapter 3 we extend some of these results to the case of a negative based numeration system.

Another widely studied aspect of the representations in positive non-integer bases is the ex-istence of different expansions representing the same value. When a representable value admits only one representation, its expansion is said to be *unique*. In the nineties Erdős, together with Horváth, Joó and Komornik, devoted a series of papers to the number of possible different rep-resentations in base $1 < q < 2$ and alphabet $\{0,1\}$ with particular attention to the expansions of 1. These results, together with those of the paper by Daróczy and Kátai [DK93], allowed to clarify the structure of the set of unique expansions in base $1 < q < 2$ and alphabet $\{0,1\}$. In particular when the base is less than the Golden Mean then every positive number can be repre-sented in at least two different ways, when the base is complied between the Golden Mean and the Komornik-Loreti constant there exists a countable set of values that can be represented by a unique expansion, and when the base is larger than the Komornik-Loreti constant the set of values with a unique expansion has the cardinality of the continuum. In Chapters 4 and 5 we assume the base to be a non-integer positive value and we prove the existence of a sort of "generalized Golden Mean" for arbitrary alphabets, namely we show that expansions are never unique if and only if the base is chosen below a critical value. In the case of a ternary alphabet we explicitly characterize such a critical base as well as the unique expansions for sufficiently small bases.

Organization of the chapters. This dissertation is organized as follows. Chapter 1 contains some preliminaries and the state of the art of the expansions in non-integer base. Chapter 2 deals with the representability in complex base. Chapter 3 is devoted to the study of expansions in non-integer negative bases by establishing many analogies with the classical positive case. In Chapters 4 and 5 the base is assumed to be non-integer and positive: Chapter 4 is devoted to the character-ization of the "generalized Golden Mean", also called critical base, for ternary alphabets. Finally in Chapter 5 we explicitly characterize the unique expansions with digits in ternary alphabets for a set of sufficiently small bases.

The research work leading to Chapter 2 has been realized under the supervision of Paola Loreti. Most of the results of Chapter 3 are in collaboration with Christiane Frougny and they can be found in [FL09]. Chapter 4 substantially contains a work in collaboration with Vilmos Komornik and Marco Pedicini [KLP].

CHAPTER 1

Background results on expansions in non-integer base

This chapter is devoted to two categories of preliminary results. First we introduce some basic notions and results not exclusively related to the numeration systems, with the purpose of recalling most of the theoretical tools used along this dissertation. We then focus on the state of the art of the classical expansions on non-integer base, with particular attention to those aspects that shall be generalized in the further chapters.

Organization of the chapter. In Section 1 some basic definitions and results about combinatorics on words, automata theory, symbolic dynamics, algebraic integers, sturmian words and iterated function systems are introduced. Section 2 contains a formal definition of positional number systems. In Section 3 we show some simple constructions concerning the number systems with integer base, these construction being generalized in Section 4. Section 4 also contains several results about the classical q-expansions, the symbolic dynamical systems associated to these representations, the case of a particular class of bases named Pisot numbers and the redundancy of representations in non-integer base.

1. Our toolbox

1.1. Combinatorics on words.

Alphabets and related operations. An *alphabet* is a totally ordered set. In this dissertation the alphabets are always finite subsets of $\mathbb{R}^+$ and the total order on the elements is $<$, i.e. the natural ordering on $\mathbb{R}$. The *translation* of the alphabet A of a factor $t \in \mathbb{R}$ is the alphabet $A + t := \{a + t | a \in A\}$; the *scaling* of the alphabet A of a factor $t \in \mathbb{R}$ is the alphabet $tA := \{ta | a \in A\}$. The *dual* of the alphabet A is the alphabet $D(A) := \{\min(A) + \max(A) - a | a \in A\}$.

Digits and gaps. An element of an alphabet is called *digit*. The difference between two consecutive digits is called *gap*. If $A = \{a_1, \ldots, a_J\}$, the *right gap* related to the digit a_j, with $j < J$, is the (positive) quantity $a_{j+1} - a_j$; the *left gap* related to the digit a_j, with $j > 0$, is the (positive) quantity $a_j - a_{j-1}$.

Finite and infinite words. The concatenation of a finite number of digits is called (finite) *word*. The set of the finite words with digits in A is denoted by A^*; the set of the finite words with digits in A and with length equal to n is denoted by A^n. An *infinite word*, or simply *sequence*, is a sequence $\mathbf{x}$ indexed by $\mathbb{N}$ with values in A, that is $\mathbf{x} = x_1 x_2 \cdots$ where each $x_i \in A$. When denoting infinite sequences, we sometimes use the notation $(x_i)_{i \geq 1}$ as well. The set of infinite words with digits in A is denoted by $A^{\mathbb{N}}$, namely $A^{\mathbb{N}} := \{\mathbf{x} \mid \mathbf{x} = x_1 x_2 \cdots ; x_i \in A; i \geq 1\}$. The set of finite or infinite words is $A^{\infty} := A^* \cup A^{\mathbb{N}}$. The length of a finite or infinite word v is denoted by $|v|$. Let v be a word of A^*, denote by v^n the concatenation of v to itself n times, and by v^{ω} the infinite concatenation $vvv \cdots$. A word of the form uv^{ω} is said to be *eventually periodic* with *period* $|v|$. A (purely) *periodic* word is an eventually periodic word of the form v^{ω}. If $v = \min A$ then the eventually periodic uv^{ω} is called *eventually minimal*, similarly if $v = \max A$ then uv^{ω} is called *eventually maximal*.

Factors. A finite word v is a *factor w* of a (finite or infinite) word $z \in A^\infty$ if there exist u and w such that $z = uvw$. When u is the empty word, v is a *prefix* of x. When w is empty, v is said to be a *suffix* of x. A factor of a word $z \in A^\infty$ is said to be *left special* (resp. *right special*) if aw and bw (resp. wa and wb) are factors of z for some digits $a, b \in A$, $a \neq b$. For every word $z \in A^\infty$, $F(z)$ denotes the set of factors of z; $F_n(z) := F(z) \cap A^n$, with $n \in \mathbb{N}$, is the set of factors of z length equal to n. The relation between the left special factors of an infinite sequence and the structure of such a sequence is explicited in the following result, proved in [BD09].

PROPOSITION 1.1. *A sequence $\mathbf{x}$ is purely periodic if and only if it has no left special factors of some length n.*

Operations on words. Two words u and v are said to be *right congruent modulo $H \subseteq A^*$* if, for every w, uw is in H if and only if vw is in H. The *index* of the congruence modulo H is the number of the classes of congruence modulo H.

Denote by ε the empty word and define $A^+ := A^* \setminus \{\varepsilon\}$. The words in A^+ can be ordered by the *lexicographic order* $<_{lex}$: if $v, w \in A^+$, $v <_{lex} w$ if and only if either v is a prefix of w or there exists $x \in A^*$ such that $v = xav'$ and $w = xbw'$ with $a, b \in A$ and $a < b$. The lexicographic order is extended to $A^{\mathbb{N}}$ as follows. Let $\mathbf{x} = x_1 x_2 \cdots, \mathbf{y} = y_1 y_2 \cdots \in A^\omega$, let $\mathbf{x} \neq \mathbf{y}$ and let i be the smallest index such that $x_i \neq y_i$. Then $\mathbf{x} <_{lex} \mathbf{y}$ if and only if $x_i < y_i$. If $w, z \in A^\infty \setminus \{\varepsilon\}$ satisfy $w <_{lex} z$, then w (resp. z) is said to be (lexicographically) *smaller* than z (resp. *larger* than w). When it can be evinced by the context, the subscript *lex* is omitted.

The *shift* is the map $\sigma : A^\omega \to A^\omega$ which satisfies $\sigma(\mathbf{x}) = x_2 x_3 \cdots$ for every $\mathbf{x} = x_1 x_2 \in A^\omega$. A set $U \subseteq A^\omega$ which is closed under shift is called *shift-invariant*. For every $\mathbf{x} \in A^\omega$, the *orbit* (with respect to the shift) of $\mathbf{x}$ is the set $Orb(\mathbf{x}) := \{\sigma^n(\mathbf{x}) | n = 0, 1, \dots\}$.

Let $z \in A^\infty$ and let k be a positive integer. We denote $\min(z|k) := \min F_k(z)$ (resp. $\max(z|k) := \max F_k(z)$) the smallest (resp. greatest) factor of z of length k. When z is infinite, we may also define:

$$\min z := \lim_{k \to \infty} \min(z|k) \qquad \max z := \lim_{k \to \infty} \max(z|k)$$

Remark that $\min(z|k)$ and $\max(z|k)$ are respectively a prefixes of $\min(z|k+1)$ and $\max(z|k+1)$ for every k. This "monotonicity" with respect to the prefixes implies that $\min z$ and $\max z$ are well defined.

1.2. Automata theory. We recall some definitions on automata, see [Eil74] and [Sak03] for instance. An *automaton over A*, $\mathcal{A} = (Q, A, E, I, T)$, is a directed graph labelled by elements of A. The set of vertices, traditionally called *states*, is denoted by Q, $I \subset Q$ is the set of *initial* states, $T \subset Q$ is the set of *terminal* states and $E \subset Q \times A \times Q$ is the set of labelled *edges*. If $(s, a, s') \in E$, we write $s \xrightarrow{a} s'$. The automaton is *finite* if Q is finite. The automaton $\mathcal{A}$ is *deterministic* if E is the graph of a (partial) function from $Q \times A$ into Q, and if there is a unique initial state. A subset H of A^* is said to be *recognizable by a finite automaton*, or *regular*, if there exists a finite automaton $\mathcal{A}$ such that H is equal to the set of labels of paths starting in an initial state and ending in a terminal state. Finite automata can be partially described by their *adjacency matrix*, whose definition is given below.

DEFINITION 1.1 (Adjacency matrix). *Let $\mathcal{A}$ be a finite automaton and denote by $Q = \{s_1, \dots, s_n\}$ the (finite) set of states. The adjacency matrix of $\mathcal{A}$ is the square matrix $(a_{i,j})_{i,j \leq n}$ where for every $i, j \leq n$ the element $a_{i,j}$ is the number of the edges from the state s_i to the state s_j.*

A recognizable by a finite automaton set H is characterized by index of the congruence modulo H, as stated in the following result.

THEOREM 1.1. *H is recognizable by a finite automaton if and only if the congruence modulo H has finite index.*

Let A and A' be two alphabets. A *transducer* is an automaton $\mathcal{T} = (Q, A^* \times A'^*, E, I, T)$ where the edges of E are labelled by couples in $A^* \times A'^*$. It is said to be *finite* if the set Q of states and the set E of edges are finite. If $(s, (u, v), s') \in E$, we write $s \xrightarrow{u|v} s'$. The *input automaton* (resp. *output automaton*) of such a transducer is obtained by taking the projection of edges on the first (resp. second) component. A transducer is said to be *sequential* if its input automaton is deterministic.

The same notions can be defined for automata and transducer processing words from right to left : they are called *right* automata or transducers.

1.3. Symbolic dynamics. Fix an alphabet A. A *bi-infinite* sequence over the alphabet A is a sequence indexed in $\mathbb{Z}$ with digits in A.

The *shift* σ is defined in $A^{\mathbb{Z}}$ by:

$$\sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}. \tag{2}$$

A set $S \subseteq A^{\mathbb{Z}}$ (or $S \subseteq A^{\omega}$) is a *symbolic dynamical system*, or *subshift*, if it is shift-invariant and closed for the product topology on $A^{\mathbb{Z}}$. The trivial subshift $S = A^{\mathbb{Z}}$ is called *full shift*.

A bi-infinite word z *avoids* a set of word $X \subset A^*$ if no factor of z is in X. The set of all words which avoid X is denoted S_X. A classical result establishes the following relation between subshifts and avoided sets.

PROPOSITION 1.2. *A set $S \subseteq A^{\mathbb{Z}}$ is a subshift if and only if S is of the form S_X for some $X \subset A^*$.*

Let $F(S)$ be the set of factors of elements of S, let $I(S) = A^+ \setminus F(S)$ be the set of words avoided by S, and let $X(S)$ be the set of elements of $I(S)$ which have no proper factor in $I(S)$. The subshift S is *sofic* if and only if $F(S)$ is recognizable by a finite automaton, or equivalently if $X(S)$ is recognizable by a finite automaton. The subshift S is of *finite type* if $S = S_X$ for some finite set X, or equivalently if $X(S)$ is finite.

The *topological entropy* of a subshift S is

$$h(S) = \lim_{n \to \infty} \frac{1}{n} \log(F_n(S)) \tag{3}$$

where $F_n(S)$ is the number of elements of $F(S)$ of length n.

THEOREM 1.2. *When S is sofic, the entropy of S is equal to the logarithm of the spectral radius of the adjacency matrix of the finite automaton recognizing $F(S)$.*

REMARK 1.1. *A comprehensive introduction on symbolic dinamical systems can be found in [Lot02, Chapter 1] and [LM95].*

1.4. Algebraic integers. We recall that an *algebraic integer* is a root of a polynomial with integral coefficients and leading coefficient equal to 1. The *minimum polynomial $M_q(X)$* of an algebraic integer q is the monic polynomial in $\mathbb{Z}[X]$ with minimal degree satisfying $M_q(q) = 0$. The *algebraic conjugates*, or simply *conjugates*, of an algebraic integer q are all the (possibly complex) other roots of $M_q(X)$.

A *Pisot number* is an algebraic integer greater than 1 whose conjugates are less than 1 in modulus.

EXAMPLE 1.1. *The positive integers (except 1) and the Golden Mean G are celebrated examples of Pisot numbers.*

EXAMPLE 1.2. *The square of the Golden Mean, G^2, is a Pisot number. In fact $G^2 = \left((1 + \sqrt{5})/2\right)^2 = (3 + \sqrt{5})/2$ and its conjugate is equal to $(3 - \sqrt{5})/2 < 1$.*

The set $\mathbb{Q}(q)$, with $q \in \mathbb{C}$ is called *algebraic extension of $\mathbb{Q}$ to q* and it is the smallest field containing both $\mathbb{Q}$ and q. For example, if $q \in \mathbb{Q}$ then $\mathbb{Q}(q) = \mathbb{Q}$. The elements $\mathbb{Q}(q)$ are all in the form $P(q)/Q(q)$ where $P(X), Q(X) \in \mathbb{Z}[X]$, $Q(q) \neq 0$. The following classical result describes the structure of $\mathbb{Q}(q)$.

THEOREM 1.3. *For every $q \in \mathbb{C}$, $\mathbb{Q}(q)$ is a vector space over $\mathbb{Q}$ with base $\{1, q, q^2, \ldots, q^{d-1}\}$ if q is an algebraic integer whose minimal polynomial has degree d and with (infinite) base $\{1, q, q^2, \ldots\}$ otherwise.*

We now show some direct consequences of the definitions and of the result above.

LEMMA 1.1. *Let q be an algebraic integer and let $M_q(X) = X^d - a_1 X^{d-1} - \cdots - a_d$ be the minimal polynomial of q. Then:*

(a) *the minimal polynomial of $-q$ is $M_{-q}(X) = X^d - (-1)^{d-1} a_1 X^{d-1} - (-1)^{d-2} a_2 X^{d-2} - \cdots - a_d$;*
(b) *every $x \in \mathbb{Q}(q)$ can be expressed as $x = b^{-1} \sum_{i=0}^{d-1} c_i(-q)^i$ with b and c_i in $\mathbb{Z}$;*
(c) *if q is a zero of $P(X) \in \mathbb{Z}[X]$, then all the algebraic conjugates of q, q_i with $i = 2, \ldots, d$, also satisfy $P(q_i) = 0$.*

PROOF. The assertions (a) and (b) follow immediately by the definition of minimal polynomial and by Theorem 1.3. Then we need to prove (c). If q is a zero of $P(X) \in \mathbb{Z}[X]$ then the degree of $P(X)$ must be greater than d to not contradict the minimality of $M_q(X)$. As $\mathbb{Z}[X]$ is an Euclidean domain, we may perform the division by $M_q(X)$ and get: $P(X) = Q(X)M_q(X) + R(X)$ and with the degree of $R(X)$ lower than d. Then $P(q) = 0$ implies $R(q) = 0$ and again by the minimality of $M_q(X)$ we have $R(X) = 0$. Hence $M_q(X)$ divides $P(X)$ and thesis follows. $\square$

1.5. Sturmian sequences. Let $A = \{a, b\}$ be a binary alphabet.

DEFINITION 1.2 (Sturmian sequences). *A sequence $\mathbf{s} \in A^\omega$ is* sturmian *if there exist $\alpha, \rho \in [0, 1]$ such that for every n:*

(4) $$s_n = a \quad \text{if } \lfloor \rho + (n+1)\alpha \rfloor = \lfloor \rho + n\alpha \rfloor \qquad s_n = b \text{ otherwise}$$

or for every n:

(5) $$s_n = a \quad \text{if } \lceil \rho + (n+1)\alpha \rceil = \lceil \rho + n\alpha \rceil \qquad s_n = b \text{ otherwise.}$$

The sequence $\mathbf{s}$ is called proper sturmian *(or* aperiodic sturmian*) if α is irrational,* periodic sturmian *if α is rational,* standard sturmian *if $\rho = 0$.*

REMARK 1.2. *Classically, sturmian sequences can be defined as sequences whose factors of length n are exactly $n + 1$, namely sequences with* minimal positive complexity, *this property being equivalent to the conditions in (4) and (5) with α irrational.*

Here, following [Pir05], we admit both rational and irrational α's, since in Chapter 5 this larger class is proved to have interesting uniqueness properties in some numeration systems.

Hereafter we state the lexicographical characterization of standard sturmian sequences.

THEOREM 1.4. *A sequence $\mathbf{s} \in A^\omega$ is standard sturmian if and only if*

(6) $$a\mathbf{s} \leq \min \mathbf{s} \leq \max \mathbf{s} \leq b\mathbf{s}.$$

In particular:

(a) *if $a\mathbf{s} = \min \mathbf{s}$ and $\max \mathbf{s} = b\mathbf{s}$ then $\mathbf{s}$ is standard proper;*

(b) if $a\mathbf{s} = \min \mathbf{s}$ and $\max \mathbf{s} < b\mathbf{s}$ then $\mathbf{s}$ is standard periodic in the form $\mathbf{s} = (wba)^\omega$;

(c) if $a\mathbf{s} < \min \mathbf{s}$ and $\max \mathbf{s} = b\mathbf{s}$ then $\mathbf{s}$ is standard periodic in the form $\mathbf{s} = (wab)^\omega$;

for some $w \in A^*$.

In particular, if $\mathbf{s}$ is a periodic sturmian word, then $\max \mathbf{s} = (bwa)^\omega$ and $\min \mathbf{s} = (awb)^\omega$ for some $w \in \{a, b\}^*$.

REMARK 1.3. *The aperiodic case of Theorem 1.4 has been rediscovered by several authors, but the first version seems to be due to Veerman in the eighties ([Vee86], [Vee87]). We refer to [AG09] for a detailed history of this result. The periodic case has been proved by Pirillo, see [Pir03] and [Pir05].*

1.6. Iterated function systems. The theory of iterated function systems (IFS) is a wide field and we refer to [Fal90] for a comprehensive overview. Our interest in this argument is due to the fact that IFS's can be used to model the representations in positional number systems. We employ this relation in Chapter 2, where only the following basic notions are necessary.

DEFINITION 1.3 (Iterated function systems and attractors). *An* iterated function system $F := \{f_0, \ldots, f_m\}$ *is a finite set of contractive maps defined on a complete metric space. An* attractor *of F is a set satisfying $S = \cup_{i=0}^{m} f_i(S)$.*

THEOREM 1.5 (Uniqueness of the attractor). *For every iterated function system $F = \{f_0, \ldots, f_m\}$ there exists only one set S satisfying $S = \cup_{i=0}^{m} f_i(S)$.*

REMARK 1.4. *The uniqueness of the attractor is a consequence of the Banach's Fixed Point Theorem.*

2. Positional number systems

A *numeration system* is a mathematical notation ensuring a representation for every element of a given numerical set. For example binary and decimal numeration systems provide a representation for every non-negative real number, but not for a general complex number. In fact complex numbers are usually represented by combining the representations of real numbers into expressions in the form $a + ib$ with $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$; $\rho \cos \theta + \rho i \sin \theta$ or in the exponential form $\rho e^{i\theta}$ with $\rho > 0$ and $\theta \in [0, 2\pi)$.

An important class of numeration systems complies the *positional number systems*. A positional number system is defined starting from a *base q*, an integer or a real or a complex number with modulus larger than 1, and a finite alphabet A. A number x is *representable* in base q and alphabet A if there exists a sequence of digits in A, say $x_{-d} \cdots x_{-1} x_0 x_1 x_2 \cdots$ satisfying

$$
\begin{aligned}
x &= x_{-d} q^d + \cdots + x_{-1} q + x_0 + \frac{x_1}{q} + \frac{x_2}{q^2} + \cdots \\
&= \sum_{i=0}^{d} x_{-i} q^i + \sum_{i=1}^{\infty} \frac{x_i}{q^i}.
\end{aligned}
$$

(7)

The sum $\sum_{i=0}^{d} x_{-i} q^i$, namely the part with positive powers of the base, is called the *integer part* of x and the sum $\sum_{i=1}^{\infty} \frac{x_i}{q^i}$ is called the *fractional part*. A representation with a positional number system is usually denoted $(x_{-d} \cdots x_{-1} x_0 \cdot x_1 x_2 \cdots)_q$ with the symbol $\cdot$ dividing the integer and the fractional part. If the integer part is equal to 0 we then write $(\cdot x_1 x_2 \cdots)_q$. The equalities in (7) imply that

(8) $x = (x_{-d} \cdots x_{-1} x_0 \cdot x_1 x_2 \cdots)_q \Leftrightarrow \dfrac{x}{q} = (x_{-d} \cdots x_{-1} \cdot x_0 x_1 x_2 \cdots)_q$

In other words, the division by the base q of the represented value is equivalent to a left-shift on the digits of its representation. If we keep shifting in (8) we then get:

$$(9) \qquad x = (x_{-d} \cdots x_{-1} x_0 . x_1 x_2 \cdots)_q \Leftrightarrow \frac{x}{q^{d+1}} = (.x_{-n} \cdots x_{-1} x_0 x_1 x_2 \cdots)_q$$

and we may conclude that a number x is representable in base q and alphabet A if and only if there exists an integer d and a sequence $x_{-d+1} x_{n-2} \cdots \in A^{\mathbb{N}}$ such that

$$\frac{x}{q^d} = \sum_{i=1}^{\infty} \frac{x_{-d+i}}{q^i}.$$

If x is representable, then its representation can be constructed starting by the representation of the fractional part of an appropriate rescaling of x, namely

$$(10) \quad x \to \frac{x}{q^d} = \sum_{i=1}^{\infty} \frac{x_{-d+i}}{q^i}$$

$$\to \frac{x}{q^d} = (.x_{-d+1} \cdots x_{-1} x_0 x_1 x_2 \cdots)_q$$

$$\to x = (x_{-d+1} \cdots x_{-1} x_0 . x_1 x_2 \cdots)_q.$$

It follows by the equations above that a general system and its fractional part share the properties related to the representability and the combinatorial and dynamical properties of the representations. Hence we may restrict ourselves without loss of generality to the study of the fractional part and, where is not specified, call representable a number satisfying the equation

$$x = \sum_{i=1}^{\infty} \frac{x_i}{q^i}$$

for some sequence $x_1 x_2 \cdots \in A^{\mathbb{N}}$, named the *expansion* of x in base q.

We finally call the *numerical value* in base q of a (finite or infinite) word $\mathbf{x} = (x_i)_{i=1}^{|\mathbf{x}|}$ the map $\pi_q(\mathbf{x}) := \sum_{i=1}^{|\mathbf{x}|} \frac{x_i}{q_i}$.

3. Positional numeration systems with positive integer base

The positional number systems with integer base $b > 1$ and *canonical alphabet* $\{0, 1, \ldots, b-1\}$ are probably the most familiar, because they include the decimal and binary numeration systems. Any real number in $[0, 1)$ can be represented by b-ary expansion in the form

$$x = \sum_{i=1}^{\infty} \frac{x_i}{b^i}$$

by using the following algorithm. We consider x and we choose as first digit the greatest integer x_1 such that:

$$\frac{x_1}{b} \leq x.$$

As $x \in [0, 1)$ then $b(x - \frac{x_1}{b}) \in [0, 1)$ as well, we may reapply the algorithm. Hence we choose the greatest integer x_2 such that

$$\frac{x_2}{b} \leq b(x - \frac{x_1}{b}).$$

Note that the inequality above implies:

$$\frac{x_1}{b} + \frac{x_2}{b^2} \leq x.$$

By iterating, at step n we get:

$$\frac{x_1}{b} + \frac{x_2}{b^2} + \cdots + \frac{x_n}{b^n} \leq x.$$

Starting from the relation $q(x - \frac{x_1}{b}) \in [0,1)$ it can be recursively proved that the *remainder of order n*, namely the difference between the value x and the expansion truncated at the n-th digit, satisfies:

$$0 \le x - \left(\frac{x_1}{b} + \frac{x_2}{b^2} + \cdots + \frac{x_n}{b^n}\right) \le \frac{1}{b^n}.$$

Hence as n tends to infinity the difference above vanishes and, consequently, the sequence $x_1 x_2 \cdots$ is an expansion of x in base b.

REMARK 1.5. *We remark that at any step n the maximal digit x_n satisfying $\sum_{i=1}^n x_i/b^i \le x$ is chosen. This implies that by incrementing any digit, i.e. considering any sequence lexicographically greater, we get a sum larger than the represented value. Hence the algorithm above is* greedy, *namely it yields the lexicographical greatest expansion of a value x.*

The greedy algorithm is equivalent to the iteration of the map $T_b := bx - \lfloor bx \rfloor$. In fact we have $x_1 = \lfloor bx \rfloor = \lfloor bT_b^0(x) \rfloor$, $x_2 = \lfloor b^2(x - \frac{x_1}{b}) \rfloor = \lfloor bT_b(x) \rfloor$ and , in general, $x_n = \lfloor bT_b^{n-1}(x) \rfloor$. Finally note that the definition implies that the digits belong to $\{0,1,\ldots,b-1\}$ thus the greedy algorithm actually yields an expansion with base b and canonical alphabet.

We have just showed that every real number in $[0,1)$ can be represented by an expansion resulting by the greedy algorithm. In following classical proposition we state that the converse is not true.

PROPOSITION 1.3. *A value $x \in [0,1)$ admits at most two distinct expansions in base b and alphabet $\{0,1,\ldots,b-1\}$.*

In particular the representation of x is not unique if and only if $x = (.x_1 \cdots x_n)_b$ for some (finite) digits $x_1,\ldots,x_n$ belonging to the canonical alphabet. The two expansions of x are the coefficients in the formula:

$$x = \sum_{i=1}^n \frac{x_i}{b^i} = \frac{x_1}{b} + \cdots + \frac{x_{n-1}}{b^{n-1}} + \frac{x_n - 1}{b^n} + \frac{b-1}{b^{n+1}} + \frac{b-1}{b^{n+2}} + \cdots.$$

REMARK 1.6. *We remark that $\sum_{i=1}^\infty \frac{b-1}{b^i} = 1$, hence when the suffix $(b-1)^\omega$ occurs in expansion we can increase the last digit different from $b-1$ and substitute the tail with $(0)^\infty$. This simple procedure yields a lexicographically greater expansion. Hence if a sequence with digits in $\{0,1,\ldots,b-1\}$ is eventually maximal, it cannot be gained with the greedy algorithm.*

4. Expansions in non-integer bases

4.1. Basic definitions. In [Rén57], Rényi introduced a positional numeration system with non-integer base $q > 1$ and *canonical* alphabet $A_q := \{0,1,\ldots,\lfloor q \rfloor\}$, with $\lfloor \cdot \rfloor$ representing the lower integer part. In Rényi's numeration systems all the real numbers belonging to $[0,1)$ are represented by the so called q-expansions, defined below.

DEFINITION 1.4 (q-transformation, q-expansions). *For every $q > 1$ the q-transformation is the map from $[0,1)$ onto itself $T_q(x) = qx - \lfloor qx \rfloor$. The q-expansion of $x \in [0,1)$ is a sequence $\gamma_q(x) := x_1 x_2 \cdots \in A_q^{\mathbb{N}}$ gained by the iteration of the q-transformation:*

$$(11) \qquad\qquad x_n = \lfloor qT_q^{n-1}(x) \rfloor.$$

for every $n \ge 1$.

REMARK 1.7. *We may convince ourselves that the q-expansion $\gamma_q(x) = x_1 x_2 \cdots$ is a representation of x by applying the definition. In fact*

$$x_1 = \lfloor qx \rfloor \Rightarrow x = \frac{x_1}{q} + \frac{T_q(x)}{q};$$

and, by iterating, for every $n \geq 1$:

$$(12) \qquad x = \sum_{i=1}^{n} \frac{x_n}{q^n} + \frac{T_q^n(x)}{q^n}.$$

As $n \to \infty$, we get $x = \sum_{i=1}^{n} \frac{x_n}{q^n}$.

We now state two classical properties of q-expansions.

PROPOSITION 1.4. *Let $x \in [0,1)$ and consider $\gamma_q(x)$, the q-expansion of x.*

(a) *If there exists a sequence $x_1' x_2' \cdots$ satisfying $x = \sum_{i=1}^{\infty} \frac{x_i'}{q^i}$, then*

$$x_1' x_2' \cdots \leq_{lex} \gamma_q(x).$$

(b) *For every $y \in [0,1)$:*

$$x < y \quad \text{if and only if} \quad \gamma_q(x) <_{lex} \gamma_q(y)$$

We remark that the first part of the previous proposition states that the q-expansion is the lexicographically greatest among all the possible representations in base q of a number. For this reason q-expansions are also called *greedy expansions*. We shall use indifferently one or the other name, by privileging the name *q-expansions* when we wish to stress the dynamical properties and the represented value and the name *greedy* when we focus on lexicographical and combinatorial aspects.

4.2. Greedy and quasi-greedy algorithms and Parry's characterization of q-expansions. In this section we characterize the sequences representing a greedy expansion. In fact a sequence with digits in $\{0, 1, \ldots, \lfloor q \rfloor\}$ is not necessarily a greedy expansion.

In some case is quite easy to recognize the greedy expansions: for example in the case of $q = G$ where $G = \frac{1+\sqrt{5}}{2}$ is the Golden Mean, the study of the q-transformation yields that after any occurrence of the digit 1 must follow the digit 0 and thus a sequence of zeros and ones is a greedy expansion if and only if there are no occurrences of the "forbidden" subsequence 11.

Parry characterized the greedy expansions by mean of the lexicographical comparison with a boundary sequence, the *quasi-greedy expansion* of 1 [Par60] . We now introduce the preliminary definitions and then we state such a result.

4.2.1. **Greedy and quasi-greedy algorithms.** Greedy expansions can be equivalently defined by the so called *greedy algorithm*. We fix a real number $x \in [0,1]$ and we proceed in computing the digits as follows. For every $n \geq 1$ we define x_n as the greatest digit in A_q satisfying the inequality:

$$(13) \qquad \sum_{i=1}^{n-1} \frac{x_i}{q^i} + \frac{x_n}{q^n} \leq x.$$

By construction the resulting sequence is the lexicographically greatest sequence representing x and hence it is the greedy expansion of x. If there exists n such that an equality holds in (13), then the greedy expansion of x is in the form $x_1 \cdots x_n(0)^\omega$: it is eventually minimal because 0 is the smallest digit in A_q.

REMARK 1.8. *1. In the literature, eventually minimal expansions in base q are called* finite, *because it suffices to expand a finite number of non-zero digits to represent the value. We adopt the more precise notation of eventually minimal to avoid confusion with finite words and to deal the cases of general alphabets, where the minimal digit is not necessarily 0;*

2. *the greedy algorithm is well defined for every $x \in [0, \frac{\lfloor q \rfloor}{q-1}]$ and, in particular, for $x = 1$. Hence we may extend the definition of greedy expansions to the value 1, so that γ_q may be considered defined over the set $[0,1]$.*

Starting from an eventually minimal greedy expansion, it is possible to construct a different expansion of the same value. In fact $\gamma_q(x) = x_1 \cdots x_n(0)^\omega$ implies that

$$x = \sum_{i=1}^{n} \frac{x_i}{q_i} = \sum_{i=1}^{n-1} \frac{x_i}{q_i} + \frac{x_n - 1}{q^n} + \frac{1}{q^n};$$

hence $x_1 \cdots x_{n-1}(x_n - 1)\gamma_q(1)$ is a different expansion of x. By iterating this reasoning if necessary, i.e. when $\gamma_q(1)$ is eventually minimal, we get a *non*-eventually minimal representation of x. By construction such a sequence is the lexicographically greatest non-eventually minimal representation of x and it is called *quasi-greedy expansion* of x. Quasi greedy expansions may also be constructed by mean of the *quasi-greedy algorithm* defined below.

DEFINITION 1.5 (quasi-greedy expansions). *Fix a real number $x \in [0,1]$. The* quasi-greedy *expansion of x is the sequence $\tilde{\gamma}_q(x) = \tilde{x}_1\tilde{x}_2 \cdots$ whose digits are the greatest in A_q satisfying the* strict *inequality:*

$$(14) \qquad\qquad \sum_{i=1}^{n-1} \frac{\tilde{x}_i}{q^i} + \frac{\tilde{x}_n}{q^n} < x.$$

for every $n \geq 1$.

4.2.2. **Parry's characterization of greedy expansions.** We remark that the quasi-greedy expansion of 1 can be explicitly defined starting by the greedy expansion of 1:

$$(15) \qquad \tilde{\gamma}_q(1) = \begin{cases} \gamma_q(1) & \text{if } \gamma_q(1) \text{ is not eventually minimal} \\ (\gamma_1 \ldots \gamma_{n-1}(\gamma_n - 1))^\omega & \text{if } \gamma_q(1) = \gamma_1 \ldots \gamma_n(0)^\omega \end{cases}$$

THEOREM 1.6 (W. Parry [Par60]). *Let $q > 1$, $x \in [0,1]$ and suppose $x_1x_2 \cdots \in A_q^{\mathbb{N}}$ be a sequence satisfying $x = \sum_{i=1}^{\infty} \frac{x_i}{q^i}$. Then $x_1x_2 \cdots$ is the greedy expansion of x if and only if for every $n \geq 1$:*

$$(16) \qquad\qquad x_{n+1}x_{n+2} \cdots <_{lex} \tilde{\gamma}_q(1).$$

COROLLARY 1.1. *The set of greedy expansions is shift-invariant.*

4.3. The q-shift. Since in Corollary 1.1 we stated that the set of q-expansions is shift invariant, *the closure of the set of the greedy expansions is a subshift*: it is denoted by S_q and it is called *q-shift*. We may characterize the set S_q by rewriting Theorem 1.6 as follows.

THEOREM 1.7. *Let $q > 1$ be a real number. A sequence $\mathbf{x} \in A^{\mathbb{Z}}$ belongs to S_q if and only if for all $n \in \mathbb{Z}$*

$$(17) \qquad\qquad x_{n+1}x_{n+2} \cdots \leq_{lex} \tilde{\gamma}_q(1).$$

EXAMPLE 1.3. *By taking as base the Golden Mean $G := \frac{1+\sqrt{5}}{2}$, we have $A_G = \{0,1\}$, $\gamma_G(1) = 11$ and, by (15), $\tilde{\gamma}_G(1) = (10)^\infty$. Then any sequence $(x_i)_{i\in\mathbb{Z}}$ satisfying (17) is such that:*

$$(18) \qquad\qquad \text{for all } n \in \mathbb{Z}, \quad x_{n+1}x_{n+2} \cdots \leq_{lex} (10)^\infty.$$

which is equivalent to

$$(19) \qquad\qquad \text{for all } n \in \mathbb{Z}, \text{ if } x_{n+1} = 1 \text{ then } x_{n+2} \neq 1.$$

This, together with Theorem 1.7, implies that any sequence in S_G does not countain any occurrence of the word 11. Namely, S_G avoids the finite set $\{11\}$ and, consequently, S_G is a set of finite type.

In previous example two features of the Golden mean are enlightened:

    - the greedy expansion of 1 is eventually minimal in base G;

- the subshift S_G is of finite type.

These two properties are actually equivalent in every non-integer base [IT74]. An analogous property has been proved in [Ber86] for the more general sofic q-shift. Hereafter we state such results.

THEOREM 1.8 (A. Bertrand [Ber86], S. Ito and Y. Takahashi [IT74]). *Let $q > 1$.*
(a) The q-shift is sofic if and only if $\gamma_q(1)$ is eventually periodic.
(b) The q-shift is of finite type if and only if $\gamma_q(1)$ is eventually minimal.

We conclude with the following result on the entropy of the q-shift.

THEOREM 1.9 (S. Ito and Y. Takahashi [IT74]). *The entropy of the q-shift is equal to $\log q$.*

4.4. Pisot bases. The numeration in Pisot base knew an increasing interest in view of the remarkable analogy with the case of expansions in integer base established in the following result.

THEOREM 1.10 (A. Bertrand [Ber77],K. Schmidt [Sch80]). *Let q be a Pisot number. Then a positive real number x has an eventually periodic greedy expansion in base q if and only if $x \in \mathbb{Q}(q)$.*

Schmidt also proved a partial converse of Theorem 1.10. Recall that a *Salem number* is an algebraic integer whose conjugates are not greater than 1 in modulus and with one equal to 1 in modulus.

THEOREM 1.11 (K. Schmidt [Sch80]). *Let q be a fixed real number. If any $x \in \mathbb{Q} \cap [0,1)$ has an eventually periodic expansion in base q then q must be Pisot or Salem.*

Another feature of the numeration in base Pisot is the realizability of some operations by mean of a finite automaton — e.g. addition, multiplication by a constant integer and digit set conversion are computable by a sequential finite automaton [Fro99, Fro03]; minimal weight expansions in Pisot base are recognizable by a finite automaton [FS08].

4.5. Redundant and unique representations. One of the main feature of non-integer base numeration systems is the *redundancy* of the representation, i.e. the existence of several expansions for the same value. The b-ary expansions have a very low redundancy: there exist exactly two different expansions only for numbers admitting a finite (eventually minimal in our notation) expansion. We generalized this property to the non-integer case in 4.2.1, but there is much more. In fact, starting from the papers [EJK90], [EJ92] and [EJK94] the cardinality of the expansions in non-integer bases has been intensively studied. Before recalling some classical results, we say that the expansion of a value x is *unique* in base q if there are no other sequences $(x_i)_{i \geq 1}$ satisfying $x = \sum_{i=1}^{\infty} x_i / q^i$. Moreover we fix the alphabet $A = \{0,1\}$ and we remark that if $q < 2$ then $A = A_q$.

(a) For every base $1 < q < G$, where G is the Golden Mean, every $x \in (0, \frac{1}{q-1})$ has a continuum of different expansions [EJK90].
(b) If $q = G$ then 1 has countable different expansions [EHJ91] and there not exist unique expansions.
(c) If $G < q < q_c$, where q_c is the Komornik-Loreti constant, then a countable many values in $[0, \frac{1}{q-1}]$ have an unique expansion [GS01].
(d) If $q = q_c$ then the (greedy) expansion of 1 is unique [KL98].
(e) If $q_c \leq q$ then a continuum of values in $[0, \frac{1}{q-1})$ has an unique expansion [GS01].
(f) For every $1 < q < 2$ the set of values with at most countable different expansions has however zero Lebesgue measure, namely almost every number in $[0, \frac{1}{q-1}]$ has a continuum of different expansions [Sid03].

REMARK 1.9. *The Golden Mean and the Komornik-Loreti constant are two critical bases respectively separating the non-existence of unique expansions, the existence of countable many unique expansions and the existence of a continuum of unique expansions. They also have particular properties related to the expansion of 1: in fact the Golden Mean is the smallest base expanding 1 in countable ways, and the Komornik-Loreti constant is the smallest base ensuring the uniqueness of the expansion of 1, namely it is the smallest univoque base.*

We recall we defined for every digit $x \in A$ the dual $\overline{x} = \max A - x$ and we also considered the dual of a sequence $\mathbf{x} = x_1 x_2 \cdots \in A^{\mathbb{N}}$: $\overline{\mathbf{x}} = \overline{x_1}\,\overline{x_2} \cdots$. The dependence between the uniqueness of an expansion and $\tilde{\gamma}_q(1)$ is explicited by the following characterizing theorem.

THEOREM 1.12 (Z. Daróczy and I. Kátai [DK93]). *Fix $q > 1$. An expansion $(x_i) \in A_q^{\mathbb{N}}$ is unique in base q if and only if for every n:*

(20) $$x_{n+1}x_{n+2}\cdots < \tilde{\gamma}_q(1) \quad \text{if } x_n \text{ is less than the first digit of } \tilde{\gamma}_q(1);$$

(21) $$\overline{x}_{n+1}\overline{x}_{n+2}\cdots < \tilde{\gamma}_q(1) \quad \text{if } x_n > 0.$$

Theorem 1.12 can be reformulated in terms of *lazy* and *quasi-lazy* expansions.

DEFINITION 1.6 (Lazy and quasi-lazy expansions). *The* lazy expansion $\lambda_q(x)$ *of $x \in [0, \frac{\lfloor q \rfloor}{q-1}]$ is the lexicographically smallest expansion of x.*

The quasi-lazy expansion $\tilde{\lambda}_q(x)$ *of x is the lexicographically smallest not eventually maximal expansion of x.*

REMARK 1.10. *Lazy expansions can be obtained in several ways: e.g. by the iteration of the L_q map. L_q is named* lazy map *and it is defined from the interval $(\frac{\lfloor q \rfloor}{q-1} - 1, \frac{\lfloor q \rfloor}{q-1}]$ onto itself: $L_q(x) = qx - \lfloor \frac{\lfloor q \rfloor}{q-1} - qx \rfloor$. The procedure is similar to the case of T_q, the q-transform.*

There also exists a lazy algorithm *for any value $x \in [0, \frac{\lfloor q \rfloor}{q-1}]$. Such an algorithm consists in choosing for every $n \geq 1$ the smallest digit $x_n \in A_q$ satisfying:*

(22) $$\sum_{i=1}^{n-1} \frac{x_i}{q^i} + \frac{x_n}{q^n} + \frac{\lfloor q \rfloor}{q^n(q-1)} \geq x.$$

By replacing $\geq$ with a strict inequality in the equation above, we get the quasi-lazy algorithm *and the resulting sequence is the quasi-lazy expansion of x.*

Finally lazy and quasi-lazy expansions can be derived by the greedy and quasi-greedy expansions as follows. Set $x \in [0, \frac{\lfloor q \rfloor}{q-1}]$. Then

(23) $$\lambda_q(x) = \overline{\gamma_q(\frac{\lfloor q \rfloor}{q-1} - x)};$$

and

(24) $$\tilde{\lambda}_q(x) = \overline{\tilde{\gamma}_q(\frac{\lfloor q \rfloor}{q-1} - x)}.$$

In view of (24) we may reformulate Theorem 1.12 as follows.

COROLLARY 1.2. *Fix $q > 1$. An expansion $(c_i) \in A_q^{\mathbb{N}}$ is unique in base q if and only if for every n:*

(25) $$c_{n+1}c_{n+2}\cdots < \tilde{\gamma}_q(1) \quad \text{if } c_n \text{ is less than the first digit of } \gamma_q(1);$$

(26) $$c_{n+1}c_{n+2}\cdots > \tilde{\lambda}_q(1) \quad \text{if } c_n > 0.$$

CHAPTER 2

Expansions in complex base

In this chapter we deal with expansions with digits in arbitrary alphabets and bases in the form $pe^{\frac{2\pi i}{n}}$ with $p > 1$ and $n \in \mathbb{N}$. We study the convex hull of the set of representable numbers by giving first a geometrical description then an explicit characterization of its extremal points. We also show a characterizing condition for the convexity of the set of representable numbers.

1. Introduction

The first number systems in complex base seem to be those in base $2i$ with alphabet $\{0, 1, 2, 3\}$ and the one in base $-1 + i$ and alphabet $\{0, 1\}$, respectively introduced by Knuth in [Knu60] and by Penney in [Pen65]. After that many papers were devoted to representability with bases belonging to larger and larger classes of complex numbers, e.g. see [KS75] for the Gaussian integers in the form $-n \pm i$ with $n \in \mathbb{N}$, [KK81] for the quadratic fields and [DK88] for the general case. Loreti and Komornik pursued the work in [DK88] by introducing a greedy algorithm for the expansions in complex base with non rational argument [KL07]. In the eighties a parallel line of research was developed by Gilbert. In [Gil81] he described the fractal nature of the set of the representable numbers, e.g. the set of representable in base $-1 + i$ with digits $\{0, 1\}$ coincides with the fascinating space-filling twin dragon curves [Knu71]. Hausdorff dimension of some set of representable numbers was calculated in [Gil84] and a weaker notion of self-similarity was introduced for the study of the boundary of the representable sets [Gil87]. Complex base numeration systems and in particular the geometry of the set of representable numbers have been widely studied by the point of view of their relations with iterated function systems and tilings of the complex plane, too. For a survey on the topology of the tiles associated to bases belonging to quadratic fields we refer to [AT04].

Expansions in complex base have several applications. For example, in the context of computer arithmetics, the interesting property of these numerations systems is that they allow multiplication and division of complex base in a unified manner, without treating real and imaginary part separately — see [Knu71], [Gil84] and [FS03]. Representation in complex base have been also used in cryptography with the purpose of speeding up honerous computations such as modular exponentiations [DJW85] and multiplications over elliptic curves [Sol00]. Finally we refer to [Pic02] for a dissertation on the applications of the numerations in complex base to the compression of images on fractal tilings.

Organization of the chapter. Most of the arguments of this chapter laying on geometrical properties, in Section 2 we show some results on complex plane geometry. In Section 3 we characterize the shape and the extremal points of the convex hull of the set of representable numbers. In Section 4 we give a necessary and sufficient condition to have a *convex* set of representable numbers, this property being sufficient for a full representability of complex numbers. Finally Section 5 contains an overview of the main results and possible further developments.

2. Geometrical background

By using the isometry between $\mathbb{C}$ and $\mathbb{R}^2$ we extend to $\mathbb{C}$ some definitions which are proper of the plane geometry.

Elements of $\mathbb{C}$ are considered vectors (or sometimes points) and we endowe $\mathbb{C}$ with the scalar product $u \cdot v := |u||v|\cos(\arg u - \arg v)$. In this setting a *semiplane* $S_{v,n}$ in $\mathbb{C}$ may be defined starting by a point v and a vector n, the *normal vector* of $S_{v,n}$, with the formula $S_{v,n} := \{x \in \mathbb{C} \mid (x - v) \cdot n \geq 0\}$. A *polygon* is the bounded intersection of a finite number of semiplanes. A polygon can be equivalently be defined as the finite region of $\mathbb{C}$ contained in a closed chain of segments, the *edges*, whose endpoints are the *vertices*. If two adjacent edges belong to the same line, namely if they are adjacent and parallel, then they are called *consecutive* and their common endpoint is a *degenerate vertex*. If a vertex is not degenerate, it is called *extremal point*. Starting from an ordered set of vertices V we may construct the corresponding polygon by joining with segments the elements of V. The relation between the sets of vertices and polygons is clearly not one-to-one, e.g. by removing the degenerate vertices the polygon does not change. Thus we may call *equivalent* those (ordered) set of vertices defining the same polygon. From now on, when we say that the vertices of a given polygon P are listed in a certain set V we mean that V belongs to the equivalence class of vertices induced by P and denoted $\mathcal{V}(P)$.

The *convex hull* of a set $X \subset \mathbb{C}$ is the smallest convex set containing X. When X is finite, its convex hull is called *simplex* of X and it is a polygon whose vertices are in X. The relation below immediately follows by the definitions:

$$(27) \qquad P \text{ is a convex polygon with vertices in } X \quad + \quad X \subseteq P \quad \Leftrightarrow \quad P \text{ is the simplex of } X.$$

REMARK 2.1. *A convex polygon is the simplex of its vertices.*

NOTATION 2.1. *The convex hull of a set X is denoted by $H_{con}(X)$.*

Through this chapter we deal only with convex polygons. We now study the convex hull of the set $P \cup (P + t)$, being P a (convex) polygon, $t \in \mathbb{C}$ and $P + t := \{x + t \mid x \in P\}$.

We assume that the vertices are indexed so to result counter-clockwise ordered. The operations on the indices of the vertices are considered modulo their number. Normal vectors of the edges are assumed to have positive scalar product with any point of the polygon, namely they are "internal" to the polygon.

THEOREM 2.1. *Let P be a polygon with vertices in $V_P := \{v_0, \ldots, v_{l-1}\}$ and let t be a translation vector. Then there exists only two indices $i_1, i_2 \in \{0, \ldots, l-1\}$ such that*

$$(28) \qquad n_{i_1-1} \cdot t < 0 \quad and \quad n_{i_1} \cdot t \geq 0$$

$$(29) \qquad n_{i_2-1} \cdot t > 0 \quad and \quad n_{i_2} \cdot t \leq 0.$$

Moreover the convex hull of $P \cup (P + t)$ is a polygon whose vertices are:

$$(30) \qquad v_{i_1}, \ldots, v_{i_2}, v_{i_2} + t, \ldots, v_{i_1-1} + t, v_{i_1} + t.$$

PROOF. We divide the proof in several parts. In Part 1 we prove the first statement of the theorem, Part 2 simply contains a definition of the polygon P_t whose vertices are listed in (30). In Part 3, Part 4 and Part 5 are proved technical results ensuring $P \cup (P + t) \subseteq P_t$, this relation being proved in Part 6. We conclude the proof in Part 7. Figure 1 shows some stages of this proof.

Part 1. The indices i_1 and i_2 are well defined.

Let n_i be the normal vector to the edge vector $v_{i+1} - v_i$, for $i = 0, \dots, l-1$. First note that the convexity of P is equivalent to the relation

(31) $$\arg(n_i) \leq \arg(n_{i+1})$$

for every $i = 0, \dots, l-1$.

Since $n_i \cdot t \geq 0$ if and only if $|\arg n_i - \arg t| \leq \pi/2$, the monotonicity of the $\arg(n_i)$'s implies that there exist only two indices i_1 and i_2 respectively satisfying (28) and (29).

Part 2. Definition of P_t, the candidate convex hull of $P \cup (P+t)$.

Define the index sets:

$$I^+ := \{ i \mid n_i \cdot t \geq 0 \} \quad \text{and} \quad I^- := \{ i \mid n_i \cdot t < 0 \};$$

recall that $S_{v,n}$ is the semiplane $\{ x \in \mathbb{C} \mid (x - v) \cdot n \geq 0 \}$ and define

$$P_t := \bigcap_{i \in I^+} S_{v_i, n_i} \cap S_{v_{i_2}, t^\perp} \cap \left(\bigcap_{i \in I^-} S_{v_i+t, n_i} \right) \cap S_{v_{i_1}, -t^\perp}.$$

By construction, P_t is the (convex) polygon whose vertices are listed in (30).

Part 3. P and $P+t$ are contained in $S_{v_{i_2}, t^\perp} \cap S_{v_{i_1}, -t^\perp}$.

First simplification. As

$$(x + t) \cdot \pm t^\perp = x \cdot \pm t^\perp + t \cdot \pm t^\perp = x \cdot \pm t^\perp,$$

we just need to show that $P \subseteq S_{v_{i_2}, t^\perp} \cap S_{v_{i_1}, -t^\perp}$, because this implies $P + t \subseteq S_{v_{i_2}, t^\perp} \cap S_{v_{i_1}, -t^\perp}$.

Second simplification. First remark that

- $P = \bigcap_{i=0}^{l-1} S_{v_i, n_i} \subseteq S_{v_{i_2-1}, n_{i_2-1}} \cap S_{v_{i_2}, n_{i_2}}$;
- $S_{v_{i_2-1}, n_{i_2-1}} = S_{v_{i_2}, n_{i_2-1}}$;
- $P = \bigcap_{i=0}^{l-1} S_{v_i, n_i} \subseteq S_{v_{i_1-1}, n_{i_1-1}} \cap S_{v_{i_1}, n_{i_1}}$;
- $S_{v_{i_1-1}, n_{i_1-1}} = S_{v_{i_1}, n_{i_1-1}}$.

Hence

(32) $$S_{v_{i_2}, n_{i_2-1}} \cap S_{v_{i_2}, n_{i_2}} \subseteq S_{v_{i_2}, t^\perp}.$$

and

(33) $$S_{v_{i_1}, n_{i_1-1}} \cap S_{v_{i_1}, n_{i_1}} \subseteq S_{v_{i_1}, -t^\perp}.$$

are sufficient conditions for $P \subseteq S_{v_{i_2}, t^\perp} \cap S_{v_{i_1}, -t^\perp}$. We show only (32), because the proof of (33) is similar.

Proof of (32). Set $x \in S_{v_{i_2}, n_{i_2-1}} \cap S_{v_{i_2}, n_{i_2}}$. Then, by the definition of semiplane, $(x - v_{i_2}) \cdot n_{i_2-1} \geq 0$ and $(x - v_{i_2}) \cdot n_{i_2} \geq 0$ and we may deduce:

(34) $$\left| \arg(x - v_{i_2}) - \arg n_{i_2-1} \right| \leq \frac{\pi}{2}$$

and

(35) $$\left| \arg(x - v_{i_2}) - \arg n_{i_2} \right| \leq \frac{\pi}{2}.$$

We now distinguish the cases $\arg(x - v_{i_2}) \geq \arg t^\perp$ and $\arg(x - v_{i_2}) < \arg t^\perp$.

Suppose $\arg(x - v_{i_2}) \geq \arg t^\perp$. As (34) implies $\arg(x - v_{i_2}) - \arg n_{i_2-1} \leq \frac{\pi}{2}$ and the definition of i_2, and in particular $t \cdot n_{i_2-1} > 0$, implies $\arg n_{i_2-1} - \arg t - \frac{\pi}{2} < 0$, we have:

$$| \arg(x - v_{i_2}) - \arg t^\perp | = \arg(x - v_{i_2}) - \arg t - \frac{\pi}{2}$$

$$= \arg(x - v_{i_2}) - \arg n_{i_2-1} + \arg n_{i_2-1} - \arg t - \frac{\pi}{2} \leq \frac{\pi}{2}.$$

Hence $(x - v_{i_2}) \cdot t^\perp \geq 0$ and $x \in S_{v_{i_2}, t^\perp}$.

Suppose now $\arg(x - v_{i_2}) < \arg t^\perp$. First note that since $t \cdot n_{i_2-1} > 0$ and since it follows by (31) that $\arg n_{i_2-1} \leq \arg n_{i_2}$ we get

$$(36) \qquad\qquad\qquad \arg t - \arg n_{i_2} \leq \arg t - \arg n_{i_2-1} < \frac{\pi}{2}.$$

Now $t \cdot n_{i_2} \leq 0$ implies $| \arg t - \arg n_{i_2} | \geq \pi/2$ and this, together with (36) implies $\arg t - \arg n_{i_2} \leq -\pi/2$ and hence

$$(37) \qquad\qquad\qquad\qquad \arg t + \frac{\pi}{2} - \arg n_{i_2} \leq 0.$$

Moreover we have by (35) that $\arg n_{i_2} - \arg(x - v_{i_2}) \leq \frac{\pi}{2}$ and this, together with (37) implies:

$$| \arg(x - v_{i_2}) - \arg t^\perp | = \arg t + \frac{\pi}{2} - \arg(x - v_{i_2})$$

$$= \arg t + \frac{\pi}{2} - \arg n_{i_2} + \arg n_{i_2} - \arg(x - v_{i_2}) \leq \frac{\pi}{2}.$$

As in the previous case, we may deduce by the inequality above that $x \in S_{v_{i_2}, t^\perp}$.

Part 4. If $i \in I^-$ then $S_{v_i, n_i} \subseteq S_{v_i+t, n_i}$.

If $x \in S_{n_i, v_i}$ then $i \in I^-$ and the definition of I^- imply

$$(x - (v_i + t)) \cdot n_i = (x - v_i) \cdot n_i - t \cdot n_i \geq (x - v_i) \cdot n_i \geq 0$$

and, consequently, $x \in S_{v_i+t, n_i}$.

Part 5. If $i \in I^+$ then $S_{v_i+t, n_i} \subseteq S_{v_i, n_i}$.

Similarly to the proof in Part 4, $i \in I^+$ and the definition of I^+ imply that for every $x \in S_{v_i+t, n_i}$:

$$(x - v_i) \cdot n_i = (x - (v_i + t)) \cdot n_i + t \cdot n_i \geq (x - (v_i + t)) \cdot n_i \geq 0$$

and, consequently, $x \in S_{v_i, n_i}$.

Part 6. $P \cup (P + t) \subseteq P_t$.

We separately prove $P \subseteq P_t$ and $P + t \subseteq P_t$.

$$P \stackrel{Part\ 3}{=} P \cap \left(S_{v_{i_2}+t^\perp} \cap S_{v_{i_1},-t^\perp} \right) = \bigcap_{i=0}^{l-1} S_{v_i, n_i} \cap \left(S_{v_{i_2}+t^\perp} \cap S_{v_{i_1},-t^\perp} \right)$$

$$= \bigcap_{i \in I^+} S_{v_i, n_i} \cap \bigcap_{i \in I^-} S_{v_i, n_i} \cap \left(S_{v_{i_2}+t^\perp} \cap S_{v_{i_1},-t^\perp} \right)$$

$$\stackrel{Part\ 4}{\subseteq} \bigcap_{i \in I^+} S_{v_i, n_i} \cap \bigcap_{i \in I^-} S_{v_i+t, n_i} \cap \left(S_{v_{i_2}+t^\perp} \cap S_{v_{i_1},-t^\perp} \right) = P_t$$

and

$$P + t \overset{Part\ 3}{=} (P + t) \cap \left(S_{v_{i_2}+t^\perp} \cap S_{v_{i_1},-t^\perp} \right) = \bigcap_{i=0}^{l-1} S_{v_i+t,n_i} \cap \left(S_{v_{i_2}+t^\perp} \cap S_{v_{i_1},-t^\perp} \right)$$

$$= \bigcap_{i\in I^+} S_{v_i+t,n_i} \cap \bigcap_{i\in I^-} S_{v_i+t,n_i} \cap \left(S_{v_{i_2}+t^\perp} \cap S_{v_{i_1},-t^\perp} \right)$$

$$\overset{Part\ 5}{\subseteq} \bigcap_{i\in I^+} S_{v_i,n_i} \cap \bigcap_{i\in I^-} S_{v_i+t,n_i} \cap \left(S_{v_{i_2}+t^\perp} \cap S_{v_{i_1},-t^\perp} \right) = P_t.$$

Part 7. P_t is the convex hull of $P \cup (P + t)$.

We prove $P_t = H_{con}(P \cup (P + t))$ by double inclusion. The inclusion $\supseteq$ immediately follows by the fact that P_t is a convex set containing $P \cup (P + t)$. On the other hand $P \cup (P + t) \subset P_t$ also implies $V_P \cup (V_P + t) \subset P_t$ and the vertices of P_t, listed in (30), are in V_P or in $V_P + t$. Then it follows by the relation (27) that $H_{con}(V_P \cup (V_P + t)) = P_t$. As $P_t = H_{con}(V_P \cup (V_P + t)) \subseteq H_{con}(P \cup (P + t))$, thesis follows. □

COROLLARY 2.1. *Let P be a convex polygon with l edges and let t be a translation vector. Then $P_t := H_{con}(P \cup (P + t))$ has $l + 2$ (possibly consecutive) edges. Moreover l edges of P_t are parallel to the edges of P and 2 edges are parallel to the translation.*

PROOF. It immediately follows by the list of vertices given in (30). □

COROLLARY 2.2. *Let P be a convex polygon with e extremal points and let t be a translation vector. Then:*

(a) *if t is not parallel to any edge of P then $P_t = H_{con}(P \cup (P + t))$ has $e + 2$ extremal points;*
(b) *if t is parallel to 1 edge of P then $P_t = H_{con}(P \cup (P + t))$ has $e + 1$ extremal points;*
(c) *if t is parallel to 2 edges of P then $P_t = H_{con}(P \cup (P + t))$ has e extremal points.*

PROOF. Let us suppose that P has l edges and let us denote $d := l - e$ the number of degenerate vertices. By definition, a vertex v is degenerate if the normal vectors of its adjacent edges are equal. We denote n_i the normal vector to the edge vector $v_{i+1} - v_i$. It follows by (30) that the normal vectors of P_t are the following:

$$n_{i_1}, \dots, n_{i_2-1}, t^\perp, n_{i_2}, \dots, n_{i_1-1}, -t^\perp$$

with i_1 and i_2 satisfying:

$$n_{i_1-1} \cdot t < 0 \quad \text{and} \quad n_{i_1} \cdot t \geq 0;$$
$$n_{i_2-1} \cdot t \geq 0 \quad \text{and} \quad n_{i_2} \cdot t < 0.$$

It immediately follows by the definition of i_1 and i_2 that $n_{i_1-1} \neq n_{i_1}$ and $n_{i_2-1} \neq n_{i_2}$. Moreover $t^\perp$ is not equal to n_{i_1-1} and n_{i_2}, otherwise the scalar products $n_{i_1-1} \cdot t$ and $n_{i_2} \cdot t$ would be null. Thus by denoting l_t the number of vertices of P_t, by d_t the number of the degenerate vertices and by e_t the number of extremal points, by Corollary 2.1 we have that $l_t = l + 2$ and :

$$d_t = \begin{cases} d & \text{if } t^\perp \text{ is not parallel to } n_{i_1} \text{ and to } n_{i_2-1}; \\ d + 1 & \text{if } t^\perp \text{ is parallel to } n_{i_1} \text{ or to } n_{i_2-1}; \\ d + 2 & \text{if } t^\perp \text{ is parallel to } n_{i_1} \text{ and to } n_{i_2-1}. \end{cases}$$

Hence thesis follows by the relation $e_t = l_t - d_t$. □

(a) P and $P + t$

(b) Selection of special indices: $i_1 = 1$ and $i_2 = 3$

(c) $P \cup (P + t) \subset S_{v_{i_1}, -t^\perp} \cap S_{v_{i_2}, t^\perp}$

(d) $\bigcap_{i \in I^+} S_{v_i + t, n_i} \subseteq \bigcap_{i \in I^+} S_{v_i, n_i}$

(e) $\bigcap_{i \in I^-} S_{v_i, n_i} \subseteq \bigcap_{i \in I^-} S_{v_i + t, n_i}$

(f) P_t is the convex hull of $P \cup (P + t)$
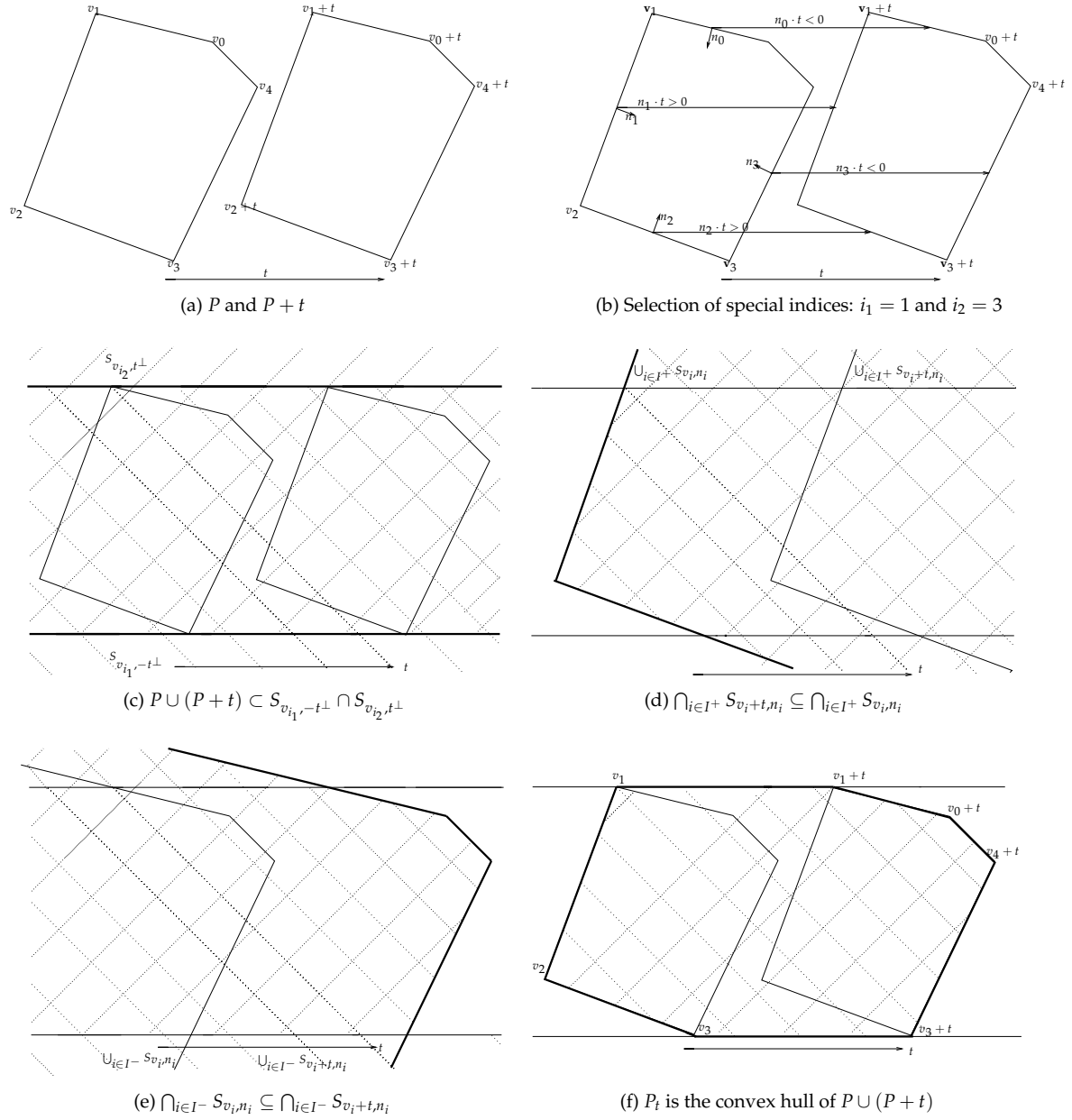
Figure 1: Various stages of the proof of Theorem 2.2: in particular (b) corresponds to Part 1, (c),(d) and (e) to Part 3,4 and 5 and (f) is a graphical representation of Part 6 and Part 7.

COROLLARY 2.3. *Let P be a polygon with $2l$ pairwise parallel edges. Set $n_{v_i}^- := n_{i-1} = (v_i - v_{i-1})^\perp$ and $n_{v_i}^+ := n_i = (v_{i+1} - v_i)^\perp$ with $i = 0, \dots, 2l - 1$ and let t be a translation vector. Moreover let i_1 be an index satisfying:*

$$(38) \qquad\qquad n_{v_{i_1}}^- \cdot t < 0 \quad \text{and} \quad n_{v_{i_1}}^+ \cdot t \geq 0$$

or

$$(39) \qquad\qquad n_{v_{i_1}}^- \cdot t = 0 \quad \text{and} \quad n_{v_{i_1}}^+ \cdot t > 0$$

If we define $P_t := H_{con}(P \cup (P + t))$, then

(40) $$\{v_{i_1}, \dots, v_{i_1+l}, v_{i_1+l} + t \dots, v_{i_1+2l}(\equiv v_{i_1}) + t\} \in \mathcal{V}(P_t).$$

PROOF. If (38) holds, it suffices to note that the parallelism between the edges implies $n_{v_{i+l}} = -n_{v_i}$, hence $n_{v_{i_1+l-1}} \cdot t > 0$ and $n_{v_{i_1+l-1}} \cdot t \leq 0$. Then thesis follows immediately by Theorem 2.1 with $i_2 := i_1 + l$.

If (39) is true, then consider the following notations. Set a the smallest integer such that $n_a \cdot t = 0$. We may assume without loss of generality that $a = i_1 - 1$ and, consequently, that $n_{v_a}^- \cdot t < 0$, $n_{v_a}^+ \cdot t = n_{v_{a+1}}^- \cdot t = 0$ and $n_{v_{a+1}}^+ \cdot t > 0$. Otherwise any other vertex complied between v_a and v_{i_1} would be degenerate and it would not act on the shape of the polygon. Now, since a satisfies (38), the reasonings in the first part of the proof imply that we may apply (40) to a and get:

(41) $$V := \{v_a, v_{a+1}, \dots, v_{a-1+l}, v_{a+l}, v_{a+l} + t, v_{a+1+l} + t, \dots, v_{a-1} + t, v_a + t\} \in \mathcal{V}(P_t).$$

The vertices v_a and, for reasons of symmetry, $v_{a+l} + t$ are degenerate vertices because:

$$(v_{a+1} - v_a)^\perp \cdot t = 0 = -t^\perp \cdot t = (v_a - (v_a + t))^\perp \cdot t.$$

Hence

$$V \equiv \{v_{a+1}, \dots, v_{a-1+l}, v_{a+l}, v_{a+1+l} + t, \dots, v_{a-1} + t, v_a + t\} \in \mathcal{V}(P_t).$$

Now let us consider the list of vertices given in (40). By recalling $a = i_1 - 1$, we get

$$V' := \{v_{a+1}, \dots, v_{a-1+l}, v_{a+l}, v_{a+1+l}, v_{a+1+l} + t, \dots, v_{a-1} + t, v_a + t, v_{a+1} + t\}$$

and, similarly to the case V, we may deduce that $v_{a+1} + t$ and v_{a+1+l} are degenerate vertices. Hence

$$V' \equiv \{v_{a+1}, \dots, v_{a-1+l}, v_{a+l}, v_{a+1+l} + t, \dots, v_{a-1} + t, v_a + t\} \equiv V \in \mathcal{V}(P_t).$$

$\square$

REMARK 2.2. *The notations in Corollary 2.3 are slightly modified with respect the previous results in view of its further application to the proof of Theorem 2.3.*

We conclude this section with the following result on the convexity of $P \cup P + t$.

LEMMA 2.1. *Let P be a convex polygon with $2l$ pairwise parallel edges and let t be a translation vector. Suppose that t is parallel to the edge e.*
Then $P \cup (P + t)$ is a convex set if and only if $|t| \leq |e|$.

PROOF. We may assume without loss of generality that the vertices adjacent to e are extremal points equal to 0 and 1 so that $e = [0, 1]$, $|e| = 1$ and, as e and t are parallel, $t \subset \mathbb{R}$. We finally assume $t \geq 0$.

Only if part.
Suppose $|t| > 1$. Since P is convex then $P \cap \mathbb{R} = [0, 1]$ and $(P + t) \cap \mathbb{R} = [|t|, 1 + |t|]$, then $(P \cup (P + t)) \cap \mathbb{R} = [0, 1] \cup [|t|, 1 + |t|]$. As $|t| > 1$ these intervals are disjoint, hence $P \cup (P + t)$ is not convex.

If part
Define $K_P := \max\{\Im(x) \mid x \in P\}$ and suppose $K_P \geq 0$: this is equivalent to assume that the polygon belongs to $\{x \in \mathbb{C} | \Im(x) \geq 0\}$ and clearly it does not imply a loss of generality . Since P is convex, for every $0 \leq k \leq K$ the set $P \cap \{x \in \mathbb{C} | \Im(x) = k\}$ is a segment in the form $I_k + i \cdot k$. As the

edges of P are pairwise parallel and P is convex, $|I_k| \geq |e| = 1$. Hence, by denoting $I_k := [a_k, b_k]$, $|t| < 1$ implies:

$$
\begin{aligned}
(I_k + i \cdot k) \cup (I_k + i \cdot k + t) &= ([a_k, b_k] \cup [a_k + t, b_k + t]) + i \cdot k \\
&= [a_k, b_k + t] + i \cdot k \\
&\subset P \cup (P + t).
\end{aligned}
$$
(42)

We want to prove that $P \cup (P + t)$ is a convex set by showing that it contains any convex combination of its points. So fix $x, y \in P \cup P + t$. If x and y are both in P or in $P + t$, the convexity of P implies the thesis. Otherwise suppose $x \in P$ and $y \in P + t$ and consider a convex combination $\lambda x + (1 - \lambda)y$, with $\lambda \in [0, 1]$. Remark that $y \in P + t$ implies that $x' := y - t \in P$ and we have:

$$
\lambda x + (1 - \lambda)y = \lambda x + (1 - \lambda)x' + (1 - \lambda)t.
$$

Since x and x' are both in P then $x'' := \lambda x + (1 - \lambda)x' \in P$ and, in particular, x'' belongs to $I_k + i \cdot k$ for some $0 \leq k \leq K$. Thus $t \geq 0$ implies

$$
a_k \leq \Re(x'') + (1 - \lambda)\Re(t) \leq b_k + \Re(t) = b_k + |t|.
$$

By (42) we have $\lambda x + (1 - \lambda)y = x'' + (1 - \lambda)t \in I_k + i \cdot k \subset P \cup (P + t)$ and this completes the proof. $\qquad\square$

3. Characterization of the convex hull of representable numbers

In this section we investigate the shape of the convex hull of the set of representable numbers in base $q_{n,p} := pe^{\frac{2\pi}{n}i}$ and with alphabet A. We adopt the following notations.

NOTATION 2.2. *We denote by $\Lambda_{n,p,A}$ the set of representable numbers in base $q_{n,p} := pe^{\frac{2\pi}{n}i}$ and with alphabet A, namely $\Lambda_{n,p,A} =: \{\sum_{k=1}^{\infty} x_k q_{n,p}^{-k} \mid x_k \in A\}$. We define $X_{n,p} := \{\sum_{j=0}^{n-1} x_j q_{n,p}^{j} \mid x_j \in \{0, 1\}\}$ and $P_{n,p}$ the convex hull of $X_{n,p}$, namely $P_{n,p} := H_{con}(X_{n,p})$. As $X_{n,p}$ is finite, $P_{n,p}$ is a polygon. When p is fixed, it is usually omitted in the subscripts.*

The following result represents a first simplification of our problem: in fact the characterization of the convex hull of the infinite set $\Lambda_{n,p,A}$ is showed to be equivalent to the study of $P_{n,p}$, namely the convex hull of the (finite) set $X_{n,p}$, i.e. $P_{n,p}$.

PROPOSITION 2.1. *For every $n \geq 1$, $p > 1$ and $q_{n,p} = pe^{\frac{2\pi}{n}i}$:*

$$
H_{con}(\Lambda_{n,p,A}) = \frac{\max A - \min A}{p^n - 1} \cdot P_{n,p} + \sum_{j=0}^{n-1} q_{n,p}^{j} \min A.
$$
(43)

PROOF. We may deduce by the relation $q_{n,p}^n = p^n$ that for any sequence $(x_k) \in A^{\omega}$:

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{x_k}{q_{n,p}^k} &= \sum_{j=0}^{n-1} \sum_{k=1}^{\infty} \frac{x_{kn-j}}{q_{n,p}^{kn-j}} \\
&= \sum_{j=0}^{n-1} q_{n,p}^{j} \sum_{k=1}^{\infty} \frac{x_{kn-j}}{q_{n,p}^{kn}} \\
&= \sum_{j=0}^{n-1} q_{n,p}^{j} \sum_{k=1}^{\infty} \frac{x_{kn-j}}{p^{kn}} \\
&= \sum_{j=0}^{n-1} q_{n,p}^{j} \tilde{x}_j
\end{aligned}
$$
(44)

with $\tilde{x}_j := \sum_{k=1}^{\infty} x_{kn-j} p^{-kn} \in \left[\frac{\min A}{p^n-1}, \frac{\max A}{p^n-1} \right]$. Hence:

$$H_{con} \left(\Lambda_{n,p,A} \right) = H_{con} \left(\left\{ \sum_{j=0}^{n-1} q_{n,p}^j \tilde{x}_j \mid \tilde{x}_j \in \left\{ \frac{\min A}{p^n-1}, \frac{\max A}{p^n-1} \right\} \right\} \right)$$

$$= \frac{\max A - \min A}{p^n - 1} H_{con}(X_{n,p}) + \sum_{j=0}^{n-1} q_{n,p}^j \min A.$$

$\square$

We now give a geometrical description of the convex hull of $\Lambda_{n,p,A}$.

THEOREM 2.2 (Convex hull of the representable numbers in complex base). *For every $n \geq 1$, $p > 1$ and $q_{n,p} = pe^{\frac{2\pi}{n}i}$, the convex hull $H_{con}(\Lambda_{n,p,A})$ is a polygon with the following properties:*
(a) *the edges are pairwise parallel to $q_{n,p}^0, \ldots, q_{n,p}^{n-1}$;*
(b) *if n is odd then $H_{con}(\Lambda_{n,p,A})$ has $2n$ extremal points;*
(c) *if n is even then $H_{con}(\Lambda_{n,p,A})$ has n extremal points.*

PROOF. First recall that we defined

(45) $$X_{n,p} = \left\{ \sum_{j=0}^{n-1} x_j q_{n,p}^j \mid x_j \in \{0,1\} \right\} \quad \text{and} \quad P_{n,p} = H_{con}(X_{n,p}).$$

Our proof is oriented to show that $P_{n,p}$ has the properties (a), (b) and (c) ; in fact being these properties invariant by rescaling and translation, Proposition 2.1 ensures the thesis.

In order to lighten the notations, the subscript p is omitted so that $q_n = q_{n,p}$, $X_n = X_{n,p}$ and $P_n = P_{n,p}$. We divide the proof in three parts.

Part 1. The edges of P_n are pairwise parallel to $q_n^0, \ldots, q_n^{n-1}$.
To the end of studying P_n, we consider the sets

$$X_{n,m} := \left\{ \sum_{j=0}^{m-1} x_j q^j \mid x_j \in \{0,1\} \right\};$$

for $m = 1, \ldots, n$. Clearly $X_{n,n} = X_n$ and the following recursive relation holds:

(46) $$\begin{cases} X_{n,1} = \{0,1\}; \\ X_{n,m} = X_{n,m-1} \cup (X_{n,m-1} + q_n^{m-1}) \end{cases}$$

which, when applied to the convex hulls $P_{n,m} := H_{con}(X_{n,m})$, becomes:

(47) $$\begin{cases} P_{n,1} = [0,1]; \\ P_{n,m} = H_{con} \left(P_{n,m-1} \cup (P_{n,m-1} + q_n^{m-1}) \right). \end{cases}$$

We remark that $P_{n,1}$ can be looked at as a polygon with two vertices and with two overlapped and parallel to q_n^0 edges. Moreover since the convex hull of a finite set is a polygon, $P_{n,m}$ fullfills the conditions of Corollary 2.1 for every $m = 1, \ldots, n$. Hence by iteratively applying Corollary 2.1 we deduce that $P_{n,m}$ has pairwise parallel edges and every couple of edges is parallel either to q_n^0 or to any of the successive translation, i.e. $q_n^1, \ldots, q_n^{m-1}$. When $m = n$ we get (a).

Part 2. If n is odd then P_n has $2n$ extremal points.
First observe that n odd implies that $q_n^0, \ldots, q_n^{n-1}$ are pairwise independent. We showed above that for every $m = 1, \ldots, n$ the edges of the polygon $P_{n,m-1}$ are parallel to $q_n^0, \ldots, q_n^{m-2}$ and, consequently, the translation q_n^{m-1} is not parallel to any edge. Hence, denoting $e_{n,m}$ the number of the

extremal points of $P_{n,m}$, the first part of Corollary 2.1 implies that $e_{n,m}$ is defined by the recursive relation:

$$\begin{cases} e_{n,1} = 2 \\ e_{n,m} = e_{n,m-1} + 2 \end{cases}$$

for every $m = 1, \ldots, n$. Hence $e_{n,n} = 2n$.

Part 3. If n is even then P_n has n extremal points.

If n is even then $q_n^{m+n/2}$ is parallel to q_n^m for every $m = 1, \ldots, n/2$. Since $P_{n,m}$ has pairwise parallel edges, we deduce by (a) and (c) in Corollary 2.2 that $e_{n,m}$ is defined by the relation:

$$\begin{cases} e_{n,0} = 0; \\ e_{n,m} = e_{n,m-1} + 2 & \text{if } m = 1, \ldots, n/2; \\ e_{n,m} = e_{n,m-1} & \text{if } m = n/2+1, \ldots, n; \end{cases}$$

hence $e_{n,n} = n$ and this concludes the proof. $\qquad\square$

EXAMPLE 2.1. *If $n = 3$ and if $p > 1$ then for every alphabet A the convex hull of $\Lambda_{n,p,A}$ is an hexagon. If $n = 4$ and if $p > 1$ then for every alphabet A the convex hull of $\Lambda_{n,p,A}$ is a rectangle.*
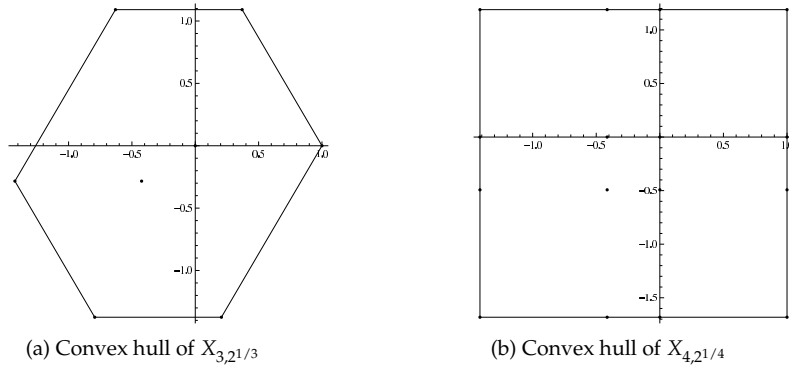

(a) Convex hull of $X_{3,2^{1/3}}$ (b) Convex hull of $X_{4,2^{1/4}}$

Figure 2: Convex hull of $X_{3,2^{1/3}}$ and of $X_{4,2^{1/4}}$. Remark that when $A = \{0,1\}$ then $P_{n,p} = H_{con}(X_{n,p})$ coincides with $H_{con}(\Lambda_{n,p,A})$.

After establishing the shape of the convex hull of $\Lambda_{n,p,A}$ we are now interested on the explicit characterization of its extremal points. By Proposition 2.1, this is equivalent to characterize the extremal points of $P_{n,p}$ and we shall focus on this problem. Let us see some examples.

EXAMPLE 2.2. *We have by a direct computation that for every $p > 1$ the set of extremal points of $P_{3,p}$, say $\mathcal{E}(P_{3,p})$, is*

$$\mathcal{E}(P_{3,p}) = \{1, 1 + q_{3,p}, q_{3,p}, q_{3,p} + q_{3,p}^2, q_{3,p}^2, q_{3,p}^2 + 1\}.$$

EXAMPLE 2.3. *We have by a direct computation that for every $p > 1$ the set of extremal points of $P_{4,p}$, say $\mathcal{E}(P_{4,p})$, is*

$$\mathcal{E}(P_{4,p}) = \{1 + q_{4,p}, q_{4,p} + q_{4,p}^2, q_{4,p}^2 + q_{4,p}^3, q_{4,p}^3 + 1\}.$$

Example 2.2 and Example 2.3 suggest that the set of extremal points of $P_{n,p}$ has an internal structure. This structure becomes more evident if we recall that the vertices of $P_{n,p}$ are element of $X_{n,p}$ and in particular they are (finite) expansions in base $q_{n,p}$ and if we focus on the sequences of binary coefficients associated to the extremal points. This point of view requires some notations.

(a) Convex hull of $\Lambda_{3,2^{1/2},\{0,1\}}$

(b) Convex hull of $\Lambda_{4,2^{1/2},\{0,1\}}$

(c) Convex hull of $\Lambda_{5,2^{1/2},\{0,1\}}$

(d) Convex hull of $\Lambda_{6,2^{1/2},\{0,1\}}$

(e) Convex hull of $\Lambda_{7,2^{1/2},\{0,1\}}$

(f) Convex hull of $\Lambda_{8,2^{1/2},\{0,1\}}$
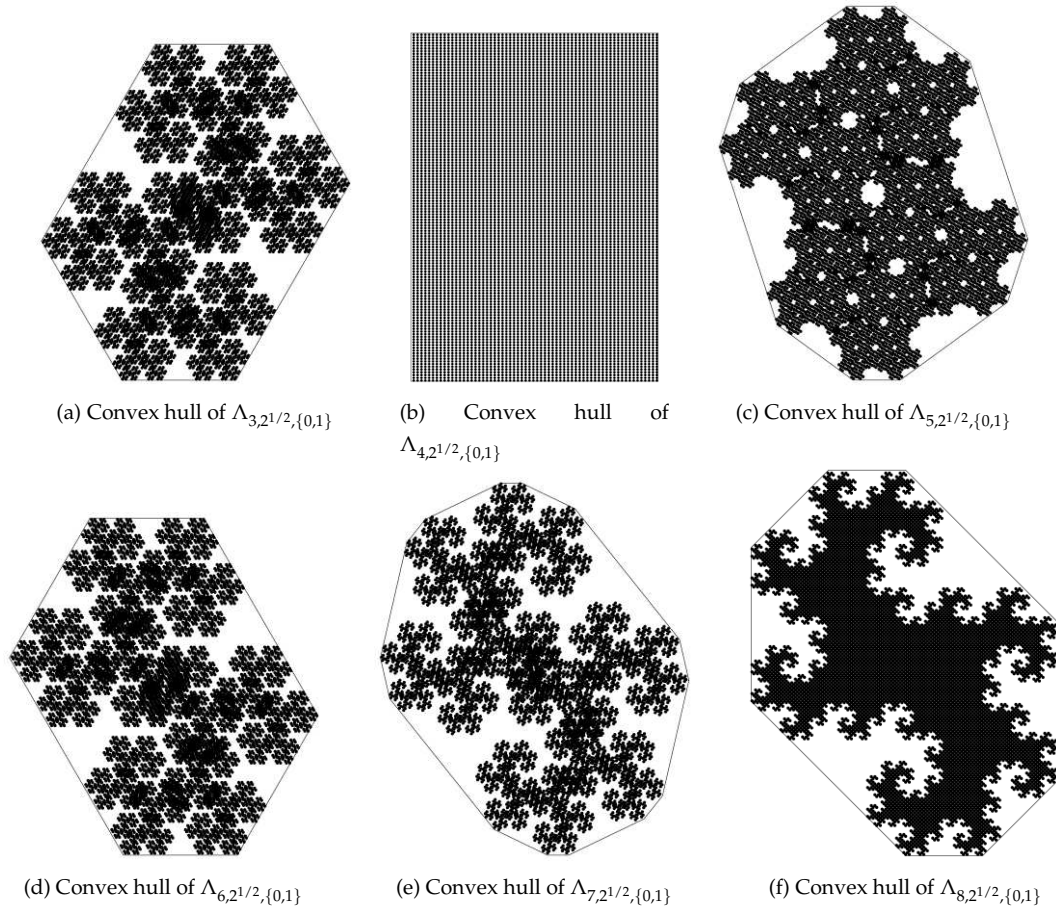
Figure 3: The set $\Lambda_{n,2^{1/2},\{0,1\}}$, with $n = 3,\ldots,9$ is approximated with the set of expansions with length 14. Remark that $q_{8,2^{1/2}} = 1 + i$ is a Gaussian integer that has been studied, for instance, in [Gil81].
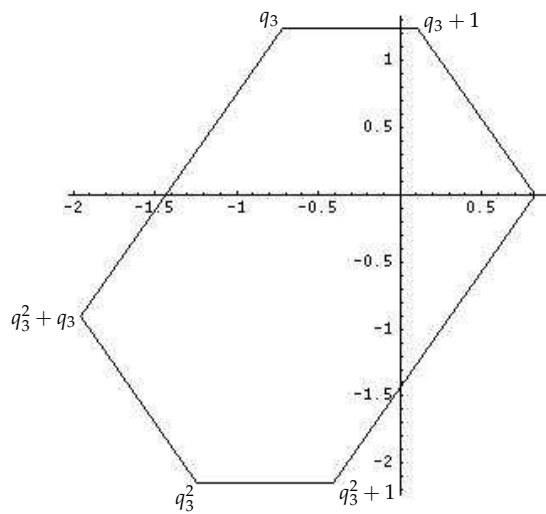

Figure 4: Extremal points of $P_{3,p}$ with $p = 2^{1/3}$. When the alphabet is $\{0,1\}$, this coincides with the convex hull of $\Lambda_{A,n,p}$.

NOTATION 2.3. *Set $(x_0 \cdots x_{n-1})$ a sequence in $\{0,1\}^n$. Define $(x_0 \cdots x_{n-1})_q := \sum_{j=0}^{n-1} x_j q^j$ and introduce the circular shift σ on the finite sequences: $\sigma(x_0 x_1 \cdots x_{n-1}) := (x_1 \cdots x_{n-1} x_0)$. The closure of $(x_0 \cdots x_{n-1})$ with respect to σ is denoted by $Orb(x_0 \cdots x_{n-1}) := \{\sigma^j(x_0 x_1 \cdots x_{n-1}) \mid j = 0, \ldots, n-1\}$. Finally define $Orb(x_0 \cdots x_{n-1})_q := \{\sigma^j(x_0 x_1 \cdots x_{n-1})_q \mid j = 0, \ldots, n-1\}$.*

EXAMPLE 2.4. *The following relations hold for every $p > 1$ and they are established by mean of a (symbolic) computer program. To lighten the notations, the subscript p is omitted: e.g. we set $q_n := q_{n,p}$.*

$$\mathcal{E}(P_3) = \{1, 1 + q_3, q_3, q_3 + q_3^2, q_3^2, q_3^2 + 1\}.$$
$$= \{(100)_{q_3}, (110)_{q_3}, (010)_{q_3}, (011)_{q_3}, (001)_{q_3}, (101)_{q_3}\}$$
$$= Orb(100)_{q_3} \cup Orb(110)_{q_3};$$

$$\mathcal{E}(P_4) = \{1 + q_4, q_4 + q_4^2, q_4^2 + q_4^3, q_4^3 + 1\}.$$
$$= \{(1100)_{q_4}, (0110)_{q_4}, (0011)_{q_4}, (1001)_{q_4}\}$$
$$= Orb(1100)_{q_4};$$

$$\mathcal{E}(P_5) = \{(11000)_{q_5}, (11100)_{q_5}, (01100)_{q_5}, (01110)_{q_5}, (00110)_{q_5},$$
$$(00111)_{q_5}, (00011)_{q_5}, (10011)_{q_5}, (10001)_{q_5}, (11001)_{q_5}\}$$
$$= Orb(1100)_{q_5} \cup Orb(11100)_{q_5};$$

$$\mathcal{E}(P_6) = \{(111000)_{q_6}, (011100)_{q_6}, (001110)_{q_6},$$
$$(000111)_{q_6}, (100011)_{q_6}, (110001)_{q_6}\}$$
$$= Orb(111000)_{q_6}.$$

In Example 2.4 the set of extremal points $\mathcal{E}(P_{n,p})$ is shown to be intimately connected with the sequences $(1^{\lfloor n/2 \rfloor} 0^{n - \lfloor n/2 \rfloor})$ and $(1^{\lceil n/2 \rceil} 0^{n - \lceil n/2 \rceil})$ when $n = 3, 5$ and with the sequence $(1^{n/2} 0^{n/2})$ when $n = 4, 6$. We now prove that this is a general result.

THEOREM 2.3. *Let $n \geq 1$, $p > 1$ and A an alphabet and denote $\mathcal{E}(\Lambda_{n,p,A})$ the set of the extremal points of $H_{con}(\Lambda_{n,p,A})$. If n is odd, then:*

$$\mathcal{E}(\Lambda_{n,p,A}) = \frac{\max A - \min A}{p^n - 1} \left(Orb(1^{\lfloor n/2 \rfloor} 0^{n - \lfloor n/2 \rfloor})_{q_{n,p}} \cup Orb(1^{\lceil n/2 \rceil} 0^{n - \lceil n/2 \rceil})_{q_{n,p}} \right)$$
$$+ \sum_{j=0}^{n-1} \min A \, q_{n,p}^j;$$

while if n is even:

$$\mathcal{E}(\Lambda_{n,p,A}) = \frac{\max A - \min A}{p^n - 1} Orb(1^{n/2} 0^{n/2})_{q_{n,p}} + \sum_{j=0}^{n-1} \min A \, q_{n,p}^j.$$

PROOF. Fix $p > 1$ and set $q_n = q_{n,p}$, $X_n = X_{n,p} = \{\sum_{j=0}^{n-1} x_j q_{n,p}^j \mid x_j \in \{0,1\}\}$; $P_n = P_{n,p} = H_{con}(X_{n,p})$. By Proposition 2.1 it suffices to characterize the extremal points of P_n and, in particular, to show that if n is odd then

(48) $$\mathcal{E}(P_n) = Orb(1^{\lfloor n/2 \rfloor} 0^{n - \lfloor n/2 \rfloor})_{q_n} \cup Orb(1^{\lceil n/2 \rceil} 0^{n - \lceil n/2 \rceil})_{q_n};$$

while if n is even:

(49) $$\mathcal{E}(P_n) = Orb(1^{n/2} 0^{n/2})_{q_n}.$$

The idea of the proof is to iteratively construct a set of vertices for P_n, say $V(P_n) \in \mathcal{V}(P_n)$, and to select from it the extremal points — see Figure 5 for the construction of $V(P_9)$. To this end we consider $X_{n,m} := \{\sum_{j=0}^{m-1} x_j q_n^j \mid x_j \in \{0,1\}\}$ and $P_{n,m} := H_{con}(X_{n,m})$ with $m = 1, \ldots, n$. Remark that $X_{n,n} = X_n$ and $P_{n,n} = P_n$. In the cases $n = 1, 2$, respectively corresponding to a positive and to a negative real base, thesis follows by a direct computation. Hence we may complete the proof by assuming $n > 2$.

For the moment we do not distinguish between the even and odd n and we organize the remaining proof in several parts. Part 1 and Part 2 are two stages of the construction of $V(P_n)$, in Part 3 the vertices of $V(P_n)$ are expressed in terms of sequences, namely the recursive formulae presented in Part 2 are explicited. In Part 4 the extremal points of P_n are selected from $V(P_n)$.

Part 1. Characterization of $V(P_{n,m})$ with $1 \leq m \leq \lceil n/2 \rceil$.

We prove by recurrence on m that

$$
\begin{aligned}
V(P_{n,m}) = \{ & (0^n)_{q_n}, (10^{n-1})_{q_n}, \ldots, (1^{m-1}0^{n-(m-1)})_{q_n}, \\
& (1^m 0^{n-m})_{q_n}, (01^{m-1}0^{n-m})_{q_n}, \ldots, (01^{m-1}0^{n-m})_{q_n} \} \\
= \{ & (0^m 0^{n-m})_{q_n}, (10^{m-1}0^{n-m})_{q_n}, \ldots, (10^{m-1}0^{n-m})_{q_n}, \\
& (1^m 0^{n-m})_{q_n}, (01^{m-1}0^{n-m})_{q_n}, \ldots, (01^{m-1}0^{n-m})_{q_n} \},
\end{aligned}
$$

the last equality being displayed to the end of underline the symmetric structure of $V(P_{n,m})$.

Now, if $m = 1$ then $X_{n,1} = \{0, 1\}$ and $P_{n,1} = [0, 1]$; hence

$$
\text{(50)} \qquad\qquad V(P_{n,1}) = \{0, 1\} = \{(0^n)_{q_n}, (10^{n-1})_{q_n}\}
$$

and this ensures the base of the induction.

Suppose now that $m > 1$ and that:

$$
\begin{aligned}
V(P_{n,m-1}) = \{ & (0^n)_{q_n}, (10^{n-1})_{q_n}, \ldots, (1^{m-2}0^{n-(m-2)})_{q_n}, \\
& (1^{m-1}0^{n-(m-1)})_{q_n}, (01^{m-2}0^{n-(m-1)})_{q_n}, \ldots, (0^{m-2}10^{n-(m-1)})_{q_n} \}.
\end{aligned}
$$

Then $P_{n,m-1}$ has $2(m-1)$ pairwise parallel edges. Our aim is to apply Corollary 2.3 with $P = P_{n,m}$, $l = m-1$ and $t = q_n^{m-1}$ so to get an explicit characterization of the vertices of $P_{n,m} = H_{con}(P_{n,m-1} \cup (P_{n,m-1}))$. To this end we need a vertex v such that setting n^- and n^+ the normal vectors to the adjacent to v edges, $n_v^- \cdot q_n^{m-1} < 0$ and $n_v^+ \cdot q_n^{m-1} \geq 0$. Our candidate is the vertex $v = (0^n)_{q_n}$. The normal vectors to the edges adjacent to v are given by the formulae:

$$
\text{(51)} \qquad\qquad n_v^- := \left((0^n)_{q_n} - (0^{m-2}10^{n-(m-1)})_{q_n} \right)^{\perp} = -(q_n^{m-2})_{q_n}^{\perp}
$$

and

$$
\text{(52)} \qquad\qquad n_v^+ := \left((10^{n-1)})_{q_n} - (0^n)_{q_n} \right)^{\perp} = (1)^{\perp}
$$

hence

$$
\begin{aligned}
n_v^- \cdot q_n^{m-1} \overset{(51)}{=} & -|q_n^{m-2}||q_n^{m-1}| \cos\left(\arg(q_n^{m-2})^{\perp} - \arg(q_n^{m-1}) \right) \\
= & -|q_n^{m-2}||q_n^{m-1}| \cos\left(\frac{m-2}{n}2\pi + \frac{\pi}{2} - \frac{m-1}{n}2\pi \right) \\
= & -|q_n^{m-2}||q_n^{m-1}| \cos\left(-\frac{1}{n}2\pi + \frac{\pi}{2} \right) < 0
\end{aligned}
$$

for every $n > 2$. Moreover

$$n_v^+ \cdot q_n^{m-1} \overset{(52)}{=} |1||q_n^{m-1}| \cos(\arg(1)^\perp - \arg(q_n^{m-1}))$$

$$= |q_n^{m-1}| \cos\left(\frac{\pi}{2} - \frac{m-1}{n}2\pi\right) \geq 0$$

because we have supposed $m \leq \lceil n/2 \rceil$. Then it follows by Corollary 2.3 that:

$$V(P_{n,m}) = \{(0^n)_{q_n}, (10^{n-1})_{q_n}, \ldots, (1^{m-2}0^{n-(m-2)})_{q_n}, (1^{m-1}0^{n-(m-1)})_{q_n},$$

$$(1^{m-1}0^{n-(m-1)})_{q_n} + q_n^{m-1}, (01^{m-2}0^{n-(m-1)})_{q_n} + q_n^{m-1},$$

$$\ldots, (0^{m-2}10^{n-(m-1)})_{q_n} + q_n^{m-1}, (0^n)_{q_n} + q_n^{m-1}\}$$

$$= \{(0^n)_{q_n}, (10^{n-1})_{q_n}, \ldots, (1^{m-2}0^{n-(m-2)})_{q_n}, (1^{m-1}0^{n-(m-1)})_{q_n},$$

$$(1^m 0^{n-(m)})_{q_n}, (01^{m-1}0^{n-m})_{q_n}, \ldots, (0^{m-2}110^n)_{q_n}, (0^{m-1}10^{n-m})_{q_n}\}.$$

and this proves at once the inductive step and (3).

Part 2. Characterization of $V(P_{n,m})$ with $\lceil n/2 \rceil \leq m \leq n$ via recursive formulae.

We prove that for every $\lceil n/2 \rceil \leq m \leq n$

(53) $$V(P_{n,m}) = \{v_j^m, u_j^m \mid j = 0, \ldots, m-1\}$$

where v_j^m and u_j^m are defined by the recursive formulae. When $m = \lceil n/2 \rceil$:

$$\begin{cases} v_j^{\lceil n/2 \rceil} = 1^j 0^{n-j} \\ u_j^{\lceil n/2 \rceil} = 0^j 1^{\lceil n/2 \rceil - j} 0^{n - \lceil n/2 \rceil} \end{cases} \quad \text{for} \quad j = 0, \ldots, \lceil n/2 \rceil - 1.$$

When $m > \lceil n/2 \rceil$:

(54) $$\begin{cases} v_j^m = v_{j+1}^{m-1} \\ u_j^m = u_{j+1}^{m-1} + q_n^{m-1} \end{cases} \quad \text{for} \quad j = 0, \ldots, m-1.$$

We complete the definition of v_j^m and u_j^m by setting $v_{m+j}^m := u_j^m$ and $u_{m+j}^m := v_j^m$, with $j = 0, \ldots, m-1$ so that:

(55) $$v_m^m \equiv u_0^m; \quad v_{m+1}^m \equiv u_1^m; \quad u_m^m \equiv v_0^m; \quad u_{m+1}^m \equiv v_1^m.$$

Note that for every $j = 0, \ldots, 2\lceil n/2 \rceil - 1$:

(56) $$v_{j+1}^{\lceil n/2 \rceil} - v_j^{\lceil n/2 \rceil} = \begin{cases} q^j & \text{if } j < \lceil n/2 \rceil; \\ -q^{j - \lceil n/2 \rceil} & \text{if } j \geq \lceil n/2 \rceil. \end{cases}$$

We prove (53) by induction on m. The case $m = \lceil n/2 \rceil$ follows by Part 1, in fact (3) implies:

$$V(P_{n, \lceil n/2 \rceil}) = \{(0^n)_{q_n}, (10^{n-1})_{q_n}, \ldots, (1^{\lceil n/2 \rceil - 1}0^{n-(\lceil n/2 \rceil - 1)})_{q_n}$$

$$(1^{\lceil n/2 \rceil}0^{n-\lceil n/2 \rceil})_{q_n}, (01^{\lceil n/2 \rceil - 1}0^{n-\lceil n/2 \rceil})_{q_n}, \ldots, (0^{\lceil n/2 \rceil - 1}10^{n-\lceil n/2 \rceil})_{q_n}\}$$

$$= \{v_j^{\lceil n/2 \rceil}, u_j^{\lceil n/2 \rceil} \mid j = 0, \ldots, \lceil n/2 \rceil - 1\}.$$

Let us assume $\lceil n/2 \rceil < m < n$ and (53) as inductive hypothesis. As in the proof of Part 1. we want to apply Corollary 2.3 to $P_{n,m+1} = H_{con}(P_{n,m} \cup (P_{n,m} + q_n^m))$ with $P = P_{n,m}$, $l = m$, $t = q_n^m$ and an appropriate vertex v satisfying either the condition $n_v^- \cdot q_n^m < 0$ and $n_v^+ \cdot q_n^m \geq 0$ or $n_v^- \cdot q_n^m = 0$ and $n_v^+ \cdot q_n^m > 0$. Our candidate is $v = v_1^m$. Set $k_m := m - \lceil n/2 \rceil$ so that $k_m \in$

$\{0, \ldots, m - \lceil n/2 \rceil\} \subset \{0, \ldots, m-1\}$. Remark that $m < n$ implies $k_m < \lceil n/2 \rceil$. By explicating the recursive relation (54) for $j = 0, 1, 2$ we have :

$$
v_0^m = v_1^{m-1} = \cdots = v_{k_m}^{\lceil n/2 \rceil} = (1^{k_m} 0^{n-k_m})_{q_n} \tag{57}
$$

$$
v_1^m = v_2^{m-1} = \cdots = v_{k_m+1}^{\lceil n/2 \rceil} = (1^{k_m+1} 0^{n-(k_m+1)})_{q_n} \tag{58}
$$

$$
v_2^m = v_3^{m-1} = \cdots = v_{k_m+2}^{\lceil n/2 \rceil}. \tag{59}
$$

We now distinguish the cases odd and even n.

If n is odd then $n/2 < \lceil n/2 \rceil$ and

$$
\begin{aligned}
n_{v_1}^- \cdot q_n^m &= (v_1^m - v_0^m)^\perp \cdot q_n^m \\
&\overset{(57)-(58)}{=} (v_{k_m+1}^{\lceil n/2 \rceil} - v_{k_m}^{\lceil n/2 \rceil})^\perp \cdot q_n^m \\
&\overset{(56)}{=} (q_n^{k_m})^\perp \cdot q_n^m \\
&= |q_n^{m-1}||q_n^m| \cos\left(\arg(q_n^{k_m})^\perp - \arg(q_n^m) \right) \\
&= |q_n^{m-1}||q_n^m| \cos\left(\frac{\pi}{2} - \frac{\lceil n/2 \rceil}{n} 2\pi \right) < 0
\end{aligned}
$$

while $q_n^n \in \mathbb{R}$ implies:

$$
\begin{aligned}
n_{v_1}^+ \cdot q_n^m &= (v_2^m - v_1^m)^\perp \cdot q_n^m \\
&\overset{(58)-(59)}{=} (v_{k_m+2}^{\lceil n/2 \rceil} - v_{k_m+1}^{\lceil n/2 \rceil})^\perp \cdot q_n^m \\
&\overset{(56)}{=} \begin{cases} (q_n^{k_m+1})^\perp \cdot q_n^m & \text{if } k_m + 1 < \lceil n/2 \rceil; \\ -(q_n^0)^\perp \cdot q_n^n & \text{if } k_m + 1 = \lceil n/2 \rceil; \end{cases} \\
&= \begin{cases} |q_n^{k_m}||q_n^m| \cos\left(\arg(q_n^{k_m+1})^\perp - \arg(q_n^m) \right) & \text{if } k_m + 1 < \lceil n/2 \rceil; \\ -|1||q_n^m| \cos\left(\arg(1)^\perp - \arg(q_n^n) \right)^\perp & \text{if } k_m + 1 = \lceil n/2 \rceil; \end{cases} \\
&= \begin{cases} |q_n^{m-1}||q_n^m| \cos\left(\frac{\pi}{2} - \frac{1}{n} 2\pi \right) \geq 0 & \text{if } k_m + 1 < \lceil n/2 \rceil; \\ -|q_n^{k_m}||q_n^m| \cos\left(\frac{\pi}{2} \right) = 0 & \text{if } k_m + 1 = \lceil n/2 \rceil. \end{cases}
\end{aligned}
$$

We then have:

$$
n_{v_1}^- \cdot q_n^m < 0 \quad \text{and} \quad n_{v_1}^+ \cdot q_n^m \geq 0. \tag{60}
$$

Suppose now that n is even. Then $k_m = m - n/2$, $(q_n^{k_m})^\perp \cdot q_n^m = 0$ and, consequently,

$$
\begin{aligned}
n_{v_1}^- \cdot q_n^m &= (v_1^m - v_0^m)^\perp \cdot q_n^m \\
&\overset{(57)-(58)}{=} (v_{k_m+1}^{\lceil n/2 \rceil} - v_{k_m}^{\lceil n/2 \rceil})^\perp \cdot q_n^m \\
&\overset{(56)}{=} (q_n^{k_m})^\perp \cdot q_n^m = 0;
\end{aligned}
$$

on the other hand $n > 2$ implies:

$$
\begin{aligned}
n_{v_1}^+ \cdot q_n^m &= (v_2^m - v_1^m)^\perp \cdot q_n^m \\
&\overset{(58)-(59)}{=} (v_{k_m+2}^{\lceil n/2 \rceil} - v_{k_m+1}^{\lceil n/2 \rceil})^\perp \cdot q_n^m \\
&\overset{(56)}{=} \begin{cases} (q_n^{k_m+1})^\perp \cdot q_n^m & \text{if } k_m + 1 < \lceil n/2 \rceil; \\ -(q_n^0)^\perp \cdot q_n^{n-1} & \text{if } k_m + 1 = \lceil n/2 \rceil; \end{cases} \\
&= \begin{cases} |q_n^{m-1}||q_n^m| \cos\left(\frac{k_m+1}{n}2\pi + \frac{\pi}{2} - \frac{m}{n}2\pi\right) > 0 & \text{if } k_m + 1 < \lceil n/2 \rceil; \\ -|q_n^{k_m}||q_n^m| \cos\left(\frac{\pi}{2} - \frac{n-1}{n}2\pi\right) > 0 & \text{if } k_m + 1 = \lceil n/2 \rceil. \end{cases}
\end{aligned}
$$

In view of the equality and of the inequality above we have $n_{v_1}^- \cdot q_n^m = 0$ and $n_{v_1}^+ \cdot q_n^m > 0$.. This, together with (60), implies that v_1 always satisfies one of the conditions of Corollary 2.3. Consequently

$$
\begin{aligned}
V(P_{n,m+1}) &= V\left(H_{con}(P_{n,m}) \cup (P_{n,m}) + q_n^m\right) \\
&= \{v_1^m, \ldots, v_{m+1}^m \equiv u_0, \\
&\quad v_{m+1}^m + q_n^m \equiv u_1^m + q_n^m, \ldots, u_{m+1}^m + q_n^m \equiv v_1^m + q_n^m\}
\end{aligned}
$$

and (53) follows.

Part 3. Explicitation of the characterizing formulae for $V(P_{n,m})$ with $\lfloor n/2 \rfloor + 1 \leq m \leq n$.

By applying the result of Part 3. to $m = n$ we get that:

(61) $$V(P_n) = \{v_j^n, u_j^n \mid j = 0, \ldots, n-1\}.$$

with v_j^n and u_j^n recursively defined in (54). In order to give an explicit expression for v_j^n and u_j^n we need to extend to u_0^m and u_1^m the formulae given in (57) and (58). Recall that we defined for every $\lceil n/2 \rceil \leq m \leq n$ the integer $k_m := m - \lceil n/2 \rceil$; we have:

(62) $u_0^m = u_1^{m-1} + q_n^{m-1} = \cdots = u_{k_m}^{\lfloor n/2 \rfloor + 1} + q_n^{m-1} + \cdots + q_n^{\lceil n/2 \rceil} = (0^{k_m} 1^{\lceil n/2 \rceil} 0^{n-m})_{q_n}$

(63) $u_1^m = u_2^{m-1} + q_n^{m-1} = \cdots = u_{k_m+1}^{\lceil n/2 \rceil} + q_n^{m-1} + \cdots + q_n^{\lfloor n/2 \rfloor + 1} = (0^{k_m+1} 1^{\lceil n/2 \rceil - 1} 0^{n-m})_{q_n}.$

Let us remark that for every couple of integers $j_1 \leq n_1$ satisfying $n_1 + j_1 = n + j$ we have $v_j^n = v_{j_1}^{n_1}$. Now, for every $j = 0, \ldots, n-1$ let $m_j := \lceil (n+j)/2 \rceil$; then

(64)
$$
\begin{aligned}
v_j^n = v_{j+1}^{n-1} = \ldots &= \begin{cases} v_{m_j-1}^{m_j} & \text{if } n + j \text{ is odd} \\ v_{m_j}^{m_j} & \text{if } n + j \text{ is even} \end{cases} \\
&\equiv \begin{cases} u_1^{m_j-1} & \text{if } n + j \text{ is odd} \\ u_0^{m_j} & \text{if } n + j \text{ is even} \end{cases} \\
&= \begin{cases} (0^{k_{m_j}} 1^{\lceil n/2 \rceil - 1} 0^{n-(m_j-1)})_{q_n} & \text{if } n + j \text{ is odd} \\ (0^{k_{m_j}} 1^{\lceil n/2 \rceil} 0^{n-m_j})_{q_n} & \text{if } n + j \text{ is even} \end{cases}
\end{aligned}
$$

and

$$
u_j^n = u_{j+1}^{n-1} + q_n^{n-1} = \ldots = \begin{cases} u_{m_j-1}^{m_j} + q_n^{m_j} + \cdots + q_n^{n-1} & \text{if } n+j \text{ is odd} \\ u_{m_j}^{m_j} + q_n^{m_j} + \cdots + q_n^{n-1} & \text{if } n+j \text{ is even} \end{cases}
$$

$$
(65) \qquad \equiv \begin{cases} v_1^{m_j-1} + q_n^{m_j} + \cdots + q_n^{n-1} & \text{if } n+j \text{ is odd} \\ v_0^{m_j} + q_n^{m_j} + \cdots + q_n^{n-1} & \text{if } n+j \text{ is even} \end{cases}
$$

$$
= \begin{cases} (1^{k_{m_j}} 0^{\lceil n/2 \rceil - 1} 1^{n-(m_j-1)})_{q_n} & \text{if } n+j \text{ is odd} \\ (1^{k_{m_j}} 0^{\lceil n/2 \rceil} 1^{n-m_j})_{q_n} & \text{if } n+j \text{ is even}. \end{cases}
$$

We now distinguish the cases n odd and n even.

If n is odd then for every $j = 0, \ldots, n-1$ we have $k_{m_j} = \lceil (n+j)/2 \rceil - \lceil n/2 \rceil = \lfloor j/2 \rfloor$, hence we may rewrite (64):

$$
v_j^n = \begin{cases} (0^{\lfloor j/2 \rfloor} 1^{\lceil n/2 \rceil - 1} 0^{n-(\lfloor j/2 \rfloor + \lceil n/2 \rceil - 1)})_{q_n} & \text{if } j \text{ is even} \\ (0^{\lfloor j/2 \rfloor} 1^{\lceil n/2 \rceil} 0^{n-(\lfloor j/2 \rfloor + \lceil n/2 \rceil)})_{q_n} & \text{if } j \text{ is odd}. \end{cases}
$$

Hence:

$$
(v_j^n)_{0 \le j \le n-1} = (1^{\lceil n/2 \rceil - 1} 0^{n-(\lceil n/2 \rceil - 1)})_{q_n}, (1^{\lceil n/2 \rceil} 0^{n-\lceil n/2 \rceil})_{q_n},
$$

$$
\ldots, (0^{\lfloor (n-1)/2 \rfloor} 1^{\lceil n/2 \rceil - 1} 0^{n-(\lfloor (n-1)/2 \rfloor + \lceil n/2 \rceil - 1)})_{q_n}
$$

$$
(66) \qquad = (1^{\lceil n/2 \rceil - 1} 0^{\lceil n/2 \rceil})_{q_n}, (1^{\lceil n/2 \rceil} 0^{\lceil n/2 \rceil - 1})_{q_n}, \ldots, (0^{\lceil n/2 \rceil - 1} 1^{\lceil n/2 \rceil - 1} 0)_{q_n}.
$$

Similarly, we may deduce by (65) that:

$$
u_j^n = \begin{cases} (1^{\lfloor j/2 \rfloor} 0^{\lceil n/2 \rceil - 1} 1^{n-(\lfloor j/2 \rfloor + \lceil n/2 \rceil - 1)})_{q_n} & \text{if } j \text{ is even} \\ (1^{\lfloor j/2 \rfloor} 0^{\lceil n/2 \rceil} 1^{n-(\lfloor j/2 \rfloor + \lceil n/2 \rceil)})_{q_n} & \text{if } j \text{ is odd} \end{cases}
$$

and that

$$
(u_j^n)_{0 \le j \le n-1} = (0^{\lceil n/2 \rceil - 1} 1^{n-(\lceil n/2 \rceil - 1)})_{q_n}, (0^{\lceil n/2 \rceil} 1^{n-\lceil n/2 \rceil})_{q_n},
$$

$$
\ldots, (1^{\lfloor (n-1)/2 \rfloor} 0^{\lceil n/2 \rceil - 1} 1^{n-(\lfloor (n-1)/2 \rfloor + \lceil n/2 \rceil - 1)})_{q_n}
$$

$$
(67) \qquad = (0^{\lceil n/2 \rceil - 1} 1^{\lceil n/2 \rceil})_{q_n}, (0^{\lceil n/2 \rceil} 1^{\lceil n/2 \rceil - 1})_{q_n}, \ldots, (1^{\lceil n/2 \rceil - 1} 0^{\lceil n/2 \rceil - 1} 1)_{q_n}.
$$

If n is even then for every $j = 0, \ldots, n-1$ we have $k_{m_j} = \lceil (j+n)/2 \rceil - \lceil n/2 \rceil = \lceil j/2 \rceil$. Similarly to the previous case, by performing appropriate substitutions in (64) and (65) we get

$$
(68) \qquad (v_j^n)_{0 \le j \le n-1} = (1^{n/2} 0^{n/2})_{q_n}, (01^{n/2-1} 0^{n/2})_{q_n}, \ldots, (0^{n/2} 1^{n/2-1} 0)_{q_n}
$$

and

$$
(69) \qquad (u_j^n)_{0 \le j \le n-1} = (0^{n/2} 1^{n/2})_{q_n}, (10^{n/2-1} 1^{n/2})_{q_n}, \ldots, (1^{n/2} 0^{n/2-1} 1)_{q_n}.
$$

Part 4. Selection of the extremal points in $V(P_n)$.

We begin by supposing that n is odd. As by construction the vertices of P_n listed in (66) and (67) are (counter-clockwise) ordered, by subtracting any couple of adjacent vertices we may neatly list the vector edges of P_n and get that

$$
(70) \qquad \begin{aligned} &v_1^n - v_0^n = q_{n-1}^{\lceil n/2 \rceil - 1}; v_2^n - v_1^n = -1; \ldots; v_n^n \equiv u_0^n - v_{n-1}^n = q_n^{n-1}; \\ &u_1^n - u_0^n = -q_n^{\lceil n/2 \rceil - 1}; u_2^n - u_1^n = 1; \ldots; v_n^n \equiv u_0^n - v_{n-1}^n = -q_n^{n-1}. \end{aligned}
$$

Since n is odd, all the powers of q_n are pairwise independent, thus (70) implies that P_n has no consecutive edges and all the vertices v_j and u_j, with $j = 0, \ldots, n-1$, are extremal points. Observe that $\sigma(v_j^n) = v_{j+2}^n (\equiv u_{j+2-n}^n$ if $j+2 \geq n)$. Then, since $v_0^n = (1^{\lceil n/2 \rceil -1} 0^{\lceil n/2 \rceil}) = (1^{\lfloor n/2 \rfloor} 0^{n-\lfloor n/2 \rfloor})$ and $v_1^n = (1^{\lceil n/2 \rceil} 0^{\lceil n/2 \rceil -1}) = (1^{\lceil n/2 \rceil} 0^{n-\lceil n/2 \rceil})$, we get (48).

If n is even we may use (68) and (69) to see that the (ordered) the vector edges of P_n are:

$$v_1^n - v_0^n = 1; \qquad v_2^n - v_1^n = -q_n^{n/2}; \ldots; \quad v_n^n \equiv u_0^n - v_{n-1}^n = -q_n^{n-1};$$
$$u_1^n - u_0^n = -1; \quad u_2^n - u_1^n = q_n^{n/2}; \ldots; \quad v_n^n \equiv u_0^n - v_{n-1}^n = q_n^{n-1}.$$

As $q_n^{n/2} = -p^{n/2}$ is parallel to -1, the couples of vector edges $(\pm q_n^j, \mp q_n^{n/2+j})$ are consecutive. Hence the common vertices to these edges, namely v_{2j}^n and u_{2j}^n with $j = 0, \ldots, n/2-1$, are degenerate vertices. As $\sigma(v_{2j+1}^n) = v_{2j+3}^n (\equiv u_{j+2-n}^n$ if $j+2 \geq n)$, we deduce by $v_1^n = (1^{n/2} 0^{n/2})_{q_n}$ that

$$\mathcal{E}(P_n) = Orb(1^{n/2} 0^{n/2})_{q_n}$$

and the proof is complete.   □

(a) $P_{9,1}$.   (b) $P_{9,2}$.   (c) $P_{9,3}$.   (d) $P_{9,4}$.

(e) $P_{9,5}$.

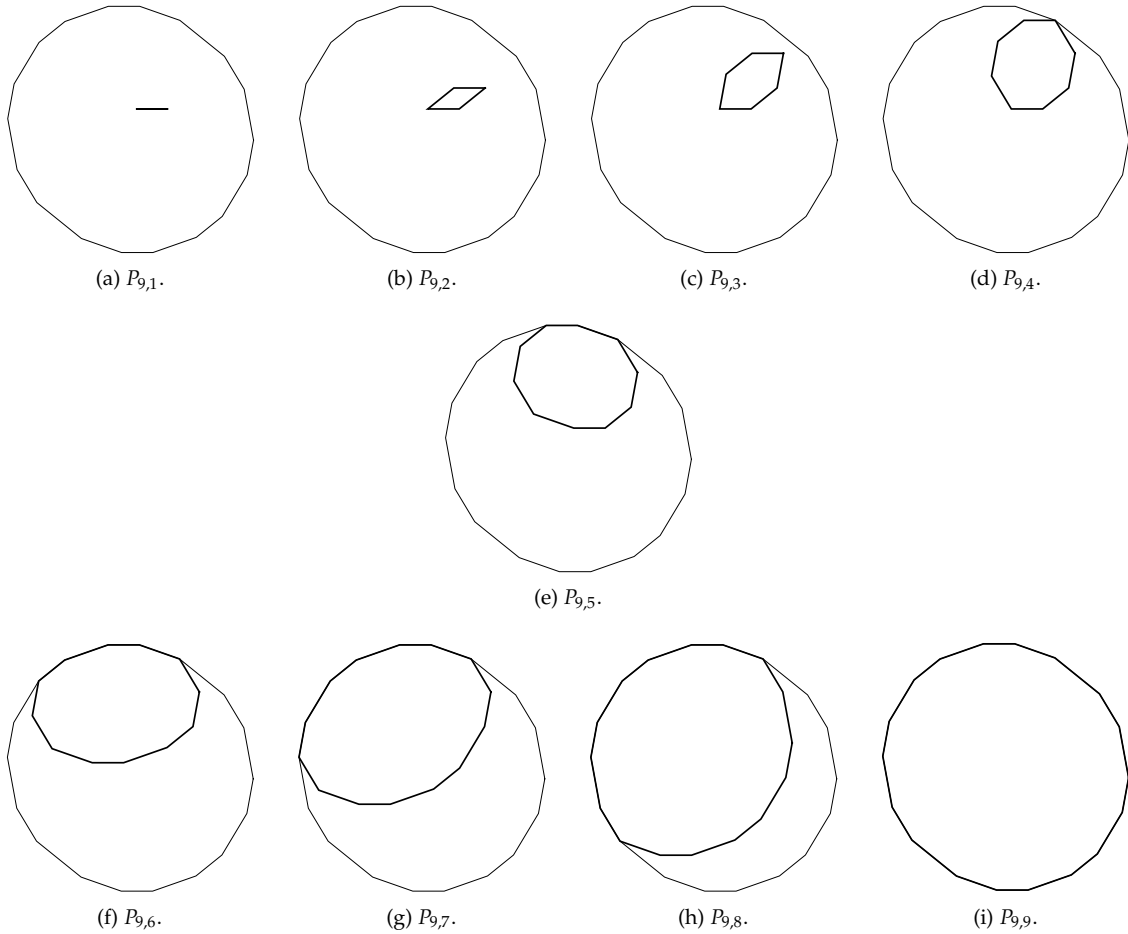(f) $P_{9,6}$.   (g) $P_{9,7}$.   (h) $P_{9,8}$.   (i) $P_{9,9}$.

Figure 5: Various stages of the construction of $P_9(= P_{9,9})$ with $p = 2^{1/9}$. Remark how $P_{9,5} = P_{9,\lceil 9/2 \rceil}$ represents a conjunction between the structures of $P_{9,1}, \ldots, P_{9,4}$ and $P_{9,6}, \ldots, P_{9,9}$.

4. Representability in complex base

In this section we give a characterization of the convexity of $\Lambda_{n,p,A}$.

THEOREM 2.4. *The set of representable numbers in base $q_{n,p}$ and alphabet $A = \{a_1, \ldots, a_J\}$ is convex if and only if*

$$\max_{j=1,\ldots,J-1} a_{j+1} - a_j \leq \frac{\max A - \min A}{p^n - 1}. \tag{71}$$

PROOF. We assume without loss of generality that $\min A = 0$ and we divide the proof in several parts.

Part 1. $\Lambda_{n,p,A}$ is the attractor of an appropriate iterated function system F_A.

Let $F_A := \{f_0, f_1, \ldots, f_m\}$ the iterated function system defined on $H_{con}(\Lambda_{n,p,A})$ whose basic functions are $f_j(x) := \frac{1}{q}(x + a_j)$ — see Figure 6. For every $x = \sum_{i=1}^{\infty} \frac{x_i}{q^i} \in \Lambda_{n,p,A}$ we have

$$f_j(x) = \frac{1}{q}x + \frac{1}{q}a_j = \sum_{i=2}^{\infty} \frac{x_{i-1}}{q^i} + \frac{a_j}{q} \in \Lambda_{n,p,A}.$$

Moreover if $x_1 = a_j$ then

$$f_j^{-1}(x) = qx - a_j = x_1 + q\sum_{i=2}^{\infty} \frac{x_i}{q^i} - a_j = \sum_{i=1}^{\infty} \frac{x_{i+1}}{q^i} \in \Lambda_{n,p,A}$$

and, consequently $\cup_{j=0}^{m} f_j(\Lambda_{n,p,A}) = \Lambda_{n,p,A}$. Hence $\Lambda_{n,p,A}$ is the attractor of F_A.

Part 2. First simplification: $\Lambda_{n,p,A}$ is a convex set if and only if $F_A(H_{con}(\Lambda_{n,p,A}))$ is a convex set.

Since Proposition 2.1 ensures $H_{con}(\Lambda_{n,p,A}) = \frac{\max A}{p^n - 1} P_{n,p}$, we have

$$\begin{aligned} H_{con}(F_A(H_{con}(\Lambda_{n,p,A}))) &= \frac{1}{q} H_{con}\left(\bigcup_{j=0}^{m} P_{n,p} \frac{\max A}{p^n - 1} + a_j\right) \\ &= \frac{1}{q} H_{con}\left(P_{n,p} \frac{\max A}{p^n - 1} \cup \left(P_{n,p} \frac{\max A}{p^n - 1} + \max A\right)\right) \\ &= \frac{1}{q} \frac{\max A}{p^n - 1} H_{con}\left(P_{n,p} \cup (P_{n,p} + p^n - 1)\right). \end{aligned} \tag{72}$$

Now we proved in Theorem 2.3 that $P_{n,p} = \{v_j, u_j | j = 0, \ldots, n - 1\}$ with v_j and u_j satisfying the following relations:

$$v_{j+n} = u_j \quad \text{and} \quad u_{j+n} = v_j \quad \text{for every } j = 0, \ldots, n - 1;$$

and, by explicit formulas given in (66) and (67) for n odd and (68) and (69) for n even:

$$v_2 - v_1 = 1; \tag{73}$$

$$v_j = qv_{j+2}; \qquad u_j - 1 = qu_{j+2}; \tag{74}$$

$$v_0 - \frac{1}{q}v_1 = -\frac{1}{q}; \qquad \frac{1}{q}v_1 - u_0 = -\frac{p^n - 1}{q} \tag{75}$$

By applying Corollary 2.3 to $P = P_{n,p}$, $l = n$ and $t = 1$ and $v_{i_1} = v_1$, see (73), we may continue the chain of equalities in (72) and get

$$H_{con}(F_A(H_{con}(\Lambda_{n,p,A}))) = \frac{1}{q}\frac{\max A}{p^n - 1}H_{con}(\{v_1, \ldots, v_{n-1}, u_0, u_1, u_1 + p^n - 1,$$

$$\ldots, u_{n-1} + p^n - 1, v_0 + p^n - 1, v_1 + p^n - 1\})$$

$$= \frac{\max A}{p^n - 1}\mathcal{H}(\{\frac{1}{q}v_1, \ldots, \frac{1}{q}v_{n-1}, \frac{1}{q}u_0, \frac{1}{q}u_1\frac{1}{q}(u_1 + p^n - 1),$$

$$\ldots, \frac{1}{q}(u_{n-1} + p^n - 1), \frac{1}{q}(v_0 + p^n - 1), \frac{1}{q}(v_1 + p^n - 1)\})$$

$$\overset{(74)}{=} \frac{\max A}{p^n - 1}H_{con}\left(\{\frac{1}{q}v_1, v_0, \ldots, v_{n-1}, \frac{1}{q}u_1 + p^n - 1, u_0, \ldots, u_{n-1}\}\right)$$

$$\overset{(75)}{=} \frac{\max A}{p^n - 1}H_{con}\left(\{v_0, \ldots, v_{n-1}, u_0, \ldots, u_{n-1}\}\right)$$

$$= \frac{\max A}{p^n - 1}P_n.$$

Consequently

(76)
$$H_{con}(F_A(H_{con}(\Lambda_{n,p,A}))) = H_{con}(\Lambda_{n,p,A})$$

and

$$\Lambda_{n,p,A} \text{ is convex} \Leftrightarrow \Lambda_{n,p,A} = H_{con}(\Lambda_{n,p,A})$$

$$\Leftrightarrow H_{con}(\Lambda_{n,p,A}) \text{ is the attractor of } F_A$$

$$\Leftrightarrow H_{con}(\Lambda_{n,p,A}) = F_A(H_{con}(\Lambda_{n,p,A}))$$

$$\overset{(76)}{\Leftrightarrow} F_A(H_{con}(\Lambda_{n,p,A})) \text{ is convex.}$$

Part 3. Second simplification: $F_A(H_{con}(\Lambda_{n,p,A}))$ is a convex set if and only if (71) holds.

As $F_A\left(H_{con}(\Lambda_{n,p,A})\right) = \frac{1}{q}\left(\bigcup_{j=0}^m P_{n,p}\frac{\max A}{p^n - 1} + a_j\right)$ the convexity of $F_A(H_{con}(\Lambda_{n,p,A}))$ is equivalent to the one of $\bigcup_{j=0}^m P_{n,p}\frac{\max A}{p^n - 1} + a_j$. By a geometrical evidence $\bigcup_{j=0}^m P_{n,p}\frac{\max A}{p^n - 1} + a_j$ is convex if and only if for every $j = 0, \ldots, m - 1$

(77) $$(P_{n,p}\frac{\max A}{p^n - 1} + a_j) \cup (P_{n,p}\frac{\max A}{p^n - 1} + a_{j+1}) = a_j + (P_{n,p}\frac{\max A}{p^n - 1} \cup (P_{n,p}\frac{\max A}{p^n - 1} + a_{j+1} - a_j))$$

is convex. It follows by Theorem 2.2 that $P_{n,p}\frac{\max A}{p^n - 1}$ has a couple of edges which are parallel to the translation $a_{j+1} - a_j$ and by (73) length of such edges is equal to $\frac{\max A}{p^n - 1}$. Hence thesis follows by Lemma 2.1. $\square$

COROLLARY 2.4. *Let $A = \{a_1, \ldots, a_J\}$. If $\max_{j=1,\ldots,J-1} a_{j+1} - a_i \leq \frac{\max A - \min A}{p^n - 1}$ then every $x \in \frac{\max A - \min A}{p^n - 1}P_{n,p} + \sum_{k=0}^{n-1}\min A q_{n,p}^k$ has a representation in base $q_{n,p}$ and alphabet A.*

PROOF. Immediate. $\square$

EXAMPLE 2.5. *If $p = 2^{1/n}$ and $A = \{0, 1\}$ then $\Lambda_{n,p,A}$ is a convex set coinciding with $P_{n,p}$. In particular $\Lambda_{n,p,A}$ is an 2n-gon if n is odd and it is an n-gon if n is even.*

EXAMPLE 2.6. *If $A = \{0, 1, \ldots, \lfloor p^n \rfloor\}$ then $\Lambda_{n,p,A}$ is a convex set or, equivalently, $\frac{\lfloor p^n \rfloor}{p^n - 1}P_{n,p}$ is completely representable.*

(a)   (b)   (c)

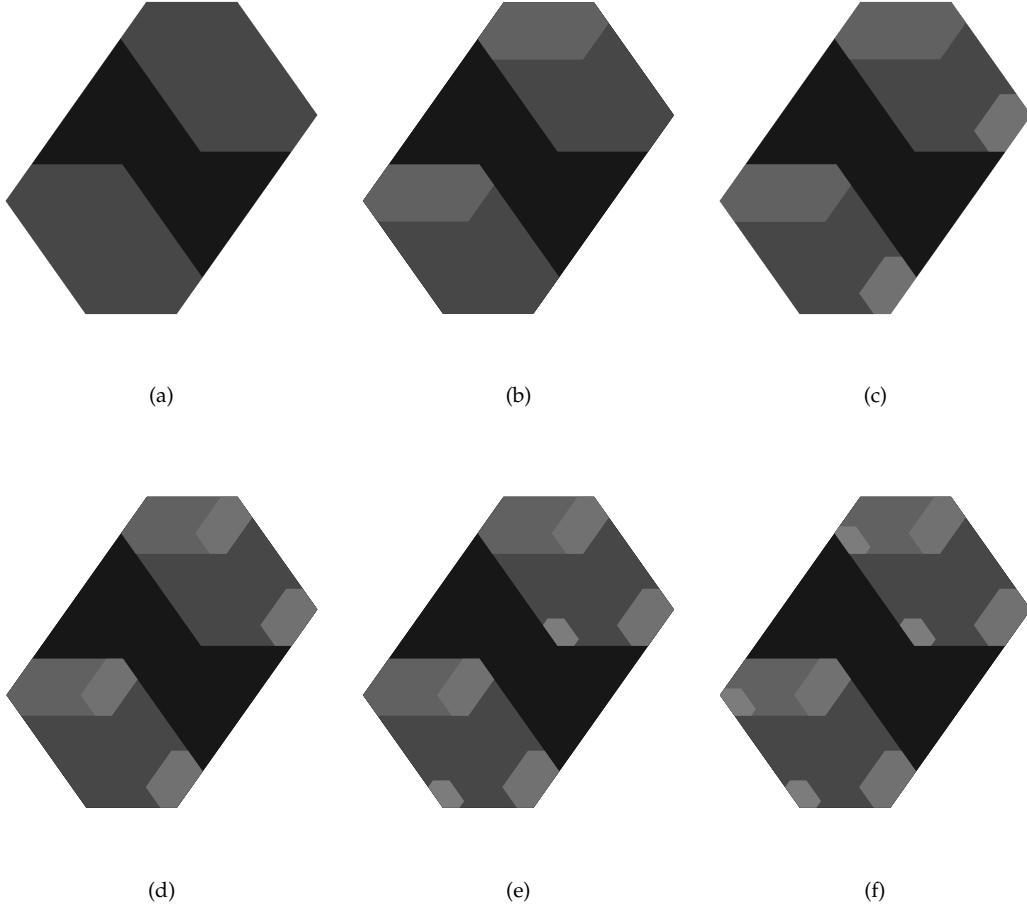(d)   (e)   (f)

Figure 6: First 6 iterations of $F_{\{0,1\}} = \{x/q_{3,2}, 1/q_{3,2}(x+1)\}$ over $H_{conv}(\Lambda_{3,2,\{0,1\}})$.

5. Overview of original contributions, conclusions and further developments

Characterization of the convex hull of representable numbers. First, a geometrical tool has been proved.

THEOREM. *Let P be a polygon with vertices in $V_P := \{v_0, \ldots, v_{l-1}\}$ and let t be a translation vector. Then there exists only two indices $i_1, i_2 \in \{0, \ldots, l-1\}$ such that $n_{i_1-1} \cdot t < 0$ and $n_{i_1} \cdot t \geq 0$ and $n_{i_2-1} \cdot t > 0$ and $n_{i_2} \cdot t \leq 0$. Moreover the convex hull of $P \cup (P+t)$ is a polygon whose vertices are:*

(78) $$v_{i_1}, \cdots, v_{i_2}, v_{i_2} + t, \ldots, v_{i_1-1} + t, v_{i_1} + t.$$

The corollaries of the result above have been applied in the following results.

THEOREM. *For every $n \geq 1$, $p > 1$ and $q_{n,p} = pe^{\frac{2\pi}{n}i}$, the convex hull $H_{con}(\Lambda_{n,p,A})$ is a polygon with the following properties:*

(a) the edges are pairwise parallel to $q_{n,p}^0, \ldots, q_{n,p}^{n-1}$;
(b) if n is odd then $H_{con}(\Lambda_{n,p,A})$ has 2n extremal points;
(c) if n is even then $H_{con}(\Lambda_{n,p,A})$ has n extremal points.

(a) $\Lambda_{3,2^{1/3},\{0,1\}}$ (b) $\Lambda_{3,2^{1/3}+0.1,\{0,1\}}$ (c) $\Lambda_{3,2^{1/3}+0.2,\{0,1\}}$

(d) $\Lambda_{3,2^{1/3}+0.3,\{0,1\}}$ (e) $\Lambda_{3,2^{1/3}+0.4,\{0,1\}}$ (f) $\Lambda_{3,2^{1/3}+0.5,\{0,1\}}$
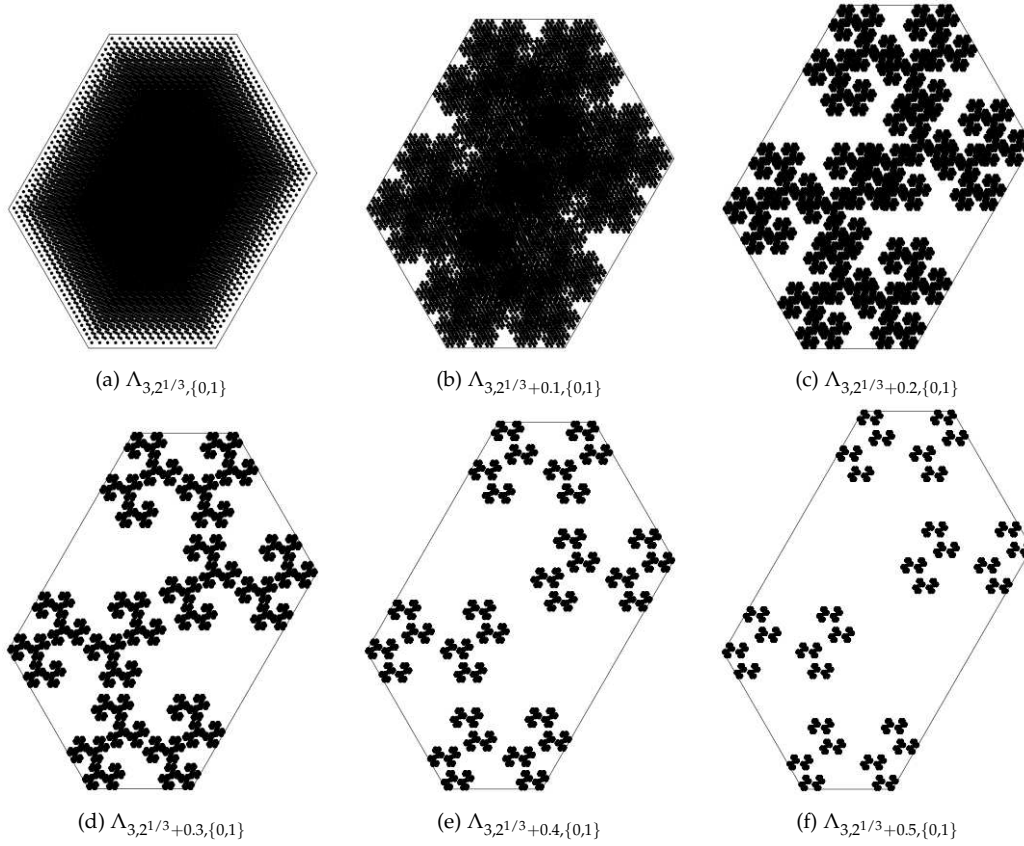
Figure 7: $\Lambda_{3,2^{1/3}+0.1k,\{0,1\}}$, with $k = 0,\ldots,5$, is approximated with the set of expansions with length 14.

THEOREM. *Let $n \geq 1$, $p > 1$ and A an alphabet and denote $\mathcal{E}(\Lambda_{n,p,A})$ the set of the extremal points of $H_{con}(\Lambda_{n,p,A})$. If n is odd, then:*

$$\mathcal{E}(\Lambda_{n,p,A}) = \frac{\max A - \min A}{p^n - 1} \left(Orb(1^{\lfloor n/2 \rfloor} 0^{n-\lfloor n/2 \rfloor})_{q_{n,p}} \cup Orb(1^{\lceil n/2 \rceil} 0^{n-\lceil n/2 \rceil})_{q_{n,p}} \right)$$
$$+ \sum_{j=0}^{n-1} \min A \, q_{n,p}^j;$$

while if n is even:

$$\mathcal{E}(\Lambda_{n,p,A}) = \frac{\max A - \min A}{p^n - 1} Orb(1^{n/2} 0^{n/2})_{q_{n,p}} + \sum_{j=0}^{n-1} \min A \, q_{n,p}^j;$$

Representability in complex base. We proved the following characterization of convex representable sets.

THEOREM. *The set of representable numbers in base $q_{n,p}$ and alphabet $A = \{a_1, \ldots, a_J\}$ is convex if and only if $\max_{j=1,\ldots,J-1} a_{j+1} - a_j \leq \frac{\max A - \min A}{p^n - 1}$.*

The result above leads to the following representability result.

COROLLARY. *Let $A = \{a_1, \ldots, a_J\}$. If $\max_{j=1,\ldots,J-1} a_{j+1} - a_j \leq \frac{\max A - \min A}{p^n - 1}$ then every $x \in \frac{\max A - \min A}{p^n - 1} P_{n,p} + \sum_{k=0}^{n-1} \min A q_{n,p}^k$ has a representation in base $q_{n,p}$ and alphabet A.*

Conclusions and further developments. The set of representable numbers can be viewed as the attractor of an appropriate (linear) iterated function system, say $F_{q,A}$, depending on the base and on the alphabet. The characterization of the convex hull of the set of representable numbers gives an operative method for defining a bounded domain for $F_{q,A}$.

Here a (sufficient) condition for a full Hausdorff dimension for the set of representable numbers has been given, but the general problem is open. The relations established in this chapter could be useful for an answer — at least for bases with rational argument. The definition of a global greedy algorithm and a partial characterization based on digit-by-digit comparison could be encouraged by these arguments, as well.

Expansions in negative base

Ito and Sadahiro recently introduced and characterized expansions in non-integer negative base $-q$ in [IS09]. They have also shown that the $(-q)$-shift is sofic if and only if the $(-q)$-expansion of the number $-\frac{q}{q+1}$ is eventually periodic. The aim of this chapter is to pursue their work, by showing that many properties of the positive base numeration systems extend to the negative base case, the main difference being the sets of numbers that are representable in the two different cases.

1. Introduction

Expansions in integer negative base $-b$, where $b \geq 2$, seem to have been introduced by Grünwald in [Gru85], and rediscovered by several authors, see the historical comments given by Knuth [Knu71]. The choice of a negative base $-b$ and of the alphabet $\{0, \ldots, b-1\}$ is interesting, because it provides a signless representation for every number (positive or negative). In this case it is easy to distinguish the sequences representing a positive integer from the ones representing a negative one: denoting $(w.)_{-b} := \sum_{i=0}^{k} w_k (-b)^i$ for any $w = w_k \cdots w_0$ in $\{0, \ldots, b-1\}^*$ with no leading 0's, we have $\mathbb{N} = \{(w.)_{-b} \mid |w| \text{ is odd}\}$. The classical monotonicity between the lexicographical ordering on words and the represented numerical values does not hold anymore in negative base, for instance $3 = (111.)_{-2}$, $4 = (100.)_{-2}$ and $111 >_{lex} 100$. Nevertheless it is possible to restore such a correspondence by introducing an appropriate ordering on words, in the sequel denoted by $\prec$ and called the *alternate order*.

Representations in negative base also appear in some complex base number systems, for instance base $q = 2i$ where $q^2 = -4$ (see [Fro99] for a study of their properties from an automata theoretic point of view).

Organization of the chapter. In Section 2 the Ito and Sadahiro's $(-q)$-expansions and the alternate order are introduced. Section 3 is devoted to the proof of a general result which is not related to numeration systems but to the alternate order, and which is of interest in itself. We define a symbolic dynamical system associated with a given infinite word s satisfying some properties with respect to the alternate order on infinite words. We design an infinite countable automaton recognizing it. We then are able to characterize the case when the symbolic dynamical system is sofic (resp. of finite type). Using this general construction we can prove in Section 4 that the $(-q)$-shift is a symbolic dynamical system of finite type if and only if the $(-q)$-expansion of $-\frac{q}{q+1}$ is purely periodic.

In Section 5 we show that the entropy of the $(-q)$-shift is equal to $\log q$.

We then focus on the case where q is a Pisot number, that is to say, an algebraic integer greater than 1 such that the modulus of its Galois conjugates is less than 1. The natural integers and the Golden Mean are Pisot numbers. We extend all the results known to hold true in the Pisot case for q-expansions to the $(-q)$-expansions. In particular we prove that, if q is a Pisot number, then every number from $\mathbb{Q}(q)$ has an eventually periodic $(-q)$-expansion, and thus that the $(-q)$-shift is a sofic system. When q is a Pisot number, it is known that addition in base q — and more generally

normalization in base q on an arbitrary alphabet — is realizable by a finite transducer [Fro92]. We show that this is still the case in base $-q$. Finally in Section 7 we introduce an algorithm of conversion from positive to negative base.

2. Preliminaries on expansions in non-integer negative base

2.1. Definition and first properties of $(-q)$-expansions. Ito and Sadahiro [IS09] introduced a greedy algorithm to represent any real number in real base $-q$, $q > 1$, and with digits in $A_{-q} := A_q = \{0, 1, \ldots, \lfloor q \rfloor\}$. Similarly to the positive case, the definition of a $(-q)$-*expansion* lays on the iteration of an appropriate map defined from an interval of length 1 onto itself.

We start with some general definitions.

DEFINITION 3.1 ($(-q)$-representations). *Let x be a real number. A $(-q)$-representation of x is a sequence $x_{-d+1} x_{-d+2} \cdots x_0 x_1 x_2 \cdots$ satisfying:*

$$x = \sum_{i=0}^{d-1} x_{-i}(-q)^i + \sum_{i=1}^{\infty} \frac{x_i}{(-q)^i}.$$

The value $\sum_{i=0}^{d-1} x_{-i}(-q)^i$ is called the integer part *of the representation and the value $\sum_{i=1}^{\infty} \frac{x_i}{(-q)^i}$ is called the* fractional part. *We also write $x = (x_{-d+1} x_{-d+2} \cdots x_0 . x_1 x_2 \cdots)_{-q}$, with the symbol . dividing the integer and the fractional part.*

We now define the class of $(-q)$-expansions.

DEFINITION 3.2 ($(-q)$-transformation, $(-q)$-expansions). *The $(-q)$-transformation T_{-q} is defined from the interval $I_{-q} := \left[-\frac{q}{q+1}, \frac{1}{q+1} \right)$ onto itself, and for every $x \in I_{-q}$,*

$$(79) \qquad T_{-q}(x) = -qx - \lfloor -qx + \frac{q}{q+1} \rfloor.$$

For every $x \in I_{-q}$, every digit x_n with $n \geq 1$ of the $(-q)$-expansion of x is defined by:

$$(80) \qquad x_n = \lfloor -q T_{-q}^{n-1}(x) + \frac{q}{q+1} \rfloor;$$

If $x \notin I_q$ denote d the smallest integer such that $x/(-q)^d \in I_{-q}$. The $(-q)$-expansion of x is the sequence $x_{-d+1} x_{-d+2} \cdots x_0 x_1 x_2 \cdots$ where for every $n \geq 1$

$$x_{-d+n} = \lfloor -q \, T_{-q}^{n-1} \left(\frac{x}{(-q)^d} \right) + \frac{q}{q+1} \rfloor.$$

For every x such that its $(-q)$-expansion has the integer part equal to zero, we define $\gamma_{-q}(x) := x_1 x_2 \cdots$.

REMARK 3.1.

1. *For every x in I_{-q}, $\gamma_{-q}(x)$ is well defined.*
2. *Even if $\frac{1}{q+1} \notin I_{-q}$, the map γ_{-q} is well defined in $\frac{1}{q+1}$ and it satisfies $\gamma_{-q}(\frac{1}{q+1}) = 0\gamma_{-q}(-\frac{q}{q+1})$.*

   In fact, since $\frac{1}{q+1} \notin I_{-q}$, we fix $d = 1$ and consider $\frac{1}{(-q)^d} \frac{1}{q+1} = -\frac{1}{(-q)^2} \frac{q}{q+1} \in I_{-q}$. As

$$(81) \qquad \gamma_{-q} \left(-\frac{1}{(-q)^2} \cdot \frac{q}{q+1} \right) = 00\gamma_{-q}(-\frac{q}{q+1}),$$

   the $(-q)$ expansion of $\frac{1}{q+1}$ is $(0.0\gamma_{-q}(\frac{q}{q+1}))_{-q} = (.0\gamma_{-q}(\frac{q}{q+1}))_{-q}$.

3. *We explicity show that $\gamma_{-q}(x)$ is a $(-q)$-representation of $x \in I_{-q}$. In fact for every n it follows by the definitions given in (79) and in (80) that:*

(82)
$$
\begin{aligned}
x &= \lfloor -qx + \frac{q}{q+1} \rfloor \frac{1}{(-q)} + \frac{T_{-q}(x)}{(-q)} \\
&= \frac{x_1}{(-q)} + \frac{1}{(-q)} \frac{T_{-q}(\lfloor -qx + \frac{q}{q+1} \rfloor \frac{1}{q} + \frac{T_{-q}(x)}{q})}{q} \\
&= \frac{x_1}{(-q)} + \frac{x_2}{(-q)^2} + \frac{T_{-q}^2(x)}{(-q)^2} \\
&= \frac{x_1}{(-q)} + \frac{x_2}{(-q)^2} + \cdots + \frac{x_n}{(-q)^n} + \frac{T_{-q}^n(x)}{(-q)^n}
\end{aligned}
$$

and, by taking the limit for n to infinity, we get

$$
x = \sum_{i=1}^{\infty} \frac{x_i}{(-q)^i} = \pi_{-q}(\gamma_{-q}(x)).
$$

We now specify the second part of Remark 3.1 to the case of $q = G$, where G is the Golden Mean, and we introduce the problem of comparing the $(-q)$-expansions.

EXAMPLE 3.1. *Set $q = G$, where G is the Golden Mean. Then*

$$
\gamma_{-G}(-\frac{G}{G+1}) = 1(0)^{\omega} \quad \text{and} \quad \gamma_{-G}(\frac{1}{G+1}) = 01(0)^{\omega}.
$$

Note that $-\frac{G}{G+1} < \frac{1}{G+1}$ but $\gamma_{-G}(-\frac{G}{G+1}) >_{lex} \gamma_{-G}(\frac{1}{G+1})$.

Example 3.1 points out that by choosing a negative base $-q$ we loose the monotonicity between numerical value of a $(-q)$-expansion and lexicographic order. Nevertheless by introducing a different order, called *alternate order*, such a correspondence with the numerical values is restored.

DEFINITION 3.3 (Alternate order). *Let $\mathbf{x} = x_1x_2\cdots, \mathbf{y} = y_1y_2\cdots$ be infinite words or finite words with same length on an alphabet.*

Let $\mathbf{x} \neq \mathbf{y}$ and let i be the smallest index such that $x_i \neq y_i$. Then the alternate order $\prec$ *satisfies:*

(83)
$$
\mathbf{x} \prec \mathbf{y} \quad \text{if and only if} \quad (-1)^i(x_i - y_i) < 0.
$$

EXAMPLE 3.2. $\gamma_{-G}(-\frac{G}{G+1}) = 1(0)^{\omega} \prec 01(0)^{\omega} = \gamma_{-G}(\frac{1}{G+1})$.

The alternated order was implicitly defined in [Gru85] and lately reintroduced in [IS09]. We now show that this order inherits in the negative case the properties of $<_{lex}$ in the positive case.

PROPOSITION 3.1. *Fix $q > 1$. Let x and y be in I_{-q}. Then $x < y$ if and only if $\gamma_{-q}(x) \prec \gamma_{-q}(y)$.*

PROOF. Suppose that $\mathbf{x} = \gamma_{-q}(x) \prec \gamma_{-q}(y) = \mathbf{y}$. Then there exists $k \geq 1$ such that $x_i = y_i$ for $1 \leq i < k$ and $(-1)^k(x_k - y_k) < 0$. Suppose that k is even, $k = 2p$. Then $x_{2p} \leq y_{2p} - 1$. Thus

$$
x - y \leq -q^{-2p} + \sum_{i \geq 2p+1} x_i(-q)^{-i} - \sum_{i \geq 2p+1} y_i(-q)^{-i} < 0,
$$

since $\sum_{i \geq 1} x_{2p+i}(-q)^{-i}$ and $\sum_{i \geq 1} y_{2p+i}(-q)^{-i}$ are in I_{-q}. The case $k = 2p + 1$ is similar. The converse is immediate. $\qquad\square$

2.2. Characterization of the $(-q)$-shift.

DEFINITION 3.4 ($(-q)$-shift). *The $(-q)$-shift S_{-q} is defined as the closure of the set of the $(-q)$-expansions and it is a subshift of $A_{-q}^{\mathbb{Z}}$.*

For every $q > 1$, the S_{-q} can be characterized by a Parry-type result. To this end, define the sequence:

$$(84) \quad \gamma_{-q}^*\left(\frac{1}{q+1}\right) = \begin{cases} \gamma_{-q}\left(\frac{1}{q+1}\right) & \text{if } \gamma_{-q}\left(\frac{1}{q+1}\right) \text{ is not periodic with odd period} \\ (0d_1 \cdots d_{2p}(d_{2p+1} - 1))^\omega & \text{if } \gamma_{-q}\left(\frac{1}{q+1}\right) = (d_1 \cdots d_{2p} d_{2p+1})^\omega. \end{cases}$$

REMARK 3.2. *Even if the expansion $\gamma_{-q}^*\left(\frac{1}{q+1}\right)$ is not necessarily the quasi-greedy expansion of $\frac{1}{q+1}$, note the similarity with (15).*

THEOREM 3.1 (S. Ito and T. Sadahiro [IS09]). *Let $q > 1$ be a real number. A word $(x_i)_{i \in \mathbb{Z}}$ belongs to S_{-q} if and only if for all $n \in \mathbb{Z}$*

$$(85) \quad \gamma_{-q}\left(-\frac{q}{q+1}\right) \preceq x_n x_{n+1} \cdots \preceq \gamma_{-q}^*\left(\frac{1}{q+1}\right).$$

EXAMPLE 3.3 (follows Example 3.1). *Set $q = G$, the Golden Mean. Then $(x_i)_{i \in \mathbb{Z}}$ belongs to S_{-G} if and only if for all $n \in \mathbb{Z}$*

$$1(0)^\omega \preceq x_n x_{n+1} \cdots \preceq 01(0)^\omega.$$

Hence $(x_i)_{i \in \mathbb{Z}} \in S_{-G}$ if and only if $(x_i)_{i \in \mathbb{Z}}$ has no factors in the form $10^{2k+1}1$, with $k \in \mathbb{N}$.

REMARK 3.3. *Let us outline the analogies between Parry's result for positive base and Theorem 3.1.*

Parry's Theorem states that the sequences belonging to q-shift can be characterized using the following tools: an order on the sequences, i.e. $<_{lex}$, which preserves the order on the numerical values; the boundary sequence $\tilde{\gamma}_q(1)$, i.e. the quasi-greedy expansion 1 this being the upper bound of T_q; the iteration of the lexicographic comparison on every shift of the candidated sequence.

In the context of negative based expansions the alternate order $\prec$ plays the role of the lexicographical ordering (see Proposition 3.1). The upper bound in (85) is also in this case an expansion of the upper bound of T_{-q}, namely $\frac{1}{q+1}$. At a first glance, the condition "from below" in (85) is an asymmetry with respect to the condition (17). Nevertheless since the lower bound of T_q is 0, the adapted version of this condition to the positive case "collapses" in a comparison with $(0)^\omega$, which is always trivially satisfied. Finally the iteration of the comparison in (85) on every shift of the sequence is required, as well.

This comparison on the conditions means to show that the structures of the q-expansions and of the $(-q)$-expansion are in a certain sense similar: this intuitive argument is formalized in Section 5 where S_q and S_{-q} are shown to have the same entropy.

We conclude this section by restating Theorem 3.1 in a form which is more orientated to the theory of symbolic dynamical systems.

LEMMA 3.1. *Let $\gamma_{-q}\left(-\frac{q}{q+1}\right) = d_1 d_2 \cdots$ and let*

$$S = \{(w_i)_{i \in \mathbb{Z}} \in A_q^{\mathbb{Z}} \mid \forall n, \ d_1 d_2 \cdots \preceq w_n w_{n+1} \cdots \} :$$

(a) if $\gamma_{-q}\left(-\frac{q}{q+1}\right)$ is not a periodic sequence with odd period, then $S_{-q} = S$;

(b) if $\gamma_{-q}\left(-\frac{q}{q+1}\right)$ is a periodic sequence of odd period, $\gamma_{-q}\left(-\frac{q}{q+1}\right) = (d_1 \cdots d_{2p+1})^\omega$, then $S_{-q} = S \cap S'$ where

$$S' = \{(w_i)_{i \in \mathbb{Z}} \in A_q^{\mathbb{Z}} \mid \forall n, \ w_n w_{n+1} \cdots \preceq (0d_1 \cdots d_{2p}(d_{2p+1} - 1))^\omega \}.$$

3. Symbolic dynamical systems and the alternate order

In this section we study the properties of the subshift S, defined as follows. Let $\mathbf{s} = s_1 s_2 \cdots$ be a word in $A^{\mathbb{N}}$ such that $s_1 = \max A$ and for each $n \geq 1$, $\mathbf{s} \preceq s_n s_{n+1} \cdots$. Denote by S the subshift:

$$S := \{\mathbf{w} = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n, \ \mathbf{s} \preceq w_n w_{n+1} \cdots \}.$$

REMARK 3.4. *The interest on S is motivated by Lemma 3.1. In fact, as we prove in next section, the properties of S together with Lemma 3.1 provide a generalization of Theorem 1.8.*

First of all we prove that $F(S)$, namely the set of factors of the subshift S, can be recognized by an automaton. To this end, we construct a countable infinite automaton $\mathcal{A}_S$ as follows (see Fig.1). The set of states is $\mathbb{N}$. For each state $i \geq 0$, there is an edge $i \xrightarrow{s_{i+1}} i + 1$. Thus the state i is the name corresponding to the path labelled $s_1 \cdots s_i$. If i is even, then for each a such that $0 \leq a \leq s_{i+1} - 1$, there is an edge $i \xrightarrow{a} j$, where j is such that $s_1 \cdots s_j$ is the suffix of maximal length of $s_1 \cdots s_i a$. If i is odd, then for each b such that $s_{i+1} + 1 \leq b \leq s_1 - 1$, there is an edge $i \xrightarrow{b} j$ where j is maximal such that $s_1 \cdots s_j$ is a suffix of $s_1 \cdots s_i b$; and if $s_{i+1} < s_1$ there is one edge $i \xrightarrow{s_1} 1$. By construction, the deterministic automaton $\mathcal{A}_S$ recognizes exactly the words w such that every suffix of w is lower than or equal to $\mathbf{s}$ with respect to $\preceq$ and the result below follows.
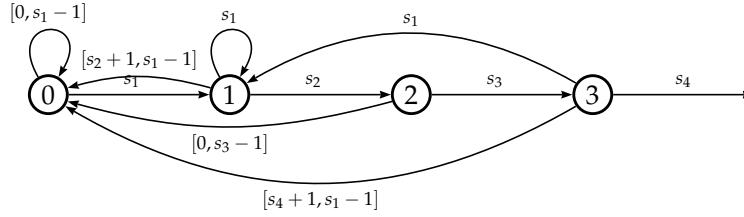

Figure 1: The automaton $\mathcal{A}_S$: $[a, b]$ denotes $\{a, a + 1, \ldots, b\}$ if $a \leq b$, ε else. In this figure it is assumed that $s_1 > s_j$ for $j \geq 2$.

PROPOSITION 3.2. *The subshift $S = \{w = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n, \ \mathbf{s} \preceq w_n w_{n+1} \cdots \}$ is recognizable by the countable infinite automaton $\mathcal{A}_S$.*

EXAMPLE 3.4. *Let us consider the sequence $\mathbf{s} = 1(0)^{\omega}$. By definition, $\mathcal{A}_S$ is constructed as follows (see Fig.2). The set of states is $\mathbb{N}$. There are an edge $0 \xrightarrow{1} 1$ and a looping edge $0 \xrightarrow{0} 0$. For each state $i > 0$, there is an edge $i \xrightarrow{0} i + 1$. If i is odd, then there is also an edge $i \xrightarrow{1} 1$.*
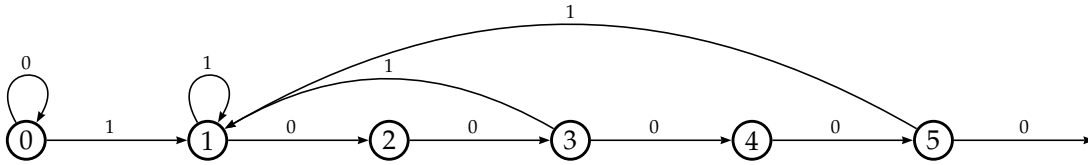
Figure 2: The automaton $\mathcal{A}_S$ with $S = \{w = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n, \ 1(0)^{\omega} \preceq w_n w_{n+1} \cdots \}$.

The automaton $\mathcal{A}_S$ accepts any finite word excepting those in the form $10^{2k+1}1$, if fact there are no edges labeled 1 leaving the even states. In other words, $\mathcal{A}_S$ recognizes $A^{\mathbb{N}} \setminus \{10^{2k+1}1 \mid k \in \mathbb{N}\}$.

On the other hand, the condition $\mathbf{x} \notin S$ is equivalent to $\sigma^n(\mathbf{x}) = 10^{2k+1}1\mathbf{x}'$, for some $k, n \geq 0$ and some $\mathbf{x}' \in A^{\omega}$. This implies that a word is in the form $10^{2k+1}1t$ if and only if it is not a factor of S. Then $F(S) = A^* \setminus \{10^{2k+1}1 \mid n \in \mathbb{N}\}$ and $\mathcal{A}_S$ recognizes $F(S)$.

PROPOSITION 3.3. *The subshift* $S = \{w = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n, \mathbf{s} \preceq w_n w_{n+1} \cdots \}$ *is sofic if and only if s is eventually periodic.*

PROOF. By definition, the subshift S is sofic if and only if the set of its finite factors $F(S)$ is recognizable by a finite automaton. Given a word u of A^*, denote by $[u]$ the right class of u modulo $F(S)$. Then in the automaton $\mathcal{A}_S$, for each state $i \geq 1, i = [s_1 \cdots s_i]$, and $0 = [\varepsilon]$. Suppose that $\mathbf{s}$ is eventually periodic, $\mathbf{s} = s_1 \cdots s_m (s_{m+1} \cdots s_{m+p})^\omega$, with m and p minimal. Thus, for each $k \geq 0$ and each $0 \leq i \leq p-1, s_{m+pk+i} = s_{m+i}$.
Case 1: p is even. Then $m + i = [s_1 \cdots s_{m+i}] = [s_1 \cdots s_{m+pk+i}]$ for every $k \geq 0$ and $0 \leq i \leq p-1$. Then the set of states of $\mathcal{A}_S$ is $\{0, 1, \ldots, m + p - 1\}$.
Case 2: p is odd. Then $m + i = [s_1 \cdots s_{m+i}] = [s_1 \cdots s_{m+2pk+i}]$ for every $k \geq 0$ and $0 \leq i \leq 2p - 1$. The set of states of $\mathcal{A}_S$ is $\{0, 1, \ldots, m + 2p - 1\}$.

Conversely, suppose that $\mathbf{s}$ is not eventually periodic. Then there exists an infinite sequence of indices $i_1 < i_2 < \cdots$ such that the sequences $s_{i_k} s_{i_k+1} \cdots$ are all different for all $k \geq 1$. Take any pair $(i_j, i_\ell), j, \ell \geq 1$. If i_j and i_ℓ do not have the same parity, then $s_1 \cdots s_{i_j}$ and $s_1 \cdots s_{i_\ell}$ are not right congruent modulo $F(S)$. If i_j and i_ℓ have the same parity, there exists $k \geq 0$ such that $s_{i_j} \cdots s_{i_j+k-1} = s_{i_\ell} \cdots s_{i_\ell+k-1} = v$ and, for instance, $(-1)^{i_j+k}(s_{i_j+k} - s_{i_\ell+k}) > 0$ (with the convention that, if $k = 0$ then $v = \varepsilon$). Then $s_1 \cdots s_{i_j-1} v s_{i_j+k} \in F(S), s_1 \cdots s_{i_\ell-1} v s_{i_\ell+k} \in F(S)$, but $s_1 \cdots s_{i_j-1} v s_{i_\ell+k}$ does not belong to $F(S)$. Hence $s_1 \cdots s_{i_j}$ and $s_1 \cdots s_{i_\ell}$ are not right congruent modulo $F(S)$, so the number of right congruence classes is infinite. By Theorem 1.1, $F(S)$ is not recognizable by a finite automaton. $\square$

EXAMPLE 3.5. *As in Example 3.4, let us consider the sequence* $s = 1(0)^\omega$. *The general algorithm for the construction of* $\mathcal{A}_S$ *yields the automaton in Fig.2. Nevertheless the automaton in Fig.3 is a* finite *automaton recognizing* $F(S) = A^* \setminus \{10^{2k+1}1 \mid k \in \mathbb{N}\}$ *thus, by definition, S is sofic.*

Figure 3: The finite automaton $\mathcal{A}'_S$, recognizing S

REMARK 3.5. *A celebrated example of subshift is the* even-shift. *Given a binary alphabet* $A = \{a, b\}$, *the even shift is the set of the bi-infinite sequences on A such that the number of occurrences of a between two consecutive b is even. In view o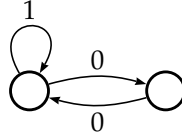f Example 3.4,* $S = \{w = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n, 1(0)^\omega \preceq w_n w_{n+1} \cdots \}$ *turns out to be the even shift with alphabet $\{0, 1\}$ and in $\mathcal{A}'_S$ can be derived by the standard construction given, for distance, in [Lot02, Chapter 1].*

In following result we establish a necessary and sufficient condition on S to be of finite type.

PROPOSITION 3.4. *The subshift* $S = \{w = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n, \mathbf{s} \preceq w_n w_{n+1} \cdots \}$ *is a subshift of finite type if and only if s is purely periodic.*

PROOF. Suppose that $\mathbf{s} = (s_1 \cdots s_p)^\omega$. Consider the finite set $X = \{s_1 \cdots s_{n-1} b \mid b \in A, (-1)^n(b - s_n) < 0, 1 \leq n \leq p\}$. We show that $S = S_X$. If w is in S, then w avoids X, and conversely. Now, suppose that S is of finite type. It is thus sofic, and by Proposition 3.3 $\mathbf{s}$ is

eventually periodic. If it is not purely periodic, then $\mathbf{s} = s_1 \cdots s_m (s_{m+1} \cdots s_{m+p})^\omega$, with m and p minimal, and $s_1 \cdots s_m \neq \varepsilon$. Let $I = \{s_1 \cdots s_{n-1} b \mid b \in A, \ (-1)^n (b - s_n) < 0, \ 1 \leq n \leq m\} \cup \{s_1 \cdots s_m (s_{m+1} \cdots s_{m+p})^{2k} s_{m+1} \cdots s_{m+n-1} b \mid b \in A, \ k \geq 0, (-1)^{m+2kp+n}(b - s_{m+n}) < 0, \ 1 \leq n \leq 2p\}$. Then $I \subset A^+ \setminus F(S)$. First, suppose there exists $1 \leq j \leq p$ such that $(-1)^j(s_j - s_{m+j}) < 0$ and $s_1 \cdots s_{j-1} = s_{m+1} \cdots s_{m+j-1}$. For $k \geq 0$ fixed, let $w^{(2k)} = s_1 \cdots s_m (s_{m+1} \cdots s_{m+p})^{2k} s_1 \cdots s_j \in I$. We have $s_1 \cdots s_m (s_{m+1} \cdots s_{m+p})^{2k} s_{m+1} \cdots s_{m+j-1} \in F(S)$. On the other hand, for $n \geq 2$, $s_n \cdots s_m (s_{m+1} \cdots s_{m+p})^{2k}$ is $\succ$ than the prefix of $\mathbf{s}$ of same length, thus

$$s_n \cdots s_m (s_{m+1} \cdots s_{m+p})^{2k} s_1 \cdots s_j \in F(S).$$

Hence any strict factor of $w^{(2k)}$ is in $F(S)$. Therefore for any $k \geq 0$, $w^{(2k)} \in X(S)$, and $X(S)$ is thus infinite: S is not of finite type. Now, if such a j does not exist, then for every $1 \leq j \leq p$, $s_j = s_{m+j}$, and $\mathbf{s} = (s_1 \cdots s_m)^\omega$ is purely periodic. $\qquad \square$

EXAMPLE 3.6. *Since the sequence $\mathbf{s} = 1(0)^\omega$ is not purely periodic, by Proposition 3.4, the subshift $S = \{\mathbf{w} = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n, \ 1(0)^\omega \preceq w_n w_{n+1} \cdots \}$ is not of finite type. This result agrees with the fact that S avoids the infinite set $\{10^{2k+1}1 \mid k \in \mathbb{N}\}$.*

EXAMPLE 3.7. *Consider the alphabet $A = \{0, 1, 2\}$ and $\mathbf{s} = (21)^\omega$. By Proposition 3.4, the subshift $S = \{\mathbf{w} = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n, \ \mathbf{s} \preceq w_n w_{n+1} \cdots \}$ is of finite type. This result agrees with the explicit computation of the minimal set of forbidden words, i.e. the finite set $\{20\}$.*

REMARK 3.6. *Let $\mathbf{s}' = s_1' s_2' \cdots$ be a word in $A^{\mathbb{N}}$ such that $s_1' = \min A$ and, for each $n \geq 1$, $s_n' s_{n+1}' \cdots \preceq \mathbf{s}'$. Let $S' = \{\mathbf{w} = (w_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n, \ w_n w_{n+1} \cdots \preceq \mathbf{s}'\}$. The statements in Propositions 3.2, 3.3 and 3.4 are also valid for the subshift S' (with the automaton $\mathcal{A}_{S'}$ constructed accordingly).*

4. A characterization of sofic $(-q)$-shifts and $(-q)$-shifts of finite type

This section is devoted to the generalization to the negative case of Theorem 1.8. In their paper, Ito and Sadahiro characterize the sofic $(-q)$-shifts by a condition on the $(-q)$-expansion of $-\frac{q}{q+1}$, namely the lower bound of T_{-q}. We now give a new proof based on the results of previous section.

THEOREM 3.2 (S. Ito and T. Sadahiro [IS09]). *The $(-q)$-shift is a sofic system if and only if $\gamma_{-q}(-\frac{q}{q+1})$ is eventually periodic.*

PROOF. First of all remark that the first digits of $\gamma_{-q}(-\frac{q}{q+1})$ and of $\gamma_{-q}(\frac{1}{q+1})$ are respectively equal to $\lfloor q \rfloor = \max A_{-q}$ and to $0 = \min A_{-q}$.

Suppose now $\gamma_{-q}(-\frac{q}{q+1})$ to be not purely periodic with an odd period. Then by Lemma 3.1 S_{-q} is equal to S, where $\mathbf{s} = \gamma_{-q}(-\frac{q}{q+1}) = s_1 s_2 \cdots$ and $s_1 = \max A_{-q}$. Then it follows by Proposition 3.3 that S_{-q} is sofic if and only if $\gamma_{-q}(-\frac{q}{q+1})$ is eventually periodic.

If $\gamma_{-q}(-\frac{q}{q+1})$ is purely periodic with an odd period. Then by Lemma 3.1 S_{-q} is equal to $S \cap S'$, where $\mathbf{s} = \gamma_{-q}(-\frac{q}{q+1})$, S' is defined in Remark 3.6 and $\mathbf{s}' = \gamma_{-q}(\frac{1}{q+1}) = s_1' s_2' \cdots$ and $s_1' = \min A_{-q}$. Since the intersection of two regular sets is regular, it follows that S_{-q} is sofic if and only if both S and S' are sofic, and by Proposition 3.3 this is equivalent to $\gamma_{-q}(-\frac{q}{q+1})$ eventually periodic. $\qquad \square$

We now prove a generalization to the negative case of the second part of Theorem 1.8.

THEOREM 3.3. *The $(-q)$-shift is a system of finite type if and only if $\gamma_{-q}(-\frac{q}{q+1})$ is purely periodic.*

PROOF. If $\gamma_{-q}(-\frac{q}{q+1})$ is purely periodic with an even period, the result follows from Theorem 3.1, Lemma 3.1 and Proposition 3.4. If $\gamma_{-q}(-\frac{q}{q+1})$ is purely periodic with an odd period, the result follows from Theorem 3.1, Lemma 3.1, Proposition 3.4, Remark 3.6, and the fact that the intersection of two finite sets is finite.

$\square$

EXAMPLE 3.8. *Let $G = \frac{1+\sqrt{5}}{2}$; then $\gamma_{-G}(-\frac{G}{G+1}) = 10^\omega$, and the $(-G)$-shift is a sofic system which is not of finite type.*

EXAMPLE 3.9. *Let $q = G^2 = \frac{3+\sqrt{5}}{2}$. Then $\gamma_{-q}(-\frac{q}{q+1}) = (21)^\omega$ and the $(-q)$-shift is of finite type: the set of minimal forbidden factors is $X(S_{-q}) = \{20\}$.*

EXAMPLE 3.10. *The automaton in Fig. 4 (right) recognizing the $(-G)$-shift is obtained by minimizing the result of the construction of Proposition 3.2.*


Figure 4: Finite automata for the G-shift (left) and for the $(-G)$-shift (right)

5. Entropy of the $(-q)$-shift

The considerations in Remark 3.3 suggest that the q-shift and the $(-q)$-shift have the same entropy, that is $\log q$. This intuition is supported by the following examples.

EXAMPLE 3.11. *In view of Fig 4 we may conclude that the entropy of the G-shift and of the $-G$-shift is the same. In fact the automata recognizing $F(S_G)$ and $F(S_{-G})$ share the same adjacency matrix and, by Theorem 1.2, the same entropy. By a direct computation, we have that greatest eigenvalue of the adjacency matrix is G, hence $h(S_{-G}) = \log G$.*

EXAMPLE 3.12. *The entropy of S_{-G^2} is $\log G^2$. This can be verified by recalling that $F(S_{-G^2}) = A^*_{-G^2} \setminus \{02\}$, with $A^*_{-G^2} = \{0, 1, 2\}$ (see Example 3.9). Then the minimal automaton recognizing $F(S_{-G^2})$ is:*
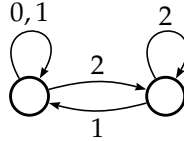
with adjacency matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and greatest eigenvalue $G^2 = 3 + \sqrt{5}/2$. Hence $h(S_{-G^2}) = \log G^2$.

A standard technique for computing the entropy of a subshift S is to construct a (not necessarily finite) automaton recognizing $F(S)$. Then the submatrices of the adjacency matrix are taken into account and for every n the greatest eigenvalue λ_n of the submatrix of order n is computed. A result proved in [Hof79] ensures that the limit λ of the sequence λ_n exists and it satisfies $h(S) = \log \lambda$. Unfortunately the construction of an automaton recognizing S_{-q} and the explicit computation of the λ_n's turned out to be very complicated.

We then decided an indirect approach. Our tools are the following:

  - the notion of topological entropy for one-dimensional dynamical systems, a one-dimensional dynamical system being a couple consisting in a bounded interval I and a piecewise continuous transformation $T : I \mapsto I$;

- a result by Y. Takahashi establishing the relation between topological entropies of one-dimensional and symbolic dynamical systems;
- a result by F. Shultz on the topological entropy of some one-dimensional dynamical systems.

Let us begin with the definition of topological entropy for one-dimensional dynamical systems.

DEFINITION 3.5 (topological entropy for one-dimensional dynamical systems). *Let (I, T) be a dynamical system.*

For every finite cover of I, say $\mathcal{C}$, set:

$$H(T, \mathcal{C}) := \limsup \frac{1}{n} \log N \left(\bigvee_{m=0}^{n-1} T^{-m} \mathcal{C} \right)$$

with $\bigvee$ denoting the finest common refinement and $N(\mathcal{C})$ denoting the number of elements of the smallest subcover of $\mathcal{C}$, a subcover of $\mathcal{C}$ being a subfamily of $\mathcal{C}$ still covering I.

The topological entropy *of (I, T) is given by the formula*

(86) $$h(I, T) := \sup H(T, \mathcal{C}).$$

In [Tak80] Takahashi proved the equality between the topological entropy of a piecewise continuous dynamical system and the topological entropy of an appropriate subshift. Before stating such a result we need a definition.

DEFINITION 3.6 (lap intervals). *Let $T : I \mapsto I$ be a piecewise continuous map on the interval I. The lap intervals $I_0, \ldots, I_l$ of T are closed intervals satisfying the following conditions:*

(a) $I_0 \cup \cdots \cup I_l = I$;

(b) T is monotone on each interval I_i, $i = 0, \ldots, l$;

(c) the number l is minimal under the conditions (a) and (b).

The number l is called lap number *and it is denoted $lap(T)$.*

REMARK 3.7. *If the map T is piecewise linear then the lap intervals are unique and they coincide with the intervals of continuity of T.*

THEOREM 3.4 (Y. Takahashi [Tak80]). *Let T be a piecewise continuous transformation over the closed interval I on itself. Let $\gamma_T : I \mapsto A_T^{\mathbb{N}}$ be the map defined by the relation $x \mapsto x_1 x_2 \cdots$ with x_n satisfying $T^n(x) \in I_{x_n}$. Define the subshift $X_T := \overline{\gamma_T(I)}$ in $A^{\mathbb{N}}$.*

If $lap(T)$ is finite then:

(87) $$h(X_T) = h(X_T, \sigma) = h(I, T).$$

The entropy in the very particular case of a piecewise linear map with constant slope is explicitly given in the following result.

PROPOSITION 3.5 (F. Shultz [Shu07, Proposition 3.7]). *Let T be a piecewise linear map with slope $\pm q$. Then the topological entropy of T is equal to $\log q$.*

We now prove our result.

THEOREM 3.5. *The topological entropy of S_{-q} is equal to $\log q$.*

PROOF. Consider the dynamical system (I_{-q}, T_{-q}), where T_{-q} is the $(-q)$-transform defined in (79). We extend the map T_{-q} to the closure of I_{-q} to fullfill the conditions of Theorem 3.4. By definition of $(-q)$-expansion, the subshift X_{T_q} coincides with the closure of the set of the $(-q)$-expansions in $A^{\mathbb{N}}$, whose entropy is the same as $S_{-q} \subset A^{\mathbb{Z}}$. As T_{-q} is piecewise linear, the lap

intervals coincide with the (finite) number of continuity intervals. Then by Theorem 3.4 and by Proposition 3.5 $h(S_{-q}) = h(I_{-q}, T_{-q}) = \log q$. $\qquad\qquad\square$

6. The Pisot case

In 4.4 we stated some results about the relation between the Pisot numbers and the eventually periodic expansions and about the arithmetics on Pisot base. Here we take in account both of these aspects by proposing some generalizations to the negative Pisot case.

6.1. Periodic $(-q)$-expansions and sofic $(-q)$-shifts. First of all we prove that Theorem 1.10 is still valid for the base $-q$.

THEOREM 3.6. *If q is a Pisot number, then every element of $\mathbb{Q}(q) \cap I_{-q}$ has an eventually periodic $(-q)$-expansion.*

REMARK 3.8. *Note that $\mathbb{Q}(q) = \mathbb{Q}(-q)$.*

PROOF OF THEOREM 3.6. First we introduce some notations. Denote by $M_q(X) = X^d - a_1 X^{d-1} - \cdots - a_d$ the minimal polynomial of q, by $q = q_1, \ldots, q_d$ the conjugates of q and by Q the matrix $((-q_j)^{-i})_{1 \le i, j \le d}$. Let x be arbitrarily fixed in $\mathbb{Q}(q) \cap I_{-q}$. Since $\mathbb{Q}(q) = \mathbb{Q}(-q)$, we can write

$$x = b^{-1} \sum_{i=0}^{d-1} c_i (-q)^i,$$

with b and c_i in $\mathbb{Z}$, $b > 0$ as small as possible in order to have uniqueness.

Let $(x_i)_{i \ge 1}$ be the $(-q)$-expansion of x and define

$$r_n = r_n^{(1)} = r_n^{(1)}(x) := \frac{x_{n+1}}{-q} + \frac{x_{n+2}}{(-q)^2} + \cdots = (-q)^n \left(x - \sum_{k=1}^{n} x_k (-q)^{-k} \right);$$

while for $2 \le j \le d$, let

$$r_n^{(j)} = r_n^{(j)}(x) = (-q_j)^n \left(b^{-1} \sum_{i=0}^{d-1} c_i (-q_j)^i - \sum_{k=1}^{n} x_k (-q_j)^{-k} \right).$$

Moreover consider the vector $R_n = (r_n^{(1)}, \ldots, r_n^{(d)})$.

The proof is now splitted in three steps.

Step 1. The sequence $(R_n)_{n \ge 1}$ is uniformly bounded.
We first remark that the last equality in (82) implies that $r_n(x) = T_{-q}^n(x)$, thus $|r_n| \le \frac{q}{q+1}$. It remains to study $r_n^{(j)}$ when $2 \le j$. Let $\eta = \max\{|q_j| \mid 2 \le j \le d\}$: since q is a Pisot number, then $\eta < 1$; this, together with $x_k \le \lfloor q \rfloor$, implies

$$| r_n^{(j)} | \le b^{-1} \sum_{i=0}^{d-1} |c_i| \eta^{n+i} + \lfloor q \rfloor \sum_{k=0}^{n-1} \eta^k < C$$

for some positive constant C.

Step 2. $R_n = b^{-1} Z_n Q$ for an uniquely determined $Z_n \in \mathbb{Z}^d$.
First we remark that if there exists a $(z_n^{(1)}, \ldots, z_n^{(d)})$ in $\mathbb{Z}^d$ such that $-q$ satisfies the equation with integer coefficients:

(88) $$r_n = b^{-1} \sum_{k=1}^{d} z_n^{(k)} (-q)^{-k}$$

then Lemma 1.1 implies that (88) is satisfied by any conjugate of q, say q_j, and we also have

$$r_n^{(j)} = b^{-1} \sum_{k=1}^{d} z_n^{(k)} (-q_j)^{-k}.$$

Hence we only need to show that for every n there exists $(z_n^{(1)}, \ldots, z_n^{(d)})$ in $\mathbb{Z}^d$ satisfying (88). The proof is by induction on n.

Set $n = 1$. Since $M_q(X) = X^d - a_1 X^{d-1} - \cdots - a_d$ is the minimal polynomial of q, we have

$$(89) \qquad\qquad 1 = -a_1(-q)^{-1} + a_2(-q)^{-2} + \cdots + (-1)^d a_d(-q)^{-d}.$$

Thus, by definition of r_1 and by (89):

$$r_1 = b^{-1} \left(\sum_{i=0}^{d-1} c_i(-q)^{i+1} - bx_1 \sum_{i=0}^{d-1} (-1)^i a_i(-q)^i \right) = b^{-1} \left(\frac{z_1^{(1)}}{-q} + \cdots + \frac{z_1^{(d)}}{(-q)^d} \right)$$

with $z_1^{(i)} = c_{i-1} - bx_1(-1)a_i$, for $i = 1, \ldots, d$.

Now, let us prove the inductive step. Since $r_{n+1} = -qr_n - x_{n+1}$ and since $z_n^{(1)} - bx_{n+1} \in \mathbb{Z}$, we may deduce by (89) that:

$$r_{n+1} = b^{-1} \left(z_n^{(1)} + \frac{z_1^{(2)}}{-q} + \cdots + \frac{z_n^{(d)}}{(-q)^{d-1}} - bx_{n+1} \right) = b^{-1} \left(\frac{z_{n+1}^{(1)}}{-q} + \cdots + \frac{z_{n+1}^{(d)}}{(-q)^d} \right),$$

for an appropriate vector of integers $(z_{n+1}^{(1)}, \ldots, z_{n+1}^{(d)})$ and this completes the proof of (88).

Step 3. x has an eventually periodic $(-q)$-expansion.

As the matrix Q is invertible, we may deduce by Step 1 and by Step 2 that $(Z_n)_{n \geq 1}$ is an uniformly bounded sequence in $\mathbb{Z}^d$. Hence there exist p and $m \geq 1$ such that $Z_{m+p} = Z_p$ and, consequently, $T^{m+p}(x) = r_{m+p} = r_p = T^p(x)$. Thus the $(-q)$-expansion of x is eventually periodic. □

REMARK 3.9. *As for the positive case, Theorem 3.6 establishes a remarkable analogy with the case of the expansion in integer base.*

As a corollary we get the following result.

THEOREM 3.7. *If q is a Pisot number then the $(-q)$-shift is a sofic system.*

PROOF. By Theorem 3.6 the $(-q)$-expansion of $-\frac{q}{q+1} \in \mathbb{Q}(q)$ is eventually periodic and by Theorem 3.2 this implies that $(-q)$-shift is a sofic system. □

6.2. Normalization. The *normalization* in base $-q$ is the function which maps any $(-q)$-representation on an alphabet C of digits of a given number of I_{-q} onto the $(-q)$-expansion of that number.

Our purpose is to generalize the following result.

THEOREM 3.8 (Ch. Frougny [Fro92]). *If q is a Pisot number, then normalization in base q on any alphabet C is realizable by a finite transducer.*

DEFINITION 3.7. *Denote $\mathcal{A}_{-q}(2c)$ the countable infinite automaton defined as follows. The set of states $Q(2c)$ consists of all $s \in \mathbb{Z}[q] \cap [-\frac{2c}{q-1}, \frac{2c}{q-1}]$. Transitions are of the form $s \xrightarrow{e} s'$ with $e \in \{-c, \ldots, c\}$ such that $s' = -qs + e$. The state 0 is initial; every state is terminal.*

LEMMA 3.2. *Let $C = \{-c, \ldots, c\}$, where $c \geq \lfloor q \rfloor$ is an integer and consider*

$$Z_{-q}(2c) = \left\{ (z_i)_{i \geq 0} \in \{-2c, \ldots, 2c\}^{\mathbb{N}} \ \Big| \ \sum_{i \geq 0} z_i(-q)^{-i} = 0 \right\}.$$

Then $Z_{-q}(2c)$ is recognized by $\mathcal{A}_{-q}(2c)$. Moreover if q is a Pisot number then the set of states of $\mathcal{A}_{-q}(2c)$ is finite.

PROOF. If $(e_i)_{i \geq 0}$ is an infinite path on $\mathcal{A}_{-q}(2c)$, call $(s_i)_{i \geq 0}$ the corresponding sequence of states. It follows by definition of the transitions that for every $n \geq 0$:

$$\sum_{i=0}^{n} \frac{e_i}{(-q)^i} = \frac{s_n}{(-q)^n}.$$

Then, as n goes to infinity, we get $\sum_{i=0}^{\infty} \frac{e_i}{(-q)^i} = 0$, hence $(e_i)_{i \geq 0} \in Z_{-q}(2c)$. On the other hand every sequence $(e_i)_{i \geq 0}$ in $Z_{-q}(2c)$ is an accepted path of $\mathcal{A}_{-q}(2c)$, whose states are $s_n := -\sum_{i=n}^{\infty} \frac{e_i}{(-q)^i}$, $n = 0, 1, \dots$. Hence $Z_{-q}(2c)$ is recognized by $\mathcal{A}_{-q}(2c)$.

Now we show that if q is Pisot then the set of states of $\mathcal{A}_{-q}(2c)$, i.e. $Q(2c)$, is finite. To this end, we call $M_q(X)$ the minimal polynomial of q and we denote by $q = q_1, \dots, q_d$ the conjugates of q. Moreover we define a norm on the discrete lattice of rank d, $\mathbb{Z}[X]/(M_q)$, as

(90) $$\|P(X)\| = \max_{1 \leq i \leq d} |P(q_i)|.$$

Every state s in $Q(2c)$ is associated with the label of the shortest path $f_0 f_1 \cdots f_n$ from 0 to s in the automaton. Thus $s = f_0(-q)^n + f_1(-q)^{n-1} + \cdots + f_n = P(q)$, with $P(X)$ in $\mathbb{Z}[X]/(M_q)$. Since $f_0 f_1 \cdots f_n$ is a prefix of a word of $Z_{-q}(2c)$, there exists $f_{n+1} f_{n+2} \cdots$ such that $(f_i)_{i \geq 0}$ is in $Z_{-q}(2c)$. Thus $s = |P(q)| < \frac{2c}{q-1}$. For every conjugate q_i, $2 \leq i \leq d$, $|q_i| < 1$, and $|P(q_i)| < \frac{2c}{1-|q_i|}$. Thus every state of $Q(2c)$ is bounded in norm, and so there is only a finite number of them. $\square$

THEOREM 3.9. *If q is a Pisot number, then normalization in base $-q$ on any alphabet C is realizable by a finite transducer.*

PROOF. We consider the *redundancy transducer* $\mathcal{R}_{-q}(c)$ obtained as follows. Starting from $\mathcal{A}_{-q}(2c)$, we replace each transition $s \xrightarrow{e} s'$ of $\mathcal{A}_{-q}(2c)$ by a set of transitions $s \xrightarrow{a|b} s'$, with $a, b \in \{-c, \dots, c\}$ and $a - b = e$.

By definition and by Lemma 3.2, the transducer $\mathcal{R}_{-q}(c)$ recognizes the set

$$\left\{ (x_1 x_2 \cdots, y_1 y_2 \cdots) \in C^{\mathbb{N}} \times C^{\mathbb{N}} \ \Big| \ \sum_{i \geq 1} x_i (-q)^{-i} = \sum_{i \geq 1} y_i (-q)^{-i} \right\}.$$

Moreover it follows again by Lemma 3.2 that if q is a Pisot number, then $\mathcal{R}_{-q}(c)$ is finite.

The normalization is thus obtained by keeping in $\mathcal{R}_{-q}(c)$ only the outputs y belonging to the $(-q)$-shift S_{-q}. In particular, by Theorem 3.7, S_{-q} is recognized by a finite automaton $\mathcal{D}_{-q}$. The finite transducer $\mathcal{N}_{-q}(c)$ doing the normalization is obtained by making the intersection of the output automaton of $\mathcal{R}_{-q}(c)$ with $\mathcal{D}_{-q}$. $\square$

As a consequence of the previous theorem we have that the addition in negative Pisot base is realizable by a finite transducer.

COROLLARY 3.1 (Addition in negative base). *If q is a Pisot number and x, y and $x + y$ are in I_{-q}, then the addition is realizable by a finite transducer.*

PROOF. Let $\gamma_{-q}(x) = x_1 x_2 \cdots$ and $\gamma_{-q}(y) = y_1 y_2 \cdots$. Then:

$$z_i := x_i + y_i \in C := \{0, 1, \dots, 2\lfloor q \rfloor\},$$

the normalization on the alphabet C, say $\nu_{-q,C}$, yields:

$$\gamma_{-q}(x + y) = \nu_{-q,C}(z_1 z_2 \cdots)$$

and, by Theorem 3.9, $\nu_{-q,C}$ is realizable by a finite transducer. $\square$

We conclude this section by showing that when the base is Pisot the conversion from negative to positive base is realizable by a finite transducer.

PROPOSITION 3.6 (Conversion from negative to positive Pisot base). *If q is a Pisot number, then the conversion from base $-q$ to base q is realizable by a finite transducer. The result is a sequence belonging the q-shift.*

PROOF. Let $x \in I_{-q}$, $x \geq 0$, such that $\gamma_{-q}(x) = x_1 x_2 x_3 \cdots$. Denote $\bar{a}$ the signed digit $(-a)$. Then $\overline{x_1} x_2 \overline{x_3} \cdots$ is a q-representation of x on the alphabet $\widetilde{A_{-q}} = \{-\lfloor q \rfloor, \ldots, \lfloor q \rfloor\}$. Thus the conversion is equivalent to the normalization in base q on the alphabet $\widetilde{A_{-q}}$, and this follows by Theorem 3.8.   □

7. A conversion algorithm from positive to negative base

Proposition 6.2 shows the actability of the conversion from positive to negative base for a particular class of bases, i.e. the Pisot numbers, with a finite machine. Hereafter we show an on-line conversion algorithm (not necessarily performable by a finite state machine) for $1 < q < 2$ with the only assumption on the input of being a q-expansion.

7.1. Definitions and preliminary settings. Fix $1 < q < 2$ and $A = \{0, 1\}$.

DEFINITION 3.8. *A conversion from q to $-q$ is a function $\chi : S_q \subset A^{\mathbb{N}} \mapsto A^{\mathbb{N}}$ satisfying*

$$\chi(\mathbf{x}) = \mathbf{y} \quad \Rightarrow \quad \pi_q(\mathbf{x}) = q^N \pi_{-q}(\mathbf{y}) \tag{91}$$

for some non negative integer N.

We denote δ_q the smallest *even* integer satisfying:

$$1 \leq \sum_{i=1}^{\delta_q} \frac{1}{q^i} = \frac{q^{\delta_q} - 1}{q^{\delta_q}(q-1)}; \tag{92}$$

and we define the following quantities:

$$M_q := \max\{\pi_q(w) \mid w \in A^{\delta_q} \cap F(S_q)\}$$

$$M_{-q} := \max\{\pi_{-q}(w) \mid w \in A^{\delta_q}\}$$

$$m_{-q} := \min\{\pi_{-q}(w) \mid w \in A^{\delta_q}\}$$

Denote by N the smallest integer satisfying:

$$\frac{M_q}{q^{2N}} \leq M_{-q}. \tag{93}$$

REMARK 3.10. *The definition of δ_q given in (92) implies that a q-expansion cannot contain δ_q consecutive occurrences of 1, hence $M_q \leq \sum_{i=1}^{\delta_q - 1} \frac{1}{q^i}$. It follows by this inequality that $N = 0$ if q is smaller than the Golden Mean and $N = 1$ otherwise.*

Hereafter are stated some relations and properties of M_{-q} and m_{-q}.

$$M_{-q} = \sum_{i=1}^{\delta_q/2} \frac{1}{q^{2i}}; \tag{94}$$

$$m_{-q} = -q \sum_{i=1}^{\delta_q/2} \frac{1}{q^{2i}}; \tag{95}$$

$$M_{-q} - m_{-q} = \frac{q^{\delta_q} - 1}{q^{\delta_q}(q-1)}; \tag{96}$$

$$M_{-q} \frac{1}{q_q^\delta - 1} \leq \frac{1}{q^{\delta_q}(q-1)}. \tag{97}$$

Note that last inequality follows by (92) and by $\sum_{i=1}^{\delta_q} \frac{1}{q^i} = \frac{q^{\delta_q}-1}{q^{\delta_q}(q-1)}$.

The set $\pi_{-q}(A^{\delta_q})$ contains all the numerical values in base $-q$ of the words of length δ_q. The order on $\pi_{-q}(A^{\delta_q})$ induces a (possibly partial) order on A^{δ_q} so that $\max A^{\delta_q} = M_{-q}$ and $\min A^{\delta_q} = M_{-q}$. We now prove a estimation from above of the difference between two consecutive values in $\pi_{-q}(A^{\delta_q})$.

PROPOSITION 3.7. *Let δ_q be the smallest even integer such that $1 \leq \sum_{i=1}^{\delta_q} \frac{1}{q^i}$. Then for every $u \in A^{\delta_q}$, $u \neq \max A^{\delta_q}$, there exists $v \in A^{\delta_q}$ such that:*

(98) $$0 < \pi_{-q}(v) - \pi_{-q}(u) \leq \frac{1}{q^{\delta_q}}.$$

PROOF. Since $u = u_1 \cdots u_{\delta_q} \neq \max A^{\delta_q}$ we can properly define k the greatest index such that $(-1)^k u_k < (-1)^k (1 - u_k)$. The minimality of δ_q implies that either:

$$\text{a)} \sum_{i=1}^{\delta_q-2} \frac{1}{q^i} < 1 \leq \sum_{i=1}^{\delta_q-1} \frac{1}{q^i}; \qquad \text{or} \qquad \text{b)} \sum_{i=1}^{\delta_q-1} \frac{1}{q^i} < 1.$$

Then we define the word v :

$$v := \begin{cases} u_1 \cdots u_{k-1}(1-u_k) \cdots u_{\delta_q} & \text{if } k > 1 \text{ and a) holds or if b) holds} \\ (1-u_1) \cdots (1-u_{\delta_q-1})u_{\delta_q} & \text{if } k = 1 \text{ and a) holds} \end{cases}$$

Since $k > 1$ and b) imply $\sum_{i=1}^{\delta_q-k} \frac{1}{q^i} < 1$, in these cases we have:

$$\pi_{-q}(v) - \pi_{-q}(u) = \frac{1}{q^k} - \frac{1}{q^k} \sum_{i=1}^{\delta_q-k} \frac{1}{q^i} \in \left(0, \frac{1}{q^\delta}\right].$$

Finally if $k = 1$ and if a) holds we get again:

$$\pi_{-q}(v) - \pi_{-q}(u) = \frac{1}{q^1} - \frac{1}{q^1} \sum_{i=1}^{\delta_q-2} \frac{1}{q^i} \in \left(0, \frac{1}{q^{\delta_q}}\right]$$

and this completes the proof. $\qquad \square$

7.2. The algorithm. Fix $q \in (1, 2)$ and $\delta := \delta_q$. Our conversion algorithm is showed in Algorithm 1.

Algorithm 1

---

input alphabet=output alphabet=$\{0, 1\}$

input: q-expansion $\mathbf{x} = x_1 x_2 \cdots$

output: sequence $\mathbf{y} := \text{output\_block}_1 \cdot \text{output\_block}_2 \cdots$ such that $\pi_q(\mathbf{x}) = q^{2N}\pi_{-q}(\mathbf{y})$.

$$\begin{aligned}
s_0 :=& \quad \frac{1}{q^\delta}\pi_q(x_1 x_2 \cdots x_\delta); \\
\text{output\_block}_0 :=& \quad \text{empty\_word};
\end{aligned}$$

     while $j \geq 1$ **do**

        { $\text{input\_block}_j = x_{(j+1)\delta+1}x_{(j+1)\delta+2} \cdots x_{(j+2)\delta}$;

            $s_j = q^\delta\left(s_{j-1} - q^{2N}\pi_{-q}(\text{output\_block}_{j-1})\right) + \pi_q(\text{input\_block}_j)/q^\delta$;

            $\text{output\_block}_j = \max\{w \in A^\delta \mid \pi_{-q}(w) \leq s_j\frac{1}{q^{2N}} - m_{-q}\frac{1}{q^\delta-1}\}$;

            output output_block_j ;

        }

To prove that Algorithm 1 performs the conversion we need the following technical lemma.

LEMMA 3.3. *Let* s *be in the interval*

$$J_q := \left[\frac{q^{\delta+2N}}{q^\delta - 1} m_{-q}, \frac{q^{\delta+2N}}{q^\delta - 1} M_{-q} - M_q \sum_{i=2}^{\infty} \frac{1}{q^{\delta i}} \right].$$

Then there exists a word $w \in A^\delta$ *such that*

(99) $$\pi_{-q}(w) \in H_{q,\mathsf{s}} := \left[-\frac{M_{-q}}{q^\delta - 1} + \frac{M_q}{q^{2N}} \sum_{i=2}^{\infty} \frac{1}{(q^{\delta_q})^i} + \frac{\mathsf{s}}{q^{2N}}, -\frac{m_{-q}}{q^\delta - 1} + \frac{\mathsf{s}}{q^{2N}} \right].$$

PROOF. Fix $\mathsf{s} \in J_q$. We first show that

(100) $$[m_{-q}, M_{-q}] \cap H_{\mathsf{s},q} \neq \emptyset.$$

To this end we estimate the end-terms of $H_{\mathsf{s},q}$. Set

(101) $$l_q := -\frac{M_{-q}}{q^\delta - 1} + \frac{M_q}{q^{2N}} \sum_{i=2}^{\infty} \frac{1}{(q^{\delta_q})^i} \quad \text{and} \quad r_q := -\frac{m_{-q}}{q^\delta - 1};$$

so that $H_{\mathsf{s},q} = [l_q, r_q] + \frac{\mathsf{s}}{q^{2N}}$. Now we have that:

$$l_q + \frac{\mathsf{s}}{q^{2N}} \leq l_q + \frac{1}{q^{2N}} \max J_q = M_{-q};$$

and that

$$r_q + \frac{\mathsf{s}}{q^{2N}} \geq -\frac{m_{-q}}{q^\delta - 1} + \frac{1}{q^{2N}} \min J_q = m_{-q}.$$

Hence we may deduce (100).

Now we remark that $|H_{\mathsf{s},q}| \geq \frac{1}{q^\delta}$. In fact:

$$r_q - l_q = (M_{-q} - m_{-q}) \frac{1}{q^\delta - 1} - \frac{M_q}{q^{2N}} \sum_{i=2}^{\infty} \frac{1}{(q^{\delta_q})^i}$$

$$\geq (M_{-q} - m_{-q}) \frac{1}{q^\delta - 1} - M_{-q} \sum_{i=2}^{\infty} \frac{1}{(q^{\delta_q})^i}$$

$$= \frac{1}{q^\delta(q-1)} - M_{-q} \frac{1}{q^\delta(q^2 - 1)}$$

$$\geq \frac{1}{q^\delta} \left(\frac{1}{q-1} - \frac{1}{q^\delta(q-1)} \right)$$

$$\geq \frac{1}{q^\delta}.$$

Now let us distinguish the cases $H_{\mathsf{s},q} \subseteq [m_{-q}, M_{-q}]$ and $H_{\mathsf{s},q} \not\subseteq [m_{-q}, M_{-q}]$. If $H_{\mathsf{s},q} \subseteq [m_{-q}, M_{-q}]$ then Proposition 7.1 implies that there exists an element of $\pi_{-q}(A^\delta)$ in $H_{\mathsf{s},q}$ because $|H_{\mathsf{s},q}|$ is larger than the minimum distance between two consecutive values of $\pi_{-q}(A^\delta)$. On the other hand if $H_{\mathsf{s},q} \not\subseteq [m_{-q}, M_{-q}]$ then (100) implies that $H_{\mathsf{s},q}$ and $[m_{-q}, M_{-q}]$ overlap. Thus one of the two end-terms of $[m_{-q}, M_{-q}]$ belongs to $H_{\mathsf{s},q}$. Since by definition m_{-q} and M_{-q} are numerical values of A^δ, the proof is complete.

□

PROOF OF ALGORITHM 1. We divide the proof in two parts.

Part 1. The output block output_block$_\mathsf{j}$ *is well defined for every* j *and the states* $(\mathsf{s_j})_{\mathsf{j} \geq 1}$ *are bounded.*

Set J_q and $H_{q,\mathsf{s}}$ as they are defined in Lemma 3.3. Following the notation given in the proof of Lemma 3.3, let l_q and r_q be those of (101).

With these settings, the definition of output_block$_j$ can be rewritten for any $j \geq 1$ as

(102) $$\text{output\_block}_j = \max\{w \in A^\delta \mid \pi_{-q}(w) \leq r_q + \frac{s_j}{q^{2N}}\}.$$

Consequently, we may deduce by Lemma 3.3 that

(103) $$s_j \in J_q \quad \Rightarrow \quad \text{output\_block}_j \text{ is well defined and it belongs to } H_{q,s_j}$$

and it suffices to prove that for every $j \geq 1$, s_j is in J_q. If $j = 1$ then the definition of N implies:

$$0 \leq s_1 = q^\delta s_0 + \frac{1}{q^\delta} \pi_q(\text{input\_block}_1) = \pi_q(x_1 x_2 \cdots x_\delta) + \frac{1}{q^\delta} \pi_q(x_1 x_2 \cdots x_\delta)$$

$$\leq M_q + \frac{1}{q^\delta} M_q = M_q \frac{q^\delta}{q^\delta - 1} - M_q \sum_{i=2}^{\infty} \frac{1}{(q^\delta)^i} \leq \frac{q^{\delta+2N}}{q^\delta - 1} M_{-q} - M_q \sum_{i=2}^{\infty} \frac{1}{(q^\delta)^i}$$

and we deduce $s_1 \in J_q$.

Now suppose $j > 1$ and $s_{j-1} \in J_q$. Then, by Lemma 3.3, $\pi_{-q}(\text{output\_block}_{j-1}) \in [l_q, r_q] + \frac{s_{j-1}}{q^{2N}}$. By applying the definition of s_j we have:

(104) $$s_j = q^\delta \left(s_{j-1} - q^{2N} \cdot \pi_{-q}(\text{output\_block}_{j-1}) \right) + \frac{1}{q^\delta} \pi_q(\text{input\_block}_j)$$

$$\leq \frac{q^{\delta+2N}}{q^\delta - 1} M_{-q} - M_q \sum_{i=1}^{\infty} \frac{1}{(q^\delta)^i} + \frac{1}{q^\delta} \pi_q(\text{input\_block}_j)$$

$$\leq \frac{q^{\delta+2N}}{q^\delta - 1} M_{-q} - M_q \sum_{i=2}^{\infty} \frac{1}{(q^\delta)^i}.$$

Remark that the last term of the inequalities above is equal to right end-term of J_q. Similarly, by substituting the right end-term of $H_{q,s_{j-1}}$ (which contains output_block$_{j-1}$) in (104) we get:

$$s_j \geq q^\delta \left[s_{j-1} - q^{2N} \left(-m_{-q} \frac{1}{q^\delta - 1} + \frac{s_{j-1}}{q^{2N}} \right) \right] + \frac{1}{q^\delta} \pi_q(\text{input\_block}_j)$$

$$\geq \frac{q^{\delta+2N}}{q^\delta - 1} m_{-q}$$

which is the left end-term of J_q. Then $s_j \in J_q$ and this completes the proof of the inductive step.

Part 2. The output is a $(-q)$-representation of $\pi_q(\mathbf{x})$.

First we prove by induction that for every $j \geq 0$,

(105)
$$\pi_q(\mathbf{x}) = q^{\delta+2N} \left(\pi_{-q}(\text{output\_block}_0) + \frac{1}{q^\delta} \pi_{-q}(\text{output\_block}_1) + \cdots + \frac{1}{q^{\delta j}} \pi_{-q}(\text{output\_block}_j) \right)$$

$$+ \frac{s_j}{q^{\delta(j-1)}} + \sum_{i=\delta(j+1)+1}^{\infty} \frac{x_i}{q^i}.$$

Set $j = 0$. Since $\pi_{-q}(\text{output\_block}_0) = 0$ we have:

$$\pi_q(\mathbf{x}) = \pi_q(x_1 x_2 \cdots x_\delta) + \sum_{i=\delta+1}^{\infty} \frac{x_i}{q^i} = q^\delta s_0 + \sum_{i=\delta+1}^{\infty} \frac{x_i}{q^i}$$

and, consequently, (105). Now, assuming the inductive hypothesis for $j - 1 \geq 0$ we have:

$$
\begin{aligned}
\pi_q(\mathbf{x}) =&\, q^{\delta+2N} \left(\pi_{-q}(\mathsf{output\_block_0}) + \cdots + \frac{1}{(q^\delta)^{j-1}} \pi_{-q}(\mathsf{output\_block_{j-1}}) \right) \\
&+ \frac{q^{\delta+2N}}{q^{\delta j}} \pi_{-q}(\mathsf{output\_block_j}) + \frac{\mathsf{s_{j-1}}}{q^{\delta(j-2)}} - \frac{q^{\delta+2N}}{q^{\delta j}} \pi_{-q}(\mathsf{output\_block_j}) + \sum_{i=\delta j+1}^{\infty} \frac{x_i}{q^i} \\
=&\, q^{\delta+2N} \left(\pi_{-q}(\mathsf{output\_block_0}) + \cdots + \frac{1}{q^{\delta j}} \pi_{-q}(\mathsf{output\_block_j}) \right) \\
&+ \frac{\mathsf{s_{j-1}}}{q^{\delta(j-2)}} - \frac{q^{2N}}{q^{\delta(j-1)}} \pi_{-q}(\mathsf{output\_block_j}) + \frac{1}{q^{\delta j}} \pi_q(\mathsf{input\_block_j}) + \sum_{i=\delta(j+1)+1}^{\infty} \frac{x_i}{q^i} \\
=&\, q^{\delta+2N} \left(\pi_{-q}(\mathsf{output\_block_0}) + \cdots + \frac{1}{(q^\delta)^{j}} \pi_{-q}(\mathsf{output\_block_j}) \right) \\
&+ \frac{\mathsf{s_j}}{q^{\delta(j-1)}} + \sum_{i=\delta(j+1)+1}^{\infty} \frac{x_i}{q^i}.
\end{aligned}
$$

and this completes the proof of (105)

To conclude the proof we recall that: $\mathbf{y}$ is defined as the (infinite) concatenation of the blocks $\mathsf{output\_block_j}$; δ is supposed to be even so that $q^\delta = (-q)^\delta$; in the first part of the proof we showed that $\mathsf{s_j}$ is bounded. Hence by (105):

$$
\begin{aligned}
\pi_q(\mathbf{x}) - q^{2N}\pi_{-q}(\mathbf{y}) &= \lim_{j\to\infty} |\pi_q(\mathbf{x}) - q^{\delta+2N} \sum_{i=1}^{j} \frac{1}{q^{i\delta}} \pi_{-q}(\mathsf{output\_block_j})| \\
&= \lim_{j\to\infty} |\frac{\mathsf{s_j}}{q^{\delta(j-1)}} + \sum_{i=\delta(j+1)+1}^{\infty} \frac{x_i}{q^i}| = 0.
\end{aligned}
$$

$\square$

8. Overview of original contributions, conclusions and further developments

Symbolic dynamical systems and the alternate order. We studied a symbolic dynamical system S associated with a given infinite word s satisfying some properties with respect to the alternate order on infinite words.

We first constructively proved this result.

PROPOSITION. *The subshift $S = \{w = (w_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n, \ \mathbf{s} \preceq w_n w_{n+1} \cdots \}$ is recognizable by a countable infinite automaton.*

Then the following characterization has been showed.

PROPOSITION. *Consider the subshift $S = \{w = (w_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}} \mid \forall n, \ \mathbf{s} \preceq w_n w_{n+1} \cdots \}$. Then*

(a) S is a sofic subshift if and only if s is eventually periodic;
(b) S is a subshift of finite type if and only if s is purely periodic.

A characterization of sofic $(-q)$-shifts and $(-q)$-shifts of finite type. The results above have been then applied to the $(-q)$-shift.

THEOREM. *The $(-q)$-shift is a system of finite type if and only if $\gamma_{-q}(-\frac{q}{q+1})$ is purely periodic.*

Entropy of the $(-q)$-shift. We computed the topological entropy of the $(-q)$-shift, that turned out to be equal to the topological entropy of the classical q-shift.

THEOREM 3.10. *The topological entropy of the $(-q)$-shift is equal to $\log q$.*

The Pisot case. Some results about the expansions in base Pisot have been extended to the negative case.

THEOREM. *If q is a Pisot number, then every element of $\mathbb{Q}(q) \cap I_{-q}$ has an eventually periodic $(-q)$-expansion.*

THEOREM. *If q is a Pisot number, then normalization in base $-q$ on any alphabet C, the addition in base $-q$ and the conversion from base $-q$ to base q are realizable by a finite transducer.*

A conversion algorithm from positive to negative base. We finally introduced an on-line algorithm for the conversion from positive to negative base, with the assumption that the input is a q-expansion. We have no control on the output: it is not necessarily a $(-q)$-expansion.

Conclusions and further developments. The results in this chapter mean to confirm that $(-q)$-expansions are a natural generalization of the classical q-expansions, the main difference consisting on the orderings — alternate for the negative bases and lexicographical for the positive ones. It could be interesting to look for a further generalization of such orders to the complex base, with the purpose of establishing a relation between the (partial) ordering on the complex numbers and their representations.

The author also wishes the conversion algorithm in Section 7 to be extended to the case of $q \geq 2$ and this algorithm to be useful to prove that the conversion from positive to negative Pisot base is realizable by a *finite* on-line machine.

CHAPTER 4

Generalized Golden Mean for ternary alphabets

Throughout this chapter we investigate the uniqueness of the expansions in real base $q > 1$ and digits in an arbitrary alphabet $A = \{a_1, \ldots, a_J\}$. For two-letter alphabets $A = \{a_1, a_2\}$ the Golden Mean $G := (1 + \sqrt{5})/2$ plays a special role: there exist nontrivial unique expansions in base q if and only if $q > G$. Our purpose is to determine analogous critical bases for each ternary alphabet $A = \{a_1, a_2, a_3\}$.

1. Introduction

In the fifties, Rényi [Rén57] introduced a new numeration system with non-integer base q and alphabet with integer digits $\{0, 1, \ldots, \lfloor q \rfloor\}$. Since then the representations in non-integer base have been intensively studied both from a measure theoretical and number theoretical point of view. One of the most interesting features of these numeration systems is the redundancy of the representation: e.g. Sidorov proved that if $1 < q < 2$ almost every number has a continuum of distinct expansions [Sid03]. Several other works have been dedicated to the study of the unique expansions and of their topological properties, see [EJK90], [EHJ91], [DK93],[KL98] and [DVK09].

Since the uniqueness of an expansion is preserved by enlarging the base ([DK93]), there exist some boundary bases separating the possible topological structures of the set of unique expansions, denoted by U_q. For example for bases lower than the Golden Mean, the set U_q contains only two elements, called *trivial expansions* ([DK93]); while it has been proved in [GS01] that for bases complied between the Golden Mean and the Komornik-Loreti constant U_q is a denumerable set, and for bases larger than the Komornik-Loreti constant U_q has a continuum of elements.

When we consider a general alphabet, i.e. a set in which the distance between consecutive digits is not necessarily constant, many results extend: e.g. in [Ped05] the unique expansions have been lexicographically characterized.

The study of expansions with arbitrary alphabets is related to some controllability problems in robotics. For example in [CP01] the digits of an arbitrary alphabets are considered controls of an unidimensional discrete control system and the set of representable numbers is interpreted as the set of reachable points starting from the origin.

Organization of the chapter.

2. Expansions in non-integer base with alphabet with deleted digits

2.1. Basic definitions. Let A be an alphabet and $q > 1$ a real number. We denote by π_q the map from A^ω (the set of finite and infinite words with digits in A) to $\mathbb{R}$ defined by $\pi_q(z) := \sum_{i=1}^{|z|} \frac{z_i}{q^i}$. For every $z \in A^\omega$, $\pi_q(z)$ is called *numerical value* in base q of z. An *expansion* of a real number x is a sequence $\mathbf{x} = x_1 x_2 \cdots \in A^{\mathbb{N}}$ whose numerical value in base q is equal to x, i.e.

$$(106) \qquad x = \pi_q(\mathbf{x}) = \sum_{i=1}^{\infty} \frac{x_i}{q^i}.$$

An expansion $\mathbf{x} \in A^{\mathbb{N}}$ is *eventually minimal* if $\mathbf{x} = w(\min A)^\omega$ for some $w \in A^*$.

A sequence $\mathbf{x}$ is called *univoque* if for every other $\mathbf{y} \in A^{\mathbb{N}}$, $\pi_q(\mathbf{x}) \neq \pi_q(\mathbf{y})$ or, equivalently, if and only if the expansion of $\pi_q(\mathbf{x})$ is unique. The lexicographically greatest expansion of a real number x is called *greedy expansion* of x and it is denoted by $\gamma_q(x)$. The lexicographically greatest *not* eventually minimal expansion of x is called the *quasi-greedy expansion* of x and it is denoted by $\tilde{\gamma}_q(x)$. Similarly, we can consider the lexicographically smallest expansion of a real number x, named the *lazy expansion* of x and denoted by $\lambda_q(x)$. The lexicographically smallest *not* eventually maximal expansion of x is called the *quasi-lazy expansion* of x and it is denoted by $\tilde{\lambda}_q(x)$.

REMARK 4.1. *We remark some direct consequences of the definition of quasi-greedy, quasi-lazy and univoque sequences.*

1. *If the greedy (resp. lazy) expansion of a real number x is not eventually minimal (resp. maximal), then it coincides with the quasi-greedy (resp. quasi-lazy) expansion of x.*
2. *If $\mathbf{x}$ is a quasi-greedy (resp. quasi-lazy) expansion then $\max A\mathbf{x}$ (resp. $\min A\mathbf{x}$) is a quasi-greedy (resp. quasi-lazy) expansion.*
3. *If $\mathbf{x}$ is respectively quasi-greedy, quasi-lazy or univoque then any suffix of $\mathbf{x}$ is respectively quasi-greedy, quasi-lazy or univoque.*

2.2. Representability conditions. The existence of the quasi-greedy and of the quasi-lazy expansions is put in relation with the gaps of the alphabet, namely the differences between consecutive digits, by the following theorem.

THEOREM 4.1 (M. Pedicini [Ped05]). *Let $A = \{a_1, \ldots, a_J\}$. Every $x \in \left[\frac{\min A}{q-1}, \frac{\max A}{q-1} \right]$ has a quasi-greedy and a quasi-lazy expansion in base $q > 1$ if and only if for every $i = 1, \ldots, J-1$:*

(107)
$$a_{j+1} - a_j \leq \frac{\max A - \min A}{q-1}.$$

We conclude this section by stating this classical monotonicity result.

PROPOSITION 4.1. *Let $\mathbf{x}$ and $\mathbf{y}$ be both quasi-greedy expansions or both quasi-lazy expansions in base $q > 1$. Then:*

$$\mathbf{x} <_{lex} \mathbf{y} \quad \text{if and only if} \quad \pi_q(\mathbf{x}) < \pi_q(\mathbf{y}).$$

2.3. Univoqueness conditions. We recall some univoqueness conditions for general alphabets.

THEOREM 4.2 (M. Pedicini [Ped05]). *Let $A = \{a_1, \ldots, a_J\}$ and $q > 1$ such that for every $j = 1, \ldots, J-1$:*

$$a_{j+1} - a_j \leq \frac{\max A - \min A}{q-1}.$$

An expansion $\mathbf{x} = (x_i)$ is unique in base q if and only if the following conditions are satisfied:

(108)
$$\sum_{i=1}^{\infty} \frac{x_{n+i} - a_1}{q^i} < a_{j+1} - a_j \quad \text{whenever} \quad x_n = a_j < a_J;$$

(109)
$$\sum_{i=1}^{\infty} \frac{a_J - x_{n+i}}{q^i} < a_j - a_{j-1} \quad \text{whenever} \quad x_n = a_j > a_1.$$

Moreover

- $\mathbf{x}$ *is a quasi-greedy expansion if and only if $\mathbf{x}$ is not eventually minimal and it satisfies (108);*
- $\mathbf{x}$ *is a quasi-lazy expansion if and only if $\mathbf{x}$ is not eventually maximal and it satisfies (109).*

COROLLARY 4.1. *Let $\mathbf{x} \in A^{\mathbb{N}}$ be univoque in base q. Then for every $t > 0$ the sequences:*

- $\mathbf{x} + t := (x_1 + t)(x_2 + t) \cdots \in (A + t)^{\omega}$;

- $t\mathbf{x} := (tx_1)(tx_2)\cdots \in (tA)^\omega$;
- $D(\mathbf{x}) := (\max A - x_1)(\max A - x_2)\cdots \in D(A)^\omega$

are univoque in base q.

REMARK 4.2. *For every given alphabet A, the sequences $(\min A)^\omega$ and $(\max A)^\omega$ are univoque in any base. For this reason they are called* trivial. *From now on, when referring to an univoque sequence, we implicitely assume that such sequence is not trivial.*

Quasi-greedy and quasi-lazy expansions, as well as univoque ones, can also be characterized by mean of a lexicographic comparison with the quasi-greedy and the quasi-lazy expansions of the gaps of the alphabet.

THEOREM 4.3 (M. Pedicini [Ped05]). *Let $A = \{a_1, \ldots, a_J\}$ and $q > 1$ such that for every $j = 1, \ldots, J-1$:*

$$a_{j+1} - a_j \leq \frac{\max A - \min A}{q - 1}.$$

An expansion $\mathbf{x} = (x_i)$ is unique in base q if and only if the following conditions are satisfied:

(110) $(x_{n+i}) <_{lex} \tilde{\gamma}_q(a_{j+1} - a_j)$ *whenever* $x_n = a_j < a_J$;

(111) $(x_{n+i}) >_{lex} \tilde{\lambda}_q(a_j - a_{j-1})$ *whenever* $x_n = a_j > a_1$.

Moreover

- $\mathbf{x}$ *is a quasi-greedy expansion if and only if $\mathbf{x}$ is not eventually minimal and it satisfies (110);*
- $\mathbf{x}$ *is a quasi-lazy expansion if and only if $\mathbf{x}$ is not eventually maximal and it satisfies (111).*

REMARK 4.3. *The condition (107) can be rewritten by stressing the bound on the base q, namely:*

$$q \leq \frac{\max A - \min A}{\max_{j<J}\{a_{j+1} - a_j\}} =: Q_A.$$

3. Critical bases

As a consequence of Theorem 4.2 we have the following result.

THEOREM 4.4 (Existence of critical base). *For every given set $X \subset A^{\mathbb{N}}$ there exists a number*

$$1 \leq q_X \leq Q_A$$

such that

$$q > q_X \implies \text{every sequence } \mathbf{x} \in X \text{ is univoque in base } q;$$

$$1 < q < q_X \implies \text{not every sequence } \mathbf{x} \in X \text{ is univoque in base } q.$$

PROOF. If $X = \varnothing$, then we may choose $q_X = 1$. If X is nonempty, then for each sequence $\mathbf{x} \in X$, each condition of Theorem 4.2 is equivalent to an inequality of the form $q > q_\alpha$. Since we consider only bases q satisfying (107), we may assume that $q_\alpha \leq Q_A$ for every α. Then

$$q_X := \max\{1, \sup q_\alpha\}$$

has the required properties. $\square$

DEFINITION 4.1 (Critical base). *The number q_X is called the* critical base *of X. If $X = \{\mathbf{x}\}$ is a one-point set, then $q_{\mathbf{x}} := q_X$ is also called the critical base of the sequence $\mathbf{x}$.*

REMARK 4.4. *If X is a nonempty finite set of eventually periodic sequences, then the supremum $\sup q_\alpha$ in the above proof is actually a maximum. In this case not all sequences $\mathbf{x} \in X$ are univoque in base $q = q_X$.*

EXAMPLE 4.1. *Consider the ternary alphabet* $A = \{0, 1, 3\}$ *and the periodic sequence* $(x_i) = (31)^\omega$. *By the periodicity of* (x_i) *we have for each n either* $x_n = 3$ *and* $(x_{n+i}) = (13)^\omega$ *or* $x_n = 1$ *and* $(x_{n+i}) = (31)^\omega$. *According to the preceding remark Theorem 4.2 contains only three conditions on q. For* $x_n = 3$ *we have the condition*

$$\sum_{i=1}^{\infty} \frac{3 - x_{n+i}}{q^i} < 2 \iff \frac{2q}{q^2 - 1} < 2,$$

while for $x_n = 1$ *we have the following two conditions:*

$$\sum_{i=1}^{\infty} \frac{3 - x_{n+i}}{q^i} < 1 \iff \frac{2}{q^2 - 1} < 1$$

and

$$\sum_{i=1}^{\infty} \frac{x_{n+i}}{q^i} < 2 \iff \frac{3}{q - 1} - \frac{2}{q^2 - 1} < 2.$$

They are equivalent approximatively to the inequalities $q > 1.61803$, $q > 1.73205$ *and* $q > 2.18614$ *respectively, so that* $q_{\mathbf{x}} \approx 2.18614$.

It is well-known that for the alphabet $A = \{0, 1\}$ there exist nontrivial univoque sequences in base q if and only if $q > \frac{1+\sqrt{5}}{2}$. There exists a "generalized Golden Mean" for every alphabet:

COROLLARY 4.2. *There exists a number* $1 < G_A \le Q_A$ *such that*

$$q > G_A \implies \text{ there exist nontrivial univoque sequences;}$$

$$1 < q < G_A \implies \text{ there are no nontrivial univoque sequences.}$$

PROOF. If a sequence is univoque in some base, then it is also univoque in every larger base. If there exists a base satisfying (107) in which there exist nontrivial univoque sequences, then it follows that the infimum of such bases satisfies the requirements for G_A, except perhaps the strict inequality $G_A > 1$. Otherwise we may choose $G_A := Q_A$.

To show that $G_A > 1$, we prove that if $q > 1$ is sufficiently close to one, then the only univoque sequences are a_1^ω and a_J^ω. We show that it suffices to choose $q > 1$ so small that the following three conditions are satisfied:

$$(112) \qquad \frac{a_J - a_1}{q - 1} \ge a_{j+1} - a_{j-1}, \quad j = 2, \ldots, J - 1,$$

$$(113) \qquad \frac{a_j - a_1}{q} + \frac{1}{q} \cdot \frac{a_J - a_1}{q - 1} \ge (a_2 - a_1) + \frac{a_j - a_{j-1}}{q}, \quad j = 2, \ldots, J,$$

$$(114) \qquad \frac{a_J - a_j}{q} + \frac{1}{q} \cdot \frac{a_J - a_1}{q - 1} \ge (a_J - a_1) + \frac{a_{j+1} - a_j}{q}, \quad j = 1, \ldots, J - 1.$$

The proof consists of three steps. Let (x_i) be a univoque sequence in base q.

If $x_n = a_j$ for some n and $1 < j < J$, then the conditions of Theorem 4.2 imply that

$$\sum_{i=1}^{\infty} \frac{x_{n+i} - a_1}{q^i} < a_{j+1} - a_j \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{a_J - x_{n+i}}{q^i} < a_j - a_{j-1}.$$

Taking their sum we conclude that

$$\frac{a_J - a_1}{q - 1} < a_{j+1} - a_{j-1},$$

which contradicts (112). This proves that $x_n \in \{a_1, a_J\}$ for every n.

If $x_n = a_1$ and $x_{n+1} = a_j > a_1$ for some n, then applying Theorem 4.2 we obtain that

$$\sum_{i=1}^{\infty} \frac{x_{n+i} - a_1}{q^i} < a_2 - a_1 \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{a_J - x_{n+i+1}}{q^i} < a_j - a_{j-1}.$$

Dividing the second inequality by q and adding the result to the first one we obtain that

$$\frac{a_j - a_1}{q} + \frac{1}{q} \cdot \frac{a_J - a_1}{q - 1} < (a_2 - a_1) + \frac{a_j - a_{j-1}}{q},$$

which contradicts (113). This proves that $x_n = a_1$ implies $x_{n+1} = a_1$ for every n.

Finally, if $x_n = a_J$ and $x_{n+1} = a_j < a_J$ for some n, then applying Theorem 4.2 we obtain that

$$\sum_{i=1}^{\infty} \frac{a_J - x_{n+i}}{q^i} < a_J - a_{J-1} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{x_{n+i+1} - a_1}{q^i} < a_{j+1} - a_j.$$

Dividing the second inequality by q and adding the result to the first one we now obtain that

$$\frac{a_J - a_j}{q} + \frac{1}{q} \cdot \frac{a_J - a_1}{q - 1} < (a_J - a_{J-1}) + \frac{a_{j+1} - a_j}{q},$$

which contradicts (114). This proves that $x_n = a_J$ implies $x_{n+1} = a_J$ for every n. $\qquad\square$

DEFINITION 4.2. *The number G_A is called the* critical base *of the alphabet A.*

The following invariance properties of critical bases readily follow from the definitions; they will simplify our proofs.

LEMMA 4.1. *The critical base does not change if we replace the alphabet A*

- *by $t + A = \{t + a_j \mid j = 1, \ldots, m\}$ for some real number t;*
- *by $tA = \{ta_j \mid j = 1, \ldots, m\}$ for some nonzero real number t;*
- *by the* conjugate alphabet $A' := \{a_J + a_1 - a_j \mid j = 1, \ldots, J\}$.

PROOF. First we note that $Q_A = Q_{t+A} = Q_{tA} = Q_{A'}$. Fix a base $1 < q \leq Q_A$ and a sequence (x_i) of real numbers. It follows from the definitions that the following properties are equivalent:

- (x_i) is an expansion of x for the alphabet A;
- $(t + x_i)$ is an expansion of $x + \frac{t}{q-1}$ for the alphabet $t + A$;
- (tx_i) is an expansion of tx for the alphabet tA;
- $(a_J + a_1 - x_i)$ is an expansion of $\frac{a_J + a_1}{q-1} - x$ for the alphabet A'.

Hence if one of these expansions is unique, then the others are unique as well. $\qquad\square$

4. Normal ternary alphabets

The invariance properties stated in Corollary 4.1 are an important tool to simplify the characterization of the critical base in ternary alphabets. The idea is to consider *normal alphabets* in the form $A_m := \{0, 1, m\}$, with $m \geq 2$. The restriction to these alphabets does not constitute a loss of generality. In fact normal alphabets can be obtained from general alphabet by an appropriate composition of translations, scaling and (if necessary) the dual operation, the univoqueness and the generalized Golden Mean being invariant with respect to these operations. Hereafter we explicitly show how to normalize an arbitrary alphabet.

DEFINITION 4.3. *The* normalized form *of A, denoted by $N(A)$, is a normal ternary alphabet obtained by A by the composition of a translation, a rescaling and (if necessary) of the dual operation.*

Before proving the existence of $N(A)$, we give an example.

EXAMPLE 4.2. *Let us consider the alphabet $A = \{1, 4, 6\}$. A can be normalized as follows:*

$$A - 1 \overset{\text{translation}}{=} \{0, 3, 5\}$$

$$D(A - 1) \overset{\text{dual}}{=} \{0, 2, 5\}$$

$$\frac{1}{2} D(A - 1) \overset{\text{scaling}}{=} \{0, 1, 2.5\};$$

PROPOSITION 4.2. *Every ternary alphabet can be normalized using only translations, scalings and dual operations.*

PROOF. The procedure in Example 4.2 can be generalized as follows. Starting from $A = \{a_1, a_2, a_3\}$, we define:

$$A_1 := A - a_1;$$

$$A_2 := \begin{cases} A_1 & \text{if } \min\{a_2 - a_1, a_3 - a_2\} = a_2 - a_1 \\ D(A_1) & \text{otherwise} \end{cases};$$

$$A_3 := \frac{1}{\min\{a_2 - a_1, a_3 - a_2\}} A_2.$$

Remark that $\min A_3 = \min A_2 = \min A_1 = 0$; that $\max A_3 \geq \max A_2 \geq 2$ and that the middle digit of A_3 is equal to 1. Hence $N(A) = A_3 = A_m$ with m satisfying:

$$m = \max\{\frac{a_3 - a_1}{a_2 - a_1}, \frac{a_3 - a_1}{a_3 - a_2}\}.$$

$\square$

We denote by $\phi : A \mapsto N(A)$ the normalizing map described in the proof of Proposition 4.2. Fixing an alphabet A, ϕ can be splitted in the one-to-one application on the digits:

$$(115) \qquad \phi_A(a) = \begin{cases} 0 & \text{if } a = a_1 \text{ and } a_2 - a_1 \leq a_3 - a_2 \\ & \text{or} \quad a = a_3 \text{ and } a_2 - a_1 > a_3 - a_2 \\ 1 & \text{if } a = a_2 \\ m = \frac{a_3 - a_1}{\min\{a_2 - a_1, a_3 - a_2\}} & \text{if } a = a_3 \text{ and } a_2 - a_1 \leq a_3 - a_2 \\ & \text{or} \quad a = a_1 \text{ and } a_2 - a_1 > a_3 - a_2 \end{cases}.$$

With a little abuse of notation we also denote by ϕ_A the normalizing map on infinite words $\phi_A : A^{\mathbb{N}} \mapsto N(A)^{\omega}$, defined by $\phi_A(\mathbf{x}) = \phi_A(x_1)\phi_A(x_2) \cdots$.

The interest on ϕ_A, and on the normal forms in general, is motivated by the following invariance results, which are a direct consequences of Corollary 4.1 and Proposition 4.2.

PROPOSITION 4.3. *Let A be a ternary alphabet and ϕ_A be the normalizing map defined above. For every $q > 1$, $\mathbf{x} \in A^{\mathbb{N}}$ is univoque in base q if and only if $\phi_A(\mathbf{x})$ is univoque in base q. In particular, any ternary alphabet and its normal form share the same generalized Golden Mean.*

5. Critical bases for ternary alphabets

5.1. Stating of the main result. Our main result on the critical base for normal ternary alphabets is the following.

THEOREM 4.5. *There exists a continuous function $p : [2, \infty) \to \mathbb{R}$, $m \mapsto p_m$ satisfying*

$$2 \leq p_m \leq P_m := 1 + \sqrt{\frac{m}{m-1}}$$

for all m such that the following properties hold true:

(a) for each $m \geq 2$, there exist nontrivial univoque expansions if $q > p_m$ and there are no such expansions if $q < p_m$.

(b) we have $p_m = 2$ if and only if $m = 2^k$ for some positive integer k;

(c) the set $C := \{m \geq 2 : p_m = P_m\}$ is a Cantor set, i.e., an uncountable closed set having neither interior nor isolated points; its smallest element is $1 + x \approx 2.3247$ where x is the first Pisot number, i.e., the positive root of the equation $x^3 = x + 1$;

(d) each connected component (m_d, M_d) of $[2, \infty) \setminus C$ has a point μ_d such that p is strictly decreasing in $[m_d, \mu_d]$ and strictly increasing in $[\mu_d, M_d]$.

Moreover, we will determine explicitly the function p and the numbers m_d, M_d, μ_d.

5.2. Driving ideas and organization of the proof. Since the proofs are rather technical let us explain how we arrived to the above results and to the particular constructions in the proof. Using a computer program we have found univoque sequences for many particular values of m for small values $q > 2$ containing only two different digits. Trying to find an explanation, we proved that if q is sufficiently close to one, namely if $1 < q \leq P_m := 1 + \sqrt{\frac{m}{m-1}}$, then no sequence satisfying these conditions (except the trivial sequence 0^ω) can contain infinitely many zero digits (Proposition 4.6). Since by removing a finite number of initial elements a univoque sequence remains univoque, it follows that if there exists a nontrivial univoque sequence in some base $1 < q \leq P_m$, then there exists also a nontrivial univoque sequence in this base which only contains the digits 1 and m. Assuming that there are such sequences in some base $1 < q \leq P_m$, this allows us to investigate two-digit sequences instead of more complicated three-digit sequences.

In the next stage we made an extensive computer research in order to find such univoque sequences. For most integer values of $m = 2, 3, \ldots, 65536$ we have found essentially one such sequence, namely the periodic sequence $(m^{h_1}1)^\omega$ with $h_1 = [\log_2 m]$. Using the above characterization it is easy to see that this sequence can be univoque in a base q only if $q > p_m := \max\{p'_m, p''_m\}$ where p'_m and p''_m are defined by the equations

$$\pi_{p'_m}\left((m^{h_1}1)^\omega\right) = m - 1 \quad \text{and} \quad \pi_{p''_m}\left((m^{h_1}1)^\omega\right) = \frac{m}{p''_m - 1} - 1,$$

and one can prove that the condition $q > p_m$ is also *sufficient*.

However, there were seven exceptional integer values: 5, 9, 130, 258, 2051, 4099, 32772, for which we have found only univoque sequences of a more involved form, e. g. $(m^2 1 m^2 1 m 1)^\omega$ for $m = 5$ and $(m^3 1 m^2 1)^\omega$ for $m = 9$ (Example 4.5). Each such sequence provided a univoque sequence in some base $1 < q < P_m$ also for small perturbations of the integer digit m. In this way we could also cover many real numbers $m \in [2, 65536]$ but not all of them.

In order to find nontrivial univoque sequences in bases $1 < q \leq P_m$ for each real number $m \in [2, \infty)$, we have generalized the structure of the above sequences. This led to the notion of *admissible sequences*. To any admissible sequence d an interval I_d is associated. It turned out that each admissible sequence $d \neq 1^\omega$ provides a nontrivial univoque sequence in some base $1 < q \leq P_m$ for real digits m belonging to I_d. We show the intervals I_d provide a disjoint covering of $[2, \infty)$. The other properties mentioned in Theorem 4.5 were obtained by a closer investigation of the admissible sequences d and the corresponding intervals I_d.

The proof of Theorem 4.5 is organized as follows. In Section 6 we define and we investigate some lexicographical properties of the admissible sequences. In Section 7 we fix an admissible sequence d an we define a map $m \to p_m$. The map p_m depends on d, as well, but the subscript d is omitted for brevity. The monotonicity properties of p_m are then investigated in Proposition 4.4. In particular we show that for every m the the condition

(116) $$p_m \leq P_m$$

holds if and only if m belongs to an interval denoted $I_d := [m_d, M_d]$ (Proposition 4.5). For any given $m \in I_d$, the quantity p_m is our candidate critical base for the alphabet $A_m = \{0, 1, m\}$. In

Section 8 we first explain the role of P_m, by showing that for bases smaller than P_m the univoque sequences have particular restrictions (Proposition 4.6) . We then apply this result to show that if $m \in I_d$, for every $q \le p_m$ there are not univoque sequences and for $q > p_m$ an appropriate tail of the associated admissible sequence is univoque (Proposition 4.7). In Proposition 4.8 the intervals I_d are proved to partitionate $[2, \infty)$ so that p_m is well defined. Proposition 4.8, together with Proposition 4.5, also shows that the I_d's with aperiodic d form a cantor set C, whose properties have been studied in Proposition 4.8 as well.

6. Admissible sequences

This section contains some preliminary technical results.

DEFINITION 4.4. *A sequence* $d = (d_i) = d_1 d_2 \cdots$ *of zeroes and ones is* admissible *if*

(117) $$0 d_2 d_3 \cdots \le (d_{n+i}) \le d_1 d_2 d_3 \cdots$$

for all $n = 0, 1, \ldots$.

EXAMPLE 4.3.

- *The trivial sequences* 0^ω *and* 1^ω *are admissible.*
- *More generally, the sequences* $(1^N 0)^\omega$ *($N = 1, 2, \ldots$) and* $(10^N)^\omega$ *($N = 0, 1, \ldots$) are admissible.*
- *The sequence* $(11010)^\omega$ *is also admissible.*
- *The (not purely periodic) sequence* 10^ω *is admissible.*

In order to clarify the structure of admissible sequences we give an equivalent recursive definition.

DEFINITION 4.5 (Recursive definition of admissible sequences). *Given a sequence* $h = (h_i)$ *of positive integers, starting with*

$$S_h(0, 1) := 1 \quad and \quad S_h(0, 0) := 0$$

we define the blocks $S_h(j, 1)$ *and* $S_h(j, 0)$ *for* $j = 1, 2, \ldots$ *by the recursive formulae*

$$S_h(j, 1) := S_h(j-1, 1)^{h_j} S_h(j-1, 0)$$

and

$$S_h(j, 0) := S_h(j-1, 1)^{h_j - 1} S_h(j-1, 0).$$

Observe that $S_h(j, 1)$ and $S_h(j, 0)$ depend only on $h_1, \ldots, h_j$, so that they can also be defined for every finite sequence $h = (h_j)$ of length $\ge j$. We also note that $S_h(j, 0) = S_h(j-1, 0)$ whenever $h_j = 1$.

Let us denote by ℓ_j the length of $S_h(j, 1)$. Setting furthermore $\ell_{-1} := 0$, then the length of $S_h(j, 0)$ is equal to $\ell_j - \ell_{j-1}$. We observe that ℓ_j tends to infinity as $j \to \infty$.

If the sequence $h = (h_j)$ is given, we often omit the subscript h and we simply write $S(j, 1)$ and $S(j, 0)$.

Let us mention some properties of these blocks that we use in the sequel. Given two finite blocks A and B we write for brevity

- $A \to B$ or $B = \cdots A$ if B ends with A;
- $A < B$ or $A \cdots < B \cdots$ if $A a_1 a_2 \cdots < B b_1 b_2 \cdots$ lexicographically for any sequences (a_i) and (b_i) of zeroes and ones;
- $A \le B$ or $A \cdots \le B \cdots$ if $A < B$ or $A = B$.

LEMMA 4.2 (Lexicographic properties of recursively defined admissible sequences). *For any given sequence $h = (h_j)$ the blocks $S(j,1)$ and $S(j,0)$ have the following properties:*

(a) We have

(118) $$S(j,1) = 1S(1,0)\cdots S(j,0) \quad \text{for all} \quad j \geq 0;$$

(119) $$S(0,0)\cdots S(j-1,0) \to S(j,1) \quad \text{for all} \quad j \geq 1;$$

(120) $$S(0,0)\cdots S(j-1,0) \to S(j,0) \quad \text{whenever} \quad h_j \geq 2;$$

(121) $$S(j,0) < S(j,1) \quad \text{for all} \quad j \geq 0.$$

(b) If $A_j \to S(j,1)$ for some nonempty block A_j, then $A_j \leq S(j,1)$.
(c) If $B_j \to S(j,0)$ for some nonempty block B_j, then $B_j \leq S(j,0)$.
(d) The finite sequence $S(j,1)S(j,0)$ is obtained from $S(j,0)S(j,1)$ by changing one block 10 to 01.

PROOF.

(a) *Proof of* (118). For $j = 0$ we have $S(j,1) = 1$ by definition. If $j \geq 1$ and the identity is true for $j - 1$, then the identity for j follows by using the equality $S(j,1) = S(j-1,1)S(j,0)$ coming from the definition of $S(j,1)$ and $S(j-1,1)$.

Proof of (119) *and* (120). For $j = 1$ we have $S(0,0) = 0$ and $S(1,0) = 1^{h_1-1}0$, so that $S(0,0) \to S(1,0) \to S(1,1)$. (The condition $h_1 \geq 2$ is not needed here.) Proceeding by induction, if (119) holds for some $j \geq 1$, then both hold for $j + 1$ because

$$S(0,0)\cdots S(j-1,0)S(j,0) \to S(j,1)S(j,0) \to S(j+1,1),$$

and in case $h_{j+1} \geq 2$ we have also $S(j,1)S(j,0) \to S(j+1,0)$.

Proof of (121). The case $j = 0$ is obvious because the left side begins with 0 and the right side begins with 1. If $j \geq 1$ and (121) holds for $j - 1$, then we deduce from the inequality $S(j-1,0)\cdots < S(j-1,1)\cdots$ that

$$S(j,0)\cdots = S(j-1,1)^{h_j-1}S(j-1,0)\cdots < S(j-1,1)^{h_j}\cdots.$$

Since $S(j,1)$ begins with $S(j-1,1)^{h_j}$, this implies (121) for j.

b) We may assume that $A_j \neq S(j,1)$; this excludes the case $j = 0$ when we have necessarily $A_0 = S(0,1) = 1$. For $j = 1$ we have $S(j,1) = 1^{h_1}0$ and $A_j = 1^t0$ with some integer $0 \leq t < h_1$, and we conclude by observing that $1^t0\cdots < 1^{h_1}\cdots$.

Now let $j \geq 2$ and assume that the result holds for $j - 1$. Using the equality $S(j,1) = S(j-1,1)^{h_j}S(j-1,0)$ we distinguish three cases.

If $A_j \to S(j-1,0)$, then we have the implications

$$A_j \to S(j-1,0) \implies A_j \to S(j-1,1) \quad \text{and} \quad A_j \neq S(j-1,1)$$
$$\implies A_j \cdots < S(j-1,1) \cdots$$
$$\implies A_j \cdots < S(j,1) \cdots.$$

If $A_j = A_{j-1}S(j-1,1)^tS(j-1,0)$ for some $0 \leq t < h_j$, $A_{j-1} \to S(j-1,1)$ and $A_{j-1} \neq S(j-1,1)$, then

$$A_{j-1}\cdots < S(j-1,1)\cdots \implies A_j \cdots < S(j-1,1)\cdots$$
$$\implies A_j \cdots < S(j,1)\cdots.$$

Finally, if $A_j = S(j-1,1)^tS(j-1,0)$ for some $0 \leq t < h_j$, then using (121) we have

$$A_j \cdots < S(j-1,1)^{t+1}\cdots$$

and therefore

$$A_j \cdots < S(j,1) \cdots.$$

(c) Proceeding by induction, the case $j = 0$ is obvious because then we have necessarily $B_0 = S(0,0) = 0$. Let $j \geq 1$ and assume that the property holds for $j-1$ instead of j. If $h_j > 1$, then the case of j follows by applying part b) with h_j replaced by $h_j - 1$. If $h_j = 1$, then we have $S(j,0) = S(j-1,0)$ and applying b) we conclude that

$$B_j \to S(j,0) \implies B_j \to S(j-1,0) \implies B_j \leq S(j-1,0) = S(j,0).$$

(d) The assertion is obvious for $j = 0$ because $S(0,1) = 1$ and $S(0,0) = 0$. Proceeding by induction, let $j \geq 1$ and assume that the result holds for $j-1$. Comparing the expressions

$$S(j,1)S(j,0) = S(j-1,1)^{h_j}S(j-1,0)S(j-1,1)^{h_j-1}S(j-1,0)$$

and

$$S(j,0)S(j,1) = S(j-1,1)^{h_j-1}S(j-1,0)S(j-1,1)^{h_j}S(j-1,0)$$

we see that $S(j,0)S(j,1)$ is obtained from $S(j,1)S(j,0)$ by changing the first block $S(j-1,1)S(j-1,0)$ to $S(j-1,0)S(j-1,1)$.   □

The following lemma is a partial converse of (119).

LEMMA 4.3 (Other lexicographic properties of recursively defined admissible sequences). *If A is a block of length ℓ_{N-1} in some sequence $S(N,a_1)S(N,a_2) \cdots$ with $N \geq 1$ and $(a_i) \subset \{0,1\}$, then $A \geq S(0,0) \cdots S(N-1,0)$. Furthermore, we have $A = S(0,0) \cdots S(N-1,0)$ if and only if $A \to S(N,a_i)$ with some $a_i = 1$.*

PROOF. The case $N = 1$ is obvious because then $S(0,0) = 0$ implies that $A = 0$, and $S(1,1) = 1^{h_1}0$ ends with 0.

Now let $N \geq 2$ and assume by induction that the result holds for $N-1$. Writing $A = BC$ with a block B of the same length as $S(0,0) \cdots S(N-2,0)$ and applying the induction hypothesis to B in the sequence

$$S(N,a_1)S(N,a_2) \cdots = \left(S(N-1,1)^{h_N-1+a_i}S(N-1,0) \right)$$

we obtain that $B \to S(N-1,1)$ for one of the blocks on the right side and thus $B = S(0,0) \cdots S(N-2,0)$. Then it follows from our assumption that C has the same length as $S(N-1,0)$ and $C \leq S(N-1,0)$. Since $S(N-1,0) < S(N-1,1)$, the block containing B must be followed by a block $S(N-1,0)$. We conclude that $C = S(N-1,0)$ and therefore $A = BC = S(0,0) \cdots S(N-1,0)$ and

$$A \to S(N-1,1)^{h_N-1+a_i}S(N-1,0) = S(N,a_i)$$

for some $a_i = 1$.   □

LEMMA 4.4 (Lexicographic and recursive definitions of admissible sequences are equivalent). *A sequence $d = (d_i)$ is admissible if and only if one of the following three conditions is satisfied:*

  - *$d = 0^\omega$;*
  - *there exists an infinite sequence $h = (h_i)$ of positive integers such that d begins with $S_h(N,1)$ for every $N = 0, 1, \ldots$;*
  - *$d = S_h(N,1)^\omega$ with some nonnegative integer N and a finite sequence $h = (h_1, \ldots, h_N)$ of positive integers.*

PROOF. It follows from the definition that $d_1 = 1$ for all admissible sequences other than 0^ω. In the sequel we consider only admissible sequences beginning with $d_1 = 1$. We omit the subscript h for brevity.

Let $d = (d_i)$ be an admissible sequence. Setting $d_i^0 := d_i$ we have

$$d = S(0, d_1^0) S(0, d_2^0) \cdots$$

with the admissible sequence (d_i^0).

Proceeding by recurrence, assume that

$$d = S(j, d_1^j) S(j, d_2^j) \cdots$$

for some integer $j \geq 0$ with an admissible sequence (d_i^j) and positive integers $h_1, \ldots, h_j$. (We need no such positive integers for $j = 0$.)

If $(d_i^j) = 1^\omega$, then $d = S(j, 1)^\omega$. Otherwise there exists a positive integer h_{j+1} such that d begins with $S(j, 1)^{h_{j+1}} S(j, 0)$. Since the sequence (d_i^j) is admissible, we have

$$0 d_2^j d_3^j \ldots \leq d_{n+1}^j d_{n+2}^j \ldots \leq d_1^j d_2^j \ldots$$

for all $n = 0, 1, \ldots$. Since the map $(c_i) \mapsto (S(j, c_i))$ preserves the lexicographic ordering by (121), it follows that

$$S(j, 0) S(j, d_2^j) S(j, d_3^j) \cdots \leq S(j, d_{n+1}^j) S(j, d_{n+2}^j) \cdots \leq S(j, d_1^j) S(j, d_2^j) \cdots$$

for all $n = 0, 1, \ldots$. Thanks to the definition of h_{j+1} we conclude that

$$S(j, 0) S(j, 1)^{h_{j+1}-1} S(j, 0) \cdots \leq S(j, d_{n+1}^j) S(j, d_{n+2}^j) \cdots \leq S(j, 1)^{h_{j+1}} S(j, 0) \cdots$$

for all $n = 0, 1, \ldots$. This implies that each block $S(j, 0)$ is followed by at least $h_{j+1} - 1$ and at most h_{j+1} consecutive blocks $S(j, 1)$, so that

$$d = S(j + 1, d_1^{j+1}) S(j + 1, d_2^{j+1}) \cdots$$

for a suitable sequence (d_i^{j+1}) of zeroes and ones. The admissibility of (d_i^j) can then be rewritten in the form

$$(122) \quad S(j, 0) S(j + 1, 0) S(j + 1, d_2^{j+1}) S(j + 1, d_3^{j+1}) \cdots$$

$$\leq S(j, d_{n+1}^j) S(j, d_{n+2}^j) \cdots$$

$$\leq S(j + 1, 1) S(j + 1, d_2^{j+1}) S(j + 1, d_3^{j+1}) \cdots$$

for $n = 0, 1, \ldots$.

We claim that the sequence (d_i^{j+1}) is also admissible. We have $d_1^{j+1} = 1$ by the definition of h_{j+1}. It remains to show that

$$S(j + 1, 0) S(j + 1, d_2^{j+1}) S(j + 1, d_3^{j+1}) \cdots$$

$$\leq S(j + 1, d_{k+1}^{j+1}) S(j + 1, d_{k+2}^{j+1}) S(j + 1, d_{k+3}^{j+1}) \cdots$$

$$\leq S(j + 1, 1) S(j + 1, d_2^{j+1}) S(j + 1, d_3^{j+1}) \cdots$$

for $k = 0, 1, \ldots$.

The second inequality is a special case of the second inequality of (122). The first inequality is obvious for $k = 0$. For $k \geq 1$ it is equivalent to

$$S(j, 0) S(j + 1, 0) S(j + 1, d_2^{j+1}) S(j + 1, d_3^{j+1}) \cdots$$

$$\leq S(j, 0) S(j + 1, d_{k+1}^{j+1}) S(j + 1, d_{k+2}^{j+1}) S(j + 1, d_{k+3}^{j+1}) \cdots$$

and this is a special case of the first inequality of (122) because $S(j+1, d_k^{j+1})$ ends with $S(j,0)$.

It follows from the above construction that (d_i) has one of the two forms specified in the statement of the lemma.

Turning to the proof of the converse statement, first we observe that if d begins with $S(N,1)$ for some sequence $h = (h_i)$ and for some integer $N \geq 1$, then

(123) $$d_n \cdots d_{\ell_N} < d_1 \cdots d_{\ell_N-n+1} \quad \text{for} \quad n = 2, \ldots, \ell_N;$$

this is just a reformulation of Lemma 4.2 (b).

If $d_1 d_2 \cdots$ begins with $S(N,1)$ for all N, then the second inequality of (117) follows for all $n \geq 1$ by using the relation $\ell_N \to \infty$. Moreover, the inequality is strict. For $n = 0$ we have clearly equality.

If $d = S(N,1)^\omega$ for some $N \geq 0$, then d is ℓ_N-periodical so that the second inequality of (117) follows from (123) for all n, except if n is a multiple of ℓ_N; we get strict inequalities in these cases. If n is a multiple of ℓ_N, then we have obviously equality again.

It remains to prove the first inequality of (117). If $d = S(N,1)^\omega$ for some $N \geq 0$, then we deduce from Lemma 4.3 that either

$$(d_{n+i}) > S(0,0) \cdots S(N-1,0)$$

or

$$(d_{n+i}) = S(0,0) \cdots S(N-1,0)S(N,1)^\omega.$$

Since

$$0d_2 d_3 \cdots = S(0,0) \cdots S(N-1,0)S(N,0)S(N,1)^\omega,$$

we conclude in both cases the strict inequalities

$$(d_{n+i}) > 0d_2 d_3 \cdots.$$

If $d_1 d_2 \cdots$ begins with $S(N,1)$ for all N, then

$$0d_2 d_3 \cdots = S(0,0)S(1,0) \cdots S(N,0) \cdots \leq (d_{n+i})$$

by Lemma 4.3.   □

DEFINITION 4.6. *We say that an admissible sequence d is* finitely generated *if $d = 0^\omega$ or if $d = S_h(N,1)^\omega$ with some nonnegative integer N and a finite sequence $h = (h_1, \ldots, h_N)$ of positive integers. Otherwise it is said to be* infinitely generated.

We now characterize the sequence d' satisfying

(124) $$d' := \min\{(d_{n+i})_{i \geq 1} | d_n = 0; \ n \geq 1\}$$

where the minimum is taken with respect to the lexicographic order.

LEMMA 4.5 (Characterization of d'). *Let $d = (d_i) \neq 1^\omega$ be an admissible sequence.*

(a) If $(d_i) = S(N,1)^\omega$ (then $N \geq 1$ because $d \neq 1^\omega$) and $(d_i') = (d_{i+1+\ell_N-\ell_{N-1}})$, then

$$(d_{n+i}') \geq (d_i') > (d_{1+i})$$

whenever $d_n' = 0$. Moreover, we have

(125) $$(d_i') = S(1,0) \cdots S(N-1,0)S(N,1)^\omega$$

and

(126) $$(d_{1+i}) = S(1,0) \cdots S(N-1,0)S(N,0)S(N,1)^\omega.$$

(b) In the other cases the sequence $(d'_i) := (d_{1+i})$ *satisfies*

$$(d'_{n+i}) \geq (d'_i)$$

whenever $d'_n = 0$.

(c) We have $d' = d$ *if and only if* $d = \left(1^{k-1}0\right)^\omega$ *for some positive integer k, i.e.,* $d = 0^\omega$ *or* $d = S(N,1)^\omega$ *with* $N = 1$.

PROOF.

(a) The first inequality follows from Lemma 4.3; the proof also shows that we have equality if and only if n is a multiple of ℓ_N.

The relations (118) and (119) of Lemma 4.2 imply (125)–(126) and they imply the second inequality because $S(N,0) < S(N,1)$.

(b) The case $(d_i) = 0^\omega$ is obvious. Otherwise (d_i) begins with $S(N,1)$ for all $N \geq 0$ and $\ell_N \to \infty$, so that we deduce from the relation (118) of Lemma 4.2 the equality

$$0d_2d_3\cdots = S(0,0)S(1,0)\cdots.$$

On the other hand, it follows from Lemma 4.3 that for any $n \geq 0$ we have

$$(d'_{n+i}) \geq S(0,0)\cdots S(N-1,0)S(N,0)^\omega$$

for every $N \geq 0$. This implies that

$$(d'_{n+i}) \geq 0d_2d_3\cdots$$

for every $n \geq 0$. If $d'_n = 0$, then we conclude that

$$d'_n d'_{n+1} d'_{n+2} \cdots \geq 0d_2d_3\cdots$$

which is equivalent to the required inequality

$$d'_{n+1} d'_{n+2} \cdots \geq d_2d_3\cdots.$$

(c) It follows from the above proof that $d = d'$ if and only if $d = 0^\omega$ or $d = S(N,1)^\omega$ for some integer $N \geq 1$ and h such that $\ell_{N-1} = 1$. These conditions are equivalent to $d = \left(1^{k-1}0\right)^\omega$ for some positive integer k. $\square$

EXAMPLE 4.4. *By Lemma 4.4 all admissible sequences* $d \neq 0^\omega$ *are defined by a finite or infinite sequence* $h = (h_j)$. *If we add the symbol* ∞ *to the end of each finite sequence* (h_j), *then the map* $d \mapsto h$ *is increasing with respect to the lexicographic orders of sequences. It follows that if* $d = S(N,1)^\omega$ *is an admissible sequence finitely generated with* $N \geq 1$ *(i.e.,* $d \neq 1^\omega$*) and* $h_1, \ldots, h_N \geq 1$*, then there exists a smallest admissible sequence* $\tilde{d} > d$. *It is infinitely generated, corresponding to the infinite sequence* $h = h_1 \cdots h_{N-1} h_N^+ 1^\omega$ *with* $h_N^+ := 1 + h_N$. *Observe that* $\tilde{d} = S(N-1,1)d$ *and hence* $\tilde{d}' = d'$.

For $d = 0^\omega$*, there exists a smallest admissible sequence* $\tilde{d} > d$*, too. It is also infinitely generated:* $\tilde{d} = 10^\omega$*, corresponding to* $h = (1,1,\ldots)$*, and* $\tilde{d}' = d' = 0^\omega$.

LEMMA 4.6. *If* $d = (d_i)$ *is an admissible sequence finitely generated, then no sequence* (c_i) *of zeroes and ones satisfies*

(127) $$0d_2d_3\cdots < (c_{n+i}) < d_1d_2d_3\cdots$$

for all $n = 1, 2, \ldots$.

PROOF. The case $d = 0^\omega$ is obvious because then $0d_2d_3\cdots = d_1d_2d_3\cdots$. The case $d = 1^\omega$ is obvious, too, because then (c_i) cannot have any zero digit by the first condition, while $(c_i) = 1^\omega$ does not satisfy the second condition. We may therefore assume that $d = S(N,1)^\omega$ for some $N \geq 1$ and for some $h = (h_i)$. Then our assumption takes the form

$$(128) \qquad S(0,0)\cdots S(N-1,0)S(N,0)S(N,1)^\omega < (c_{n+i}) < S(N,1)^\omega.$$

First step: the sequence (c_i) cannot end with $S(K,0)^\omega$ for any $0 \leq K \leq N$.

This is true if $S(K,0) = 0$ because $0^\omega \leq 0d_2d_3\cdots$.

Otherwise we have $K \geq 1$ and there exists $1 \leq M \leq K$ such that $h_M \geq 2$ and $h_{M+1} = \cdots h_N = 1$. Then we have

$$S(M,0) = S(M+1,0) = \cdots = S(K,0)$$

and (see (120))

$$S(0,0)\cdots S(M-1,0) \to S(M,0) = S(K,0)$$

Therefore in case (c_i) ends with $S(K,0)^\omega$ there exists n such that

$$\begin{aligned}
(c_{n+i}) &= S(0,0)\cdots S(M-1,0)S(K,0)^\omega \\
&= S(0,0)\cdots S(K-1,0)S(K,0)^\omega \\
&\leq S(0,0)\cdots S(N-1,0)S(N,0)^\omega \\
&< S(0,0)\cdots S(N-1,0)S(N,0)S(N,1)^\omega,
\end{aligned}$$

contradicting the first inequality of (128).

Second step: the sequence (c_i) ends with $S(N,c_1^N)S(N,c_2^N)\ldots$ for a suitable sequence $(c_j^N) \subset \{0,1\}$.

We have $(c_i) = S(0,c_1^0)S(0,c_2^0)\ldots$ with $c_i^0 := c_i$. Now let $1 \leq M \leq N$ and assume by induction that (c_i) ends with $S(M-1,c_1^{M-1})S(M-1,c_2^{M-1})\ldots$ for a suitable sequence $(c_j^{M-1}) \subset \{0,1\}$.

Since $S(N,1)$ begins with $S(M,1) = S(M-1,1)^{h_M}S(M-1,0)$, by (128) each block $S(M-1,c_j^{M-1})$ is followed by at most h_M consecutive blocks $S(M-1,1)$. On the other hand, since the first expression in (128) begins with

$$S(0,0)\cdots S(M-2,0)S(M-1,0)S(M-1,1)^{h_M-1}S(M-1,0)$$

and since (see (119))

$$S(0,0)\cdots S(M-2,0) \to S(M-1,1)$$

(for $M = 1$ the block $S(0,0)\cdots S(M-2,0)$ is empty by definition), each block $S(M-1,1)S(M-1,0)$ in $\left(S(M-1,c_j^{M-1})\right)$ is followed by at least $h_M - 1$ consecutive blocks $S(M-1,1)$.

Since (c_i) cannot end with $S(M-1,0)^\omega$ by the first step, we conclude that (c_i) ends with $\left(S(M,c_j^M)\right)$ for a suitable sequence $(c_j^M) \subset \{0,1\}$.

Third step: the sequence (c_i) ends with $S(N,1)S(N,0)S(N,a_1)S(N,a_2)\cdots$ for a suitable sequence $(a_j) \subset \{0,1\}$.

Indeed, in view of the first two steps it suffices to observe that (c_i) cannot end with $S(N,1)^\omega$ by the second condition of (128).

Fourth step. Using the relation $S(0,0)\cdots S(N-1,0) \to S(N,1)$ (see (119)) we deduce from the preceding step that (c_i) ends with

$$S(0,0)\cdots S(N-1,0)S(N,0)\left(S(N,a_j)\right) \leq S(0,0)\cdots S(N-1,0)S(N,0)S(N,1)^\omega,$$

contradicting the first condition in (128) again. $\qquad\square$

LEMMA 4.7. *If $d = (d_i) \neq 1^\omega$ is a finitely generated admissible sequence, then no sequence (c_i) of zeroes and ones satisfies*

$$0(d'_i) < (c_{n+i}) < 1(d'_i) \tag{129}$$

for all $n = 1, 2, \ldots$.

PROOF. If $d = 0^\omega$, then $d' = 0^\omega$ and our hypothesis takes the form $0^\omega < (c_{n+i}) < 10^\omega$. Such a sequence cannot have digits 1 by the second condition, but it cannot be 0^ω either by the first condition. It remains to consider the case where $d = S(N, 1)^\omega$ for some $N \geq 1$ and $h = (h_1, \ldots, h_N)$. Then by Lemmas 4.2 and 4.5 our hypothesis may be written in the form

$$S(0,0) \cdots S(N-1,0)S(N,1)^\omega < (c_{n+i}) < S(N-1,1)S(N,1)^\omega. \tag{130}$$

Using (130) instead of (128), we may repeat the proof of the preceding proposition by keeping $h_1, \ldots, h_{N-1}$ but changing h_N to $h_N + 1$. At the end of the third step we obtain that a sequence (c_i) satisfying (130) must end with

$$S_+(N,1)S_+(N,0)S_+(N,a_1)S_+(N,a_2) \cdots$$

for a suitable sequence $(a_j) \subset \{0,1\}$, where we use the notations

$$S_+(N,1) := S(N-1,1)^{h_N+1}S(N-1,0)$$

and

$$S_+(N,0) := S(N,1) = S(N-1,1)^{h_N}S(N-1,0).$$

Since

$$S_+(N,1)S_+(N,0)S_+(N,a_1)S_+(N,a_2) \cdots \geq S_+(N,1)S_+(N,0)^\omega$$
$$= S(N-1,1)S(N,1)^\omega,$$

this contradicts the second inequality of (130). $\square$

REMARK 4.5. *It follows by Theorem 1.4 that any admissible sequences $d \neq 0^\omega$ is of the form $1s$ where s is a sturmian word. Hence results like Lemma 4.6 and Lemma 4.7 could be deduced by known results about sturmian words. We privilege the approach based on recursive admissible sequences because they yield a good description of experimental data on univoque expansions; see for instance Example 4.5.*

7. m-admissible sequences

Throughout this section we fix an admissible sequence $d = (d_i) \neq 1^\omega$ and we define the sequence $d' = (d'_i)$ as in (124). Furthermore, for any given real number $m > 1$ we denote by $\delta = (\delta_i)$ and $\delta' = (\delta'_i)$ the sequences obtained from d and d' by the substitutions $1 \to m$ and $0 \to 1$. We define the numbers $p'_m, p''_m > 1$ by the equations

$$\sum_{i=1}^\infty \frac{\delta_i}{(p'_m)^i} = m - 1 \tag{131}$$

and

$$\sum_{i=1}^\infty \frac{m - \delta'_i}{(p''_m)^i} = 1 \tag{132}$$

and we put $p_m := \max\{p'_m, p''_m\}$.

Introducing the *conjugate* of δ by the formula $\overline{\delta'_i} := m - \delta'_i$ we may also write (131) and (132) in the more economical form

$$\pi_{p'_m}(\delta) = m - 1 \quad \text{and} \quad \pi_{p''_m}\left(\overline{\delta'}\right) = 1.$$

Let us also introduce the number

$$P_m := 1 + \sqrt{\frac{m}{m-1}}.$$

A direct computation shows that $P_m > 1$ can also be defined by any of the following equivalent conditions:

$$(133) \qquad\qquad (P_m - 1)^2 = \frac{m}{m-1};$$

$$(134) \qquad\qquad \frac{m}{P_m} + \frac{1}{P_m}\left(\frac{m}{P_m - 1} - 1\right) = m - 1;$$

$$(135) \qquad\qquad (m-1)P_m - m = \frac{m}{P_m - 1} - 1;$$

$$(136) \qquad\qquad \frac{m}{P_m - 1} - (m-1) = \frac{1}{P_m}.$$

We begin by investigating the dependence of P_m, p'_m and p''_m on m.

PROPOSITION 4.4 (Monotonicity properties of p_m).

(a) The function $m \mapsto P_m$ is continuous and strictly decreasing in $(1, \infty)$.

(b) The function $m \mapsto p'_m - P_m$ is continuous and strictly decreasing in $(1, \infty)$, and it has a unique zero m_d.

(c) The function $m \mapsto p''_m - P_m$ is continuous and strictly increasing in $(1, \infty)$, and it has a unique zero M_d.

(d) The function $m \mapsto p'_m - p''_m$ is continuous and strictly decreasing in $(1, \infty)$, and it has a unique zero μ_d.

(e) The function $m \mapsto p_m$ is continuous in $(1, \infty)$, strictly decreasing in $(1, \mu_d]$ and strictly increasing in $[\mu_d, \infty)$, so that it has a strict global minimum in μ_d.

PROOF.

(a) A straightforward computation shows that P is infinitely differentiable in $(1, \infty)$ and

$$P'(m) = \frac{-1}{2(m-1)\sqrt{m(m-1)}} < 0$$

for all $m > 1$.

(b) Since $\delta_i = 1 + (m-1)d_i$, we may rewrite (131) in the form

$$(137) \qquad\qquad \frac{1}{m-1} + (p'_m - 1)\sum_{i=1}^{\infty}\frac{d_i}{(p'_m)^i} = p'_m - 1.$$

Applying the implicit function theorem it follows that the function $m \mapsto p'_m$ is C^ω.

Differentiating the last identity with respect to m, denoting the derivatives by dots and setting

$$A := 1 + (p'_m - 1)\left(\sum_{i=1}^{\infty}\frac{d_i}{i(p'_m)^{i+1}}\right) - \left(\sum_{i=1}^{\infty}\frac{d_i}{(p'_m)^i}\right),$$

we get

$$A\dot{p}'_m = \frac{-1}{(m-1)^2}.$$

Differentiating (133) we obtain that the right side is equal to $2(P_m - 1)\dot{P}_m$, so that

$$A\dot{p}'_m = 2(P_m - 1)\dot{P}_m.$$

Since $\dot{P}_m < 0$ and $2(P_m - 1) > 1$, it suffices to show that $A \in (0,1)$. Indeed, then we will have $\dot{p}'_m / \dot{P}_m > 1$ and therefore $\dot{p}'_m < \dot{P}_m (< 0)$.

The inequality $A > 0$ follows by using (137):

$$A = (p'_m - 1)\left(\sum_{i=1}^{\infty} \frac{d_i}{i(p'_m)^{i+1}}\right) + \frac{1}{(m-1)(p'_m - 1)} > 0,$$

while the proof of $A < 1$ is straightforward:

$$A \leq 1 + (p'_m - 1)\left(\sum_{i=1}^{\infty} \frac{d_i}{(p'_m)^{i+1}}\right) - \left(\sum_{i=1}^{\infty} \frac{d_i}{(p'_m)^i}\right)$$

$$= 1 - \frac{1}{p'_m}\left(\sum_{i=1}^{\infty} \frac{d_i}{(p'_m)^i}\right)$$

$$< 1.$$

It remains to show that $p'_m - P_m$ changes sign in $(1, \infty)$. It is clear from the definition that

(138) $$\lim_{m \searrow 1} P_m = \infty \quad \text{and} \quad \lim_{m \to \infty} P_m = 2.$$

Furthermore, using the equality $d_1 = 1$ it follows from (137) that

$$\frac{1}{m-1} \leq p'_m - 1 \leq 1 + \frac{1}{m-1};$$

hence

(139) $$\lim_{m \searrow 1} p'_m = \infty \quad \text{and} \quad \lim_{m \to \infty} p'_m = 1.$$

We infer from (138)–(139) that $\lim_{m \to \infty} p'_m - P_m = -1 < 0$. The proof is completed by observing that

$$p'_m - P_m \geq \frac{1}{m-1} - 1 - \sqrt{\frac{m}{m-1}} \to \infty > 0$$

if $m \searrow 1$.

(c) We may rewrite (132) in the form

(140) $$\sum_{i=1}^{\infty} \frac{1 - d'_i}{(p''_m)^i} = \frac{1}{m-1}.$$

Applying the implicit function theorem it follows from (140) that the function $m \mapsto p''_m$ is C^ω.

The last identity also shows that the function $m \mapsto p''_m$ is strictly increasing. Using (a) we conclude that the function $m \mapsto p''_m - P_m$ is strictly increasing, too.

It remains to show that $p''_m - P_m$ changes sign in $(1, \infty)$. Since $d \neq 1^\omega$, there exists an index k such that $d'_k = 0$. Therefore we deduce from (140) the inequalities

$$\frac{1}{(p''_m)^k} \leq \frac{1}{m-1} \leq \frac{1}{p''_m - 1}$$

and hence that

(141) $$\lim_{m \searrow 1} p''_m = 1 \quad \text{and} \quad \lim_{m \to \infty} p''_m = \infty.$$

We conclude from (138) and (141) that

$$\lim_{m \searrow 1} p''_m - P_m = -\infty < 0 \quad \text{and} \quad \lim_{m \to \infty} p''_m - P_m = \infty > 0.$$

(d) The proof of (b) and (c) shows that $m \mapsto p'_m$ is continuous and strictly decreasing and $m \mapsto p''_m$ is continuous and strictly increasing; hence the function $m \mapsto p'_m - p''_m$ is continuous and strictly decreasing. It remains to observe that $p'_m - p''_m$ changes sign in $(1, \infty)$ because (139) and (141) imply that

$$\lim_{m \searrow 1} p'_m - p''_m = \infty > 0 \quad \text{and} \quad \lim_{m \to \infty} p'_m - p''_m = -\infty < 0.$$

(e) This follows from the definition $p_m := \max\{p'_m, p''_m\}$ and from the fact that $m \mapsto p'_m$ is continuous and strictly decreasing and $m \mapsto p''_m$ is continuous and strictly increasing. □

The first part of the following lemma is a variant of a similar result in [EJK90].

LEMMA 4.8. *We consider expansions in some base $q > 1$ on some alphabet $\{a, b\}$ with $a < b$.*

(a) Let (c_i) be an expansion of some number $s \le b - a$. If

$$c_{n+1} c_{n+2} \cdots \le c_1 c_2 \cdots \quad \text{whenever} \quad c_n = a,$$

then

$$\frac{c_{n+1}}{q^{n+1}} + \frac{c_{n+2}}{q^{n+2}} + \cdots \le \frac{s}{q^n} \quad \text{whenever} \quad c_n = a.$$

Moreover, the inequality is strict if the sequence (c_i) is infinite and $(c_{n+i}) \ne (c_i)$.

(b) Let $c = (c_i)$ and $d = (d_i)$ be two expansions. If $q \ge 2$, then

$$(c_i) \le (d_i) \implies \pi_q(c) \le \pi_q(d).$$

Moreover, if $q > 2$, then

$$(c_i) < (d_i) \iff \pi_q(c) < \pi_q(d).$$

PROOF.

(a) Starting with $k_0 := n$ we define by recurrence a sequence of indices $k_0 < k_1 < \cdots$ satisfying for $j = 1, 2, \ldots$ the conditions

$$c_{k_{j-1}+i} = c_i \quad \text{for} \quad i = 1, \ldots, k_j - k_{j-1} - 1, \quad \text{and} \quad c_{k_j} < c_{k_j - k_{j-1}}.$$

If we obtain an infinite sequence, then we have

$$\sum_{i=n+1}^{\infty} \frac{c_i}{q^i} = \sum_{j=1}^{\infty} \sum_{i=1}^{k_j - k_{j-1}} \frac{c_{k_{j-1}+i}}{q^{k_{j-1}+i}}$$

$$\le \sum_{j=1}^{\infty} \left(\left(\sum_{i=1}^{k_j - k_{j-1}} \frac{c_i}{q^{k_{j-1}+i}} \right) - \frac{b-a}{q^{k_j}} \right)$$

$$\le \sum_{j=1}^{\infty} \left(\frac{s}{q^{k_{j-1}}} - \frac{b-a}{q^{k_j}} \right)$$

$$\le \sum_{j=1}^{\infty} \left(\frac{s}{q^{k_{j-1}}} - \frac{s}{q^{k_j}} \right)$$

$$= \frac{s}{q^n}.$$

Otherwise we have $(c_{k_N+i}) = (c_i)$ after a finite number of steps (we do not exclude the possibility that $N = 0$), and we may conclude as follows:

$$\sum_{i=n+1}^{\infty} \frac{c_i}{q^i} = \Big(\sum_{j=1}^{N} \sum_{i=1}^{k_j-k_{j-1}} \frac{c_i}{q^{k_{j-1}+i}}\Big) + \sum_{i=1}^{\infty} \frac{c_{k_N+i}}{q^{k_N+i}}$$

$$\leq \sum_{j=1}^{N} \Big(\Big(\sum_{i=1}^{k_j-k_{j-1}} \frac{c_i}{q^{k_{j-1}+i}}\Big) - \frac{b-a}{q^{k_j}}\Big) + \sum_{i=1}^{\omega} \frac{c_i}{q^{k_N+i}}$$

$$\leq \sum_{j=1}^{N} \Big(\frac{s}{q^{k_{j-1}}} - \frac{b-a}{q^{k_j}}\Big) + \frac{s}{q^{k_N}}$$

$$\leq \sum_{j=1}^{N} \Big(\frac{s}{q^{k_{j-1}}} - \frac{s}{q^{k_j}}\Big) + \frac{s}{q^{k_N}}$$

$$= \frac{s}{q^n}.$$

The last property follows from the above proof.

(b) If $c < d$, then let n be the first integer for which $c_n < d_n$. Then $c_i = d_i$ for $i < n$, $d_n - c_n = b - a$, and $d_i - c_i \geq a - b$ for $i > n$, so that

$$\pi_q(d) - \pi_q(c) \geq \frac{b-a}{q^n} - \sum_{i=n+1}^{\infty} \frac{b-a}{q^i} = \frac{b-a}{q^n} - \frac{b-a}{q^n(q-1)} \geq 0.$$

Moreover, in case $q > 2$ the last inequality is strict. $\qquad\square$

Now we investigate the mutual positions of m_d, M_d and μ_d.

PROPOSITION 4.5 (Study of the condition $p_m \leq P_m$).
(a) If d is finitely generated, then $m_d < \mu_d < M_d$, and $p_m < P_m$ for all $m_d < m < M_d$. Furthermore, $p_m \geq 2$ for all $m \in (1, \infty)$ with equality if and only if $d = \big(1^{k-1}0\big)^\omega$ and $m = 2^k$ for some positive integer k.

(b) In the other cases we have $m_d = \mu_d = M_d$ and $p_m \geq p_{\mu_d} = P_{\mu_d} > 2$ for all $m \in (1, \infty)$.

PROOF.
(a)
In view of Lemma 4.4 the first assertion will follow if we show that $p_m < P_m$ for $m := \mu_d$. If $d = 0^\infty$, then $d' = 0^\infty$ and therefore

$$m - 1 = \pi_{p'_m}(\delta) = \pi_{p''_m}(\delta') = \frac{m}{p''_m - 1} - 1 = \frac{m}{p_m - 1} - 1.$$

It follows that $p_m = 2$ and therefore $P_m = 1 + \sqrt{m/(m-1)} > p_m$.
In the other cases, using the relations (125)–(126) of Lemma 4.5 we have

$$m - 1 = \sum_{i=1}^{\infty} \frac{\delta_i}{p_m^i}$$

$$= \frac{m}{p_m} + \frac{1}{p_m} \sum_{i=1}^{\infty} \frac{\delta_{i+1}}{p_m^i}$$

$$< \frac{m}{p_m} + \frac{1}{p_m} \sum_{i=1}^{\infty} \frac{\delta_i'}{p_m^i}$$

$$= \frac{m}{p_m} + \frac{1}{p_m} \Big(\frac{m}{p_m - 1} - 1\Big).$$

In this computation the crucial inequality follows from Lemmas 4.5 and 4.8 (a). Indeed, writing $d = S(N,1)^\omega$, in view of the relations (125)–(126) of Lemma 4.5 the inequality is equivalent to

$$\pi_{p'_m}\left((\delta_{\ell_{N-1}+i})\right) < \pi_{p'_m}(\delta),$$

and this inequality follows from Lemma 4.8 a) with $c = \delta$, $q = p'_m$ and $n = \ell_{N-1}$. (The hypotheses of the lemma are satisfied because d is an admissible sequence.)

Using (134) we conclude that $p_m < P_m$ indeed.

Furthermore, for $m := \mu_d$ we deduce from the equalities

$$\pi_{p_m}(\delta) = m - 1 \quad \text{and} \quad \pi_{p_m}\left(\overline{\delta'}\right) = 1$$

that

$$\sum_{i=1}^{\infty} \frac{m - \delta'_i + \delta_i}{p_m^i} = m.$$

It follows that $p_m \geq 2$ if and only if

$$\sum_{i=1}^{\infty} \frac{m - \delta'_i + \delta_i}{2^i} \geq m$$

which is equivalent to the inequality

$$\pi_2(\delta') \leq \pi_2(\delta).$$

Since $\delta' \leq \delta$ by Lemma 4.5, this is satisfied by a well-known property of diadic expansions.

The proof also shows that we have equality if and only if $\delta' = \delta$. By Lemma 4.5 ((c) this is equivalent to $d = \left(1^{k-1}0\right)^\omega$ for some positive integer k. In this case we infer from the equations

$$\frac{m}{p'_m - 1} - \frac{m - 1}{(p'_m)^k - 1} = m - 1$$

and

$$\frac{m}{p''_m - 1} - \frac{m - 1}{(p''_m)^k - 1} = \frac{m}{p''_m - 1} - 1$$

that $p'_m = p''_m = m^{1/k} = 2$.

Since by Lemma 4.4 p_m has a global strict minimum in $m = \mu_d$, we have $p_m > 2$ for all other values of m.

(b) Putting $m = \mu_d$ and repeating the first part of the proof of (a), by Lemma 4.5 now we have an equality instead of the strict inequality; using (134) we conclude that $p_m = P_m$ and hence $p_m = p'_m = p''_m = P_m$. Applying Lemma 4.4 we conclude that $m_d = \mu_d = M_d$. $\qquad\square$

8. Characterization of critical base for ternary alphabets

In this section we determine the generalized Golden Mean for every ternary alphabet $A = \{a_1, a_2, a_3\}$. Putting

$$m := \max\left\{\frac{a_3 - a_1}{a_2 - a_1}, \frac{a_3 - a_1}{a_3 - a_2}\right\}$$

we will show that

$$2 \leq G_A \leq P_m := 1 + \sqrt{\frac{m}{m - 1}}.$$

Moreover, we will give an exact expression of G_A for each m and we will determine the values of m for which $G_A = 2$ or $G_A = P_m$.

By Lemma 4.1 we may restrict ourselves without loss of generality to the case of the alphabets $A_m = \{0, 1, m\}$ with $m \geq 2$. Condition (107) takes the form

$$1 < q \leq \frac{2m-1}{m-1};$$

under this assumption, that we assume henceforth, the results of the preceding section apply. For the sequel we fix a real number $m \geq 2$ and we consider expansions in bases $q > 1$ with respect to the ternary alphabet $A_m := \{0, 1, m\}$.

One of our main tools will be Theorem 4.2 which now takes the following special form:

LEMMA 4.9. *An expansion (x_i) is unique in base q for the alphabet A_m if and only if the following conditions are satisfied:*

(142) $$\sum_{i=1}^{\infty} \frac{x_{n+i}}{q^i} < 1 \quad \text{whenever } x_n = 0;$$

(143) $$\sum_{i=1}^{\infty} \frac{x_{n+i}}{q^i} < m - 1 \quad \text{whenever } x_n = 1;$$

(144) $$\sum_{i=1}^{\infty} \frac{x_{n+i}}{q^i} > \frac{m}{q-1} - 1 \quad \text{whenever } x_n = 1;$$

(145) $$\sum_{i=1}^{\infty} \frac{x_{n+i}}{q^i} > \frac{m}{q-1} - (m-1) \quad \text{whenever } x_n = m.$$

Moreover:

(a) *the sequence (x_i) is a quasi-greedy expansion of some x in base q if and only if (142) and (143) hold. Hence, if $\mathbf{x} = (x_i)$ is a quasi-greedy expansion in base q, then $m^n\mathbf{x}$ and (x_{n+i}) are also quasi-greedy expansions in every base $\geq q$, for every positive integer n;*

(b) *the sequence (x_i) is a quasi-lazy expansion of some x in base q if and only if (144) and (145) hold. Hence, if $\mathbf{x} = (x_i)$ is a quasi-lazy expansion in base q, then $0^n\mathbf{x}$ and (x_{n+i}) are also quasi-lazy expansions in every base $\geq q$, for every positive integer n.*

COROLLARY 4.3. *We have $G_{A_m} \geq 2$.*

PROOF. Let (x_i) be a univoque sequence in some base $1 < q \leq 2$. We infer from (143) and (144) that $x_n \neq 1$ for every n. Since $m \geq q$, then we conclude from (142) that each 0 digit is followed by another 0 digit. Therefore condition (145) implies that each m digit is followed by another m digit. For otherwise the left-hand side of (145) would be zero, while the right-hand side is greater than zero. Hence (x_i) must be equal to 0^ω or m^ω. □

PROPOSITION 4.6. *If (x_i) is a nontrivial univoque sequence in some base $1 < q \leq P_m$, then (x_i) contains at most finitely many zero digits.*

PROOF. Since a univoque sequence remains univoque in every larger base, too, we may assume that $q = P_m$. It suffices to prove that (x_i) does not contain any block of the form $m0$ or 10.

(x_i) *does not contain any block of the form $m0$.* If $x_n = m$ and $x_{n+1} = 0$ for some n, then we deduce from Lemma 4.9 that

$$\sum_{i=1}^{\infty} \frac{x_{n+i}}{P_m^i} > \frac{m}{P_m - 1} - (m-1) \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{x_{n+i+1}}{P_m^i} < 1.$$

Hence

$$\frac{m}{P_m - 1} - (m-1) < \sum_{i=1}^{\infty} \frac{x_{n+i}}{P_m^i} = \frac{1}{P_m} \sum_{i=1}^{\infty} \frac{x_{n+i+1}}{P_m^i} < \frac{1}{P_m},$$

contradicting condition (136) on P_m.

(c_i) *does not contain any block of the form* 10. If $x_n = 1$ and $x_{n+1} = 0$ for some n, then the application of Lemma 4.9 shows that

$$\sum_{i=1}^{\infty} \frac{x_{n+i}}{P_m^i} > \frac{m}{P_m - 1} - 1 \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{x_{n+i+1}}{P_m^i} < 1.$$

Since $m \geq 2$, these inequalities imply those of the preceding step, contradicting again our condition on P_m. $\qquad\square$

Next we select a particular admissible sequence for each given m. Given an admissible sequence $d \neq 1^\omega$ we set

$$(146) \qquad I_d := \begin{cases} [m_d, M_d) & \text{if } m_d < M_d, \\ \{m_d\} & \text{if } m_d = M_d. \end{cases}$$

LEMMA 4.10. *Given a real number $m \geq 2$ there exists a lexicographically largest admissible sequence $d = (d_i)$ such that using the notations of the preceding section we have*

$$(147) \qquad \sum_{i=1}^{\infty} \frac{\delta_i}{P_m^i} \leq m - 1.$$

Furthermore, we have $d \neq 1^\omega$ and $m \in I_d$.

REMARK 4.6. *The lemma and its proof remain valid for all $m \geq (1 + \sqrt{5})/2$.*

PROOF. The sequence $d = 0^\omega$ always satisfies (147) because using (133) we have

$$\sum_{i=1}^{\infty} \frac{\delta_i}{P_m^i} = \frac{1}{P_m - 1} = \sqrt{\frac{m-1}{m}} \leq m - 1;$$

the last inequality is equivalent to $m \geq (1 + \sqrt{5})/2$. If it is not the only such admissible sequence, then applying the monotonicity of the map $d \mapsto h$ mentioned in Example 4.4 we obtain the existence of a lexicographically largest finite or infinite sequence h such that the corresponding admissible sequence $d = (d_i)$ satisfies (147).

We have $d \neq 1^\omega$ because the sequence $d = 1^\omega$ does not satisfy (147): using (133) again we have

$$\sum_{i=1}^{\infty} \frac{\delta_i}{P_m^i} = \frac{m}{P_m - 1} = \sqrt{(m-1)m} > m - 1.$$

It remains to prove that $m \in I_d$. We distinguish three cases.

(a) If (d_i) is defined by an infinite sequence (h_j), then we already know that $p_m = p'_m = p''_m$ and that

$$\sum_{i=1}^{\infty} \frac{\delta_i}{P_m^i} \leq m - 1.$$

It remains to show the converse inequality

$$(148) \qquad \sum_{i=1}^{\infty} \frac{\delta_i}{P_m^i} \geq m - 1.$$

It follows from the definition of (δ_i) that if we denote by (δ_i^N) the sequence associated with the admissible sequence defined by the sequence $h := h_1, \ldots, h_{N-1}, h_N + 1, 1, 1, \ldots$, then

$$\sum_{i=1}^{\infty} \frac{\delta_i^N}{P_m^i} > m - 1.$$

Since both (d_i) and (d_i^N) begin with $S(N-1, 1)^{h_N}$ and since the length of this block tends to infinity, letting $N \to \omega$ we conclude (148).

(b) If $(d_i) = S(N, 1)^\omega$ for some $N \geq 1$, then

$$(e_i) := S(N-1, 1)^{h_N+1} S(N-1, 0) \left[S(N-1, 1)^{h_N} S(N-1, 0) \right]^\omega$$

$$= S(N-1, 1) S(N, 1)^\omega$$

does not satisfy (147), so that

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{P_m^i} > m - 1$$

where (ε_i) is obtained from (e_i) by the usual substitutions $1 \to m$ and $0 \to 1$.

Observe that now we have $e_1 e_2 \cdots = 1 d_1' d_2' \cdots$ and therefore (using the notations of the first page of the paper)

$$m - 1 < \pi_{P_m}(\varepsilon) = \frac{m}{P_m} + \frac{1}{P_m} \pi_{P_m}(\delta').$$

It follows that

$$\pi_{P_m}(\delta') > (m-1)P_m - m = \frac{m}{P_m - 1} - 1$$

which is equivalent to $\pi_{P_m}\left(\overline{\delta'}\right) < 1$. Since we have $\pi_{p_m''}\left(\overline{\delta'}\right) = 1$ by the definition of p_m'', we conclude that $P_m > p_m''$.

Finally, since we have $\pi_{P_m}(\delta) \leq m - 1 = \pi_{p_m'}(\delta)$ by the definitions of (d_i) and p_m', we have also $P_m \geq p_m'$.

(c) If $(d_i) = 0^\omega$, then we may repeat the proof of (b) with $(d_i') = 0^\omega$ and $(e_i) = 10^\omega$.   □

EXAMPLE 4.5. *Using a computer program we can determine the admissible sequences of Lemma 4.10 for all* integer *values* $2 \leq m \leq 2^{16}$. *For all but seven values the corresponding admissible sequence is infinitely generated with* $N = 1$, *more precisely* $d = (1^{h_1} 0)^\omega$ *with* $h_1 = [\log_2 m]$. *For the exceptional values* $m = 5, 9, 130, 258, 2051, 4099, 32772$ *the corresponding admissible sequence is infinitely generated with* $N = 2$ *and* $h_1 = [\log_2 m]$ *as shown in the following table:*

m	d	N	h
5	$(1^2 01^2 010)^\omega$	2	(2,2)
9	$(1^3 01^2 0)^\omega$	2	(3,1)
130	$(1^7 01^6 0)^\omega$	2	(7,1)
258	$(1^8 01^7 0)^\omega$	2	(8,1)
2051	$(1^{11} 01^{10} 0)^\omega$	2	(11,1)
4099	$(1^{12} 01^{11} 0)^\omega$	2	(12,1)
32772	$(1^{15} 01^{14} 0)^\omega$	2	(15,1)

LEMMA 4.11. *Given an admissible sequence* $d \neq 1^\omega$ *and* $m \in I_d$ *let us define the sequences* d', δ, δ' *and the numbers* p_m', p_m'', p_m *as at the beginning of Section 7.*

(a) The sequences δ, $m\delta'$ *are quasi-greedy in base* p_m.

(b) The sequences δ' *and* (δ_{1+i}) *are quasi-lazy in base* p_m.

PROOF.

(a) Using the admissibility of d and applying Lemma 4.8 (b) with $(c_i) := \delta$ and $q := p_m \geq 2$ on the alphabet $\{1, m\}$ we obtain that

$$\sum_{i=1}^{\infty} \frac{\delta_{n+i}}{p_m^i} \leq \sum_{i=1}^{\infty} \frac{\delta_i}{p_m^i}$$

for all n. Since $p_m \geq p'_m$ and

$$\sum_{i=1}^{\infty} \frac{\delta_i}{(p'_m)^i} = m - 1,$$

it follows that

$$\sum_{i=1}^{\infty} \frac{\delta_{n+i}}{p_m^i} \leq m - 1$$

for all n. Applying Lemma 4.9 we conclude that δ is a quasi-greedy expansion in base p_m. The same inequalities ensure that $m\delta'$ is also a quasi-greedy expansion in base p_m.

(b) Since $(\delta_{1+i}) = (\delta'_{k+i})$ for some $k \geq 0$, in view of Lemma 4.9 it suffices to show that

$$\sum_{i=1}^{\infty} \frac{m - \delta'_{n+i}}{p_m^i} \leq 1 \quad \text{whenever} \quad \delta'_n = 1$$

and

$$\sum_{i=1}^{\infty} \frac{m - \delta'_{n+i}}{p_m^i} \leq m - 1 \quad \text{whenever} \quad \delta'_n = m.$$

If $\delta'_n = 1$, then applying Lemma 4.5 and Lemma 4.8 (b) with $(c_i) := \delta'$ and $q := p_m \geq 2$ on the alphabet $\{1, m\}$ we obtain that

$$\sum_{i=1}^{\infty} \frac{\delta'_{n+i}}{p_m^i} \geq \sum_{i=1}^{\omega} \frac{\delta'_i}{p_m^i}.$$

Using the definition of p''_m and the inequality $p_m \geq p''_m$ hence the first property follows:

$$\sum_{i=1}^{\infty} \frac{m - \delta'_{n+i}}{p_m^i} \leq \sum_{i=1}^{\infty} \frac{m - \delta'_i}{p_m^i} \leq \sum_{i=1}^{\infty} \frac{m - \delta'_i}{(p''_m)^i} = 1.$$

If $\delta'_n = m$, then let k be the smallest positive integer satisfying $\delta'_{n+k} = 1$. Applying the first property and the inequalities $p_m \geq 2 \geq \frac{m}{m-1}$ the second property follows:

$$\sum_{i=1}^{\infty} \frac{m - \delta'_{n+i}}{p_m^i} \leq \frac{m-1}{p_m^k} + \frac{1}{p_m^k} \cdot 1 = \frac{m}{p_m^k} \leq \frac{m}{2^k} \leq \frac{m}{2} \leq m - 1. \qquad \square$$

REMARK 4.7. *Applying Lemma 4.8 (a) instead of (b) we may obtain the stronger result that δ and $m\delta'$ are quasi-greedy expansion in every base $q \geq p'_m$.*

LEMMA 4.12. *Denoting by $\tilde{\gamma} = (\tilde{\gamma}_i)$ and $\tilde{\lambda} = (\tilde{\lambda}_i)$ the quasi-greedy expansion of $m - 1$ in base p_m and the quasi-lazy expansion of $\frac{m}{p_m-1} - 1$ in base p_m, respectively, we have either*

$$(\delta_{1+i}) \leq \tilde{\lambda} \quad and \quad \tilde{\gamma} = \delta$$

or

$$\delta' = \tilde{\lambda} \quad and \quad \tilde{\gamma} \leq m\delta'.$$

PROOF. If $p'_m \geq p''_m$, then both $\tilde{\gamma}$ and δ are quasi-greedy expansions of $m - 1$ in base $p_m = p'_m$ by Lemma 4.11, so that $\tilde{\gamma} = \delta$. Since furthermore both $\hat{\delta} := (\delta_{1+i})$ and $\tilde{\lambda}$ are quasi-lazy expansions in base p_m, in view of Lemma 4.9 it remains to show only that $\pi_{p_m}(\hat{\delta}) \leq \pi_{p_m}(\tilde{\lambda})$. Since

$$m - 1 = \pi_{p_m}(\delta) = \frac{m}{p_m} + \frac{1}{p_m} \pi_{p_m}(\hat{\delta})$$

and $p_m \leq P_m$, using (135) we have

$$\pi_{p_m}(\hat{\delta}) = (m - 1)p_m - m \leq \frac{m}{p_m - 1} - 1 = \pi_{p_m}(\tilde{\lambda}).$$

If $p_m'' \geq p_m'$, then both $\tilde{\lambda}$ and δ' are quasi-lazy expansions of $\frac{m}{p_m - 1} - 1$ in base $p_m = p_m''$ by Lemma 4.11, so that $\tilde{\lambda} = \delta'$. Furthermore $m\delta'$ and $\tilde{\gamma}$ are quasi-greedy expansions in base p_m. Since $p_m \leq P_m$, using (134) we obtain that

$$
\begin{aligned}
\pi_{p_m}(m\delta') &= \frac{m}{p_m} + \frac{1}{p_m}\pi_{p_m}(\delta') \\
&= \frac{m}{p_m} + \frac{1}{p_m}\left(\frac{m}{p_m - 1} - 1\right) \\
&\geq m - 1 \\
&= \pi_{p_m}(\tilde{\gamma}).
\end{aligned}
$$

Applying Lemma 4.9 we conclude that $m\delta' \geq \tilde{\gamma}$.   □

Given $m \geq 2$ we choose an admissible sequence $d \neq 1^\omega$ satisfying $m \in I_d$ (see Lemma 4.10) and we define p_m as at the beginning of Section 7 (see Lemma 4.11). The following lemma proves Theorem 4.5 (a).

PROPOSITION 4.7.
(a) If $q > p_m$, then δ' is a nontrivial univoque sequence in base q.
(b) There are no nontrivial univoque sequences in any base $1 < q < p_m$.

PROOF.
(a) Since the sequence δ is quasi-greedy and the sequence δ' is quasi-lazy in base p_m and since δ' is obtained from δ by removing a finite initial block, δ' is both quasi-greedy and quasi-lazy in base p_m. Hence

$$
\sum_{i=1}^{\infty} \frac{\delta'_{n+i}}{p_m^i} \leq m - 1 \quad \text{whenever} \quad \delta'_n = 1,
$$

$$
\sum_{i=1}^{\infty} \frac{m - \delta'_{n+i}}{p_m^i} \leq 1 \quad \text{whenever} \quad \delta'_n = 1,
$$

$$
\sum_{i=1}^{\infty} \frac{m - \delta'_{n+i}}{p_m^i} \leq m - 1 \quad \text{whenever} \quad \delta'_n = m.
$$

Since $q > p_m$, it follows that

$$
\sum_{i=1}^{\infty} \frac{\delta'_{n+i}}{q^i} < m - 1 \quad \text{whenever} \quad \delta'_n = 1,
$$

$$
\sum_{i=1}^{\infty} \frac{m - \delta'_{n+i}}{q^i} < 1 \quad \text{whenever} \quad \delta'_n = 1,
$$

$$
\sum_{i=1}^{\infty} \frac{m - \delta'_{n+i}}{q^i} < m - 1 \quad \text{whenever} \quad \delta'_n = m.
$$

Applying Lemma 4.9 we conclude that δ' is a univoque sequence in base q.

(b) Assume first that d is finitely generated and assume on the contrary that there exists a nontrivial univoque sequence in some base $1 < q \leq p_m$. Since a univoque sequence remains univoque in every greater base and since a univoque sequence remains univoque if we remove an arbitrary finite initial block, by Lemma 4.6 it follows that there exists a univoque sequence (x_i) in base $p_m (\leq P_m)$ that contains only the digits 1 and m.

Assume on the contrary that there exists a nontrivial univoque sequence in some base $1 < q \leq p_m$. Then it is also univoque in base p_m. Furthermore, since a univoque sequence in a base $\leq P_m$ contains at most finitely many zero digits and since a univoque sequence remains univoque

if we remove an arbitrary finite initial block, the there exists also a univoque sequence (x_i) in base p_m that contains only the digits 1 and m.

It follows from the lexicographic characterization of univoque sequences that

$$x_n = 1 \Longrightarrow (\tilde{\lambda}_i) < (x_{n+i}) < (\gamma_i)$$

and therefore (using the preceding lemma) that either

$$x_n = 1 \Longrightarrow (\delta_{1+i}) < (x_{n+i}) < (\delta_i)$$

or

$$x_n = 1 \Longrightarrow (\delta_i') < (x_{n+i}) < m(\delta_i)$$

Setting $c_i = 0$ if $x_i = 1$ and $c_i = 1$ if $x_i = m$ we obtain a sequence (c_i) of zeroes and ones, satisfying either

$$(149) \qquad (d_{1+i}) < (c_{n+i}) < (d_i) \quad \text{whenever} \quad c_n = 0$$

or

$$(150) \qquad (d_i') < (c_{n+i}) < 1(d_i') \quad \text{whenever} \quad c_n = 0.$$

The second inequalities imply that (c_i) has infinitely many zero digits. By removing a finite initial block if necessary we obtain a new sequence (still denoted by (c_i)) which begins with $c_1 = 0$ and which satisfies (149) or (150).

In case of (149) we claim that

$$(151) \qquad 0d_2d_3 \cdots < (c_{n+i}) < (d_i) \quad \text{for all} \quad n \geq 0.$$

Indeed, if $c_n = 1$ for some n then there exist $m < n \leq M$ such that $c_m = c_{M+1} = 0$ and $c_{m+1} = \cdots = c_M = 1$. Using (149) it follows that

$$(c_{n+i}) \leq (c_{m+i}) < (d_i)$$

and

$$(c_{n+i}) \geq (c_{M+i}) = 0(c_{M+1+i}) > 0(d_{1+i}) = 0d_2d_3 \cdots .$$

However, (151) contradicts Lemma 4.6.

In case of (150) we claim that

$$(152) \qquad 0(d_i') < (c_{n+i}) < 1(d_i') \quad \text{for all} \quad n \geq 0.$$

Indeed, if $c_n = 1$ for some n then choosing again $m < n \leq M$ such that $c_m = c_{M+1} = 0$ and $c_{m+1} = \cdots = c_M = 1$, we have

$$(c_{n+i}) \leq (c_{m+i}) < 1(d_i')$$

and

$$(c_{n+i}) \geq (c_{M+i}) = 0(c_{M+1+i}) > 0(d_i').$$

However, (152) contradicts Lemma 4.7.

Now assume that d is infinitely generated, associated with an infinite sequence $h = (h_1, h_2, \ldots)$, and that there exists a nontrivial univoque sequence (x_i) in some base $1 < q < p_m$. (Note that $m > 2$.) We will then prove the existence of a nontrivial univoque sequence in some base $1 < q' < p_{m'}$ where $m' \in I_{d'}$ where d' is finitely generated, contradicting to what we have already established. (In this part of the proof d' does not mean the sequence defined in Lemma 4.5.)

We may assume again that $x_i \equiv 1 + (m-1)c_i$ for some sequence $(c_i) \subset \{0, 1\}$. By Lemma 4.9 we have

$$\frac{m}{q-1} - 1 < \pi_q((x_{n+i})) < m - 1 \quad \text{whenever} \quad x_n = 1$$

and

$$\pi_q\left((x_{n+i})\right) > \frac{m}{q-1} - (m-1) \quad \text{whenever } x_n = m.$$

This can be rewritten equivalently in the following form:

$$\pi_q\left((c_{n+i})\right) < 1 - \frac{1}{(q-1)(m-1)} \qquad\qquad \text{whenever } c_n = 0;$$

$$\pi_q\left((1-c_{n+i})\right) < \frac{1}{m-1} \qquad\qquad \text{whenever } c_n = 0;$$

$$\pi_q\left((1-c_{n+i})\right) < 1 \qquad\qquad \text{whenever } c_n = 1.$$

If $2 < m' < m$ and q' are defined by the equation $(q'-1)(m'-1) = (q-1)(m-1)$, then $q' > q$, so that the above three conditions remain valid by changing q to q' and m to m'. (Observe that the left sides decrease and the right sides increase.) Applying Lemma 4.9 again we conclude that the formula $x_i' := 1 + (m'-1)c_i$ defines a nontrivial univoque sequence in base q' for the alphabet $\{0, 1, m'\}$. To end the proof it remains to show that we can choose m' such that $1 < q' < p_{m'}$ and $m' \in I_{d'}$ for some d' is finitely generated. Thanks to the continuity of the maps $m' \mapsto q'$ and $m' \mapsto p_{m'}$ the first condition is satisfied for all m' sufficiently close to m.

If $h = (h_1, h_2, \ldots)$ contains infinitely many elements $h_j \geq 2$, then we may choose d' associated with the finite sequence $h = (h_1, h_2, \ldots, h_{j-1}, h_j - 1)$ for a sufficiently large index j such that $h_j \geq 2$, and an arbitrary element $m' \in I_{d'}$. If $h = (h_1, h_2, \ldots)$ has a last element $h_j \geq 2$, then m is the right endpoint of the interval $I_{d'}$ for d' associated with the finite sequence $h = (h_1, h_2, \ldots, h_{j-1}, h_j - 1)$ (see Example 4.4), and we may choose $m' \in I_{d'}$ sufficiently close to m. The only remaining case $h = (1, 1, \ldots)$ is similar: m is the right endpoint of the interval $I_{d'}$ for $d' = 0^\omega$, and we may choose $m' \in I_{d'}$ sufficiently close to m. (See Example 4.4 again.) $\square$

The following lemma completes the proof of Theorem 4.5.

PROPOSITION 4.8 (Proof of Theorem 4.5 (c)).

(a) If $d < \tilde{d} < 1^\omega$ are admissible sequences, then $M_d \leq m_{\tilde{d}}$ with equality if and only if $d = S(N, 1)^\omega$ is finitely generated and $\tilde{d} = S(N-1, 1)S(N, 1)^\omega$.

(b) The sets I_d, where d runs over all admissible sequences $d \neq 1^\omega$, form a partition of the interval $[\frac{1+\sqrt{5}}{2}, \infty)$.

(c) The set C of numbers $m > \frac{1+\sqrt{5}}{2}$ satisfying $p_m = P_m$ is a Cantor set, i.e., a nonempty closed set having neither interior, nor isolated points. Its smallest element is $1 + x \approx 2.3247$ where x is the first Pisot number, i.e., the positive root of the equation $x^3 = x + 1$.

PROOF.

(a) If d and $\tilde{d}$ are infinitely generated, then $m_d = M_d$ and $m_{\tilde{d}} = M_{\tilde{d}}$, so that it suffices to prove the inequality $M_d < M_{\tilde{d}}$. For this, it is sufficient to show that $p''_{d,m} > p''_{\tilde{d},m}$ for each $m \in (1, \infty)$ where $p''_{d,m}$ and $p''_{\tilde{d},m}$ denote the expressions p''_m of Section 7 for the admissible sequences d and $\tilde{d}$, respectively. Indeed, then we can conclude that $p''_{d,M_{\tilde{d}}} > p''_{\tilde{d},M_{\tilde{d}}} = P_{M_{\tilde{d}}}$ and therefore, since the function $m \mapsto p''_{d,m} - P_m$ is strictly increasing Lemma 4.4, $M_d < M_{\tilde{d}}$.

Assuming on the contrary that $p''_{d,m} \leq p''_{\tilde{d},m}$ for some m, in base $q := p''_{\tilde{d},m}$ we have

$$\pi_q(m - \tilde{\delta}') = 1 = \pi_{p''_{d,m}}(m - \delta') \geq \pi_q(m - \delta') \implies \pi_q(\delta') \geq \pi_q(\tilde{\delta}')$$

Since d and $\tilde{d}$ are infinitely generated, we have $\delta = m\delta'$ and $\tilde{\delta} = m\tilde{\delta}'$ by Lemma 4.5, so that the last inequality is equivalent to $\pi_q(\delta) \geq \pi_q(\tilde{\delta})$.

Since quasi-greedy expansions remain quasi-greedy in larger bases, it follows from Lemma 4.11 that both δ and $\tilde{\delta}$ are quasi-greedy expansions in base q. Therefore we deduce from the last inequality that $\delta \geq \tilde{\delta}$, contradicting our assumption.

If $d = S(N,1)^\omega$ is finitely generated and $\tilde{d}$ infinitely generated, then we recall from Example 4.4 that $\hat{d} = S(N-1,1)S(N,1)^\omega$ is the smallest admissible sequence satisfying $\hat{d} > d$, and that $m_d < M_d = m_{\hat{d}} = M_{\hat{d}}$. Since $\hat{d}$ is infinitely generated, we conclude that $M_d = M_{\hat{d}} < M_{\tilde{d}} = m_{\tilde{d}}$. The case of $d = 0^\omega$ is similar with $\hat{d} = 10^\omega$.

If d is arbitrary and $\tilde{d}$ finitely generated, then $\tilde{d}$ is associated with a finite sequence $(h_1, \ldots, h_N)$ of length $N \geq 1$. If k is a sufficiently large positive integer, then the admissible sequence d^k associated with the infinite sequence $(h_1, \ldots, h_N, k, 1, 1, \ldots)$ satisfies $d < d^k < \tilde{d}$, so that $M_d \leq m_{d^k}$. Letting $k \to \infty$ we conclude that $M_d \leq m_{\tilde{d}}$. Indeed, for any fixed $m < m_{\tilde{d}}$ we have $p'_{\tilde{d},m} - P_m > 0$ by Lemma 4.4 (b) and therefore $\pi_{P_m}(\tilde{\delta}) > m - 1$. Since the first k digits of $\tilde{\delta}$ and δ^k coincide, for $k \to \infty$ we have

$$\pi_{P_m}(\delta^k) = \sum_{i=1}^{k} \frac{\tilde{\delta}_i}{P_m^i} + \sum_{i=k+1}^{\infty} \frac{\delta_i^k}{P_m^i} = \sum_{i=1}^{k} \frac{\tilde{\delta}_i}{P_m^i} + O\left(\frac{1}{P_m^k}\right) \to \pi_{P_m}(\tilde{\delta}),$$

so that $\pi_{P_m}(\delta^k) > m - 1$ for if k is sufficiently large. Hence $p'_{d^k,m} > P_m$ and therefore $m < m_{d^k}$ by Lemma 4.4 (b). Similarly, for any fixed $m > m_{\tilde{d}}$ we have $m > m_{d^k}$ for all sufficiently large k.

(b) The sets I_d are disjoint by (a) and they cover the interval $\left[\frac{1+\sqrt{5}}{2}, \infty\right)$ by Lemma 4.10. In view of (a) the proof will be completed if we show that for the smallest admissible sequence we have

(153) $$I_{0^\omega} = \left[\frac{1+\sqrt{5}}{2}, 1 + P_1\right)$$

where $x > 1$ is the first Pisot number.

The values m_d and M_d are the solutions of the equations

$$\pi_{P_m}(\delta) = m - 1 \quad \text{and} \quad \pi_{P_m}(\delta') = \frac{m}{P_m - 1} - 1.$$

Now we have $\delta = \delta' = 1^\omega$, so that our equations take the form

$$\frac{1}{P_m - 1} = m - 1$$

and

$$\frac{1}{P_m - 1} = \frac{m}{P_m - 1} - 1.$$

Using (133) we obtain that they are equivalent to $m = (1 + \sqrt{5})/2$ and $m = 1 + P_1$, respectively.

(c) If we denote by D_1 and D_2 the set of admissible sequences $d \neq 1^\omega$ finitely and infinitely generated, respectively, then

$$C = [2, \infty) \setminus \cup_{d \in D_1} (m_d, M_d)$$

so that C is a closed set. The relation (153) shows that its smallest element is $1 + P_1$. In order to prove that it is a Cantor set, it suffices to show that

- the intervals $[m_d, M_d]$ ($d \in D_1$) are disjoint;
- for each $m \in C$ there exist two sequences $(a_N) \subset [2, \infty) \setminus C$ and $(b_N) \subset C \setminus \{m\}$, both converging to m.

The first property follows from (a). For the proof of the second property let us consider the infinite sequence $h = (h_j)$ of positive integers defining the admissible sequence d for which $m_d = m$, and set $d_N := S_h(N, 1)^\omega$, $N = 1, 2, \ldots$. This is a decreasing sequence of admissible sequences, converging pointwise to d. Using (a) we conclude that both (m_{d_N}) and (M_{d_N}) converge to $m_d = M_d$. Since $m_{d_N} \in D_1$ and $M_{d_N} \in D_2$ for every N, the proof is complete. $\qquad\square$

9. Overview of original contributions, conclusions and further developments

Critical bases. The notion of *critical base* has first been referred to a set of sequences:

THEOREM (Existence of critical base). *For every given set $X \subset A^{\mathbb{N}}$ there exists a number*

$$1 \le q_X \le Q_A$$

such that

$$q > q_X \implies \text{ every sequence } \mathbf{x} \in X \text{ is univoque in base } q;$$
$$1 < q < q_X \implies \text{ not every sequence } \mathbf{x} \in X \text{ is univoque in base } q.$$

Then we introduced G_A, namely the *critical base of the alphabet A*, whose existence has been proved in the following:

COROLLARY. *There exists a number $1 < G_A \le Q_A$ such that*

$$q > G_A \implies \text{ there exist nontrivial univoque sequences;}$$
$$1 < q < G_A \implies \text{ there are no nontrivial univoque sequences.}$$

Normal ternary alphabets. The problem of characterizing the critical base of ternary alphabets is simplified by the following results.

PROPOSITION. *Every ternary alphabet can be normalized using only translations, scalings and dual operations.*

PROPOSITION. *Let A be a ternary alphabet and ϕ_A be the normalizing map defined above. For every $q > 1$, $\mathbf{x} \in A^{\mathbb{N}}$ is univoque in base q if and only if $\phi_A(\mathbf{x})$ is univoque in base q. In particular, any ternary alphabet and its normal form share the same generalized Golden Mean.*

Critical bases for ternary alphabets. We proved that the critical base of alphabets of the form $A = \{a_1, a_2, a_3\}$ with

$$m := \max \left\{ \frac{a_3 - a_1}{a_2 - a_1}, \frac{a_3 - a_1}{a_3 - a_2} \right\}$$

is the value $p_m = G_{A_m} = G_{\{0,1,m\}}$ whose properties are stated in the following result.

THEOREM 4.6. *There exists a continuous function $p : [2, \infty) \to \mathbb{R}$, $m \mapsto p_m$ satisfying*

$$2 \le p_m \le P_m := 1 + \sqrt{\frac{m}{m-1}}$$

for all m such that the following properties hold true:

(a) for each $m \ge 2$, there exist nontrivial univoque expansions if $q > p_m$ and there are no such expansions if $q < p_m$.

(b) we have $p_m = 2$ if and only if $m = 2^k$ for some positive integer k;

(c) the set $C := \{m \ge 2 : p_m = P_m\}$ is a Cantor set, i.e., an uncountable closed set having neither interior nor isolated points; its smallest element is $1 + x \approx 2.3247$ where x is the first Pisot number, i.e., the positive root of the equation $x^3 = x + 1$;

(d) each connected component (m_d, M_d) of $[2, \infty) \setminus C$ has a point μ_d such that p is strictly decreasing in $[m_d, \mu_d]$ and strictly increasing in $[\mu_d, M_d]$.

Admissible sequences. The proof of Theorem 4.6 needs some auxiliary notions to be introduced. In fact the definition of p_m lies on the *admissible sequences*.

DEFINITION. *A sequence* $d = (d_i) = d_1 d_2 \cdots$ *of zeroes and ones is* admissible *if*

$$0 d_2 d_3 \cdots \leq (d_{n+i}) \leq d_1 d_2 d_3 \cdots$$

for all $n = 0, 1, \ldots$.

For any admissible sequence d we denote d' a particular suffix of d, satisfying:

$$d' = \min\{(d_{n+i})_{i \geq 1} | d_n = 0; \; n \geq 1\}$$

with respect to the lexicographic order. A recursive characterization of the admissible sequences allows us to prove the following results.

PROPOSITION. *If* $d = (d_i)$ *is a finitely generated admissible sequence, then no sequence* (c_i) *of zeroes and ones satisfies*

$$0 d_2 d_3 \cdots < (c_{n+i}) < d_1 d_2 d_3 \cdots$$

for all $n = 1, 2, \ldots$.

PROPOSITION. *If* $d = (d_i) \neq 1^\omega$ *is a finitely generated admissible sequence, then no sequence* (c_i) *of zeroes and ones satisfies*

$$0(d_i') < (c_{n+i}) < 1(d_i')$$

for all $n = 1, 2, \ldots$.

REMARK. *The Propositions above are used to exclude the existence of univoque sequences for bases smaller than* p_m.

m-admissible sequences. We specialize any fixed binary admissible sequence d to the corresponding *m-admissible* sequence δ by replacing any occurrence of 0 in d with 1 and any occurrence of 1 with m. We then define p_m' and p_m'' as the positive solutions of the following equations:

$$\pi_{p_m'}(\delta) = m - 1 \qquad \pi_{p_m''}\left(\overline{\delta'}\right) = 1.$$

and $p_m := \max\{p_m', p_m''\}$. By studying the monotonicity properties of p_m', p_m'', p_m and of $P_m :=$ $1 + \sqrt{\frac{m}{m-1}}$ we may deduce the existence of some values m_d and M_d satisfying:

$$p_{m_d}' = P_{m_d} \quad \text{and} \quad p_{M_d}'' = P_{M_d}$$

Moreover we also proved that there exists a value μ_d such that p_m is decreasing and coinciding with p_m' if and only if $m \leq \mu_d$ and p_m is increasing and coinciding with p_m'' if and only if $m \geq \mu_d$. We established the mutual positions of m_d, M_d and μ_d so to get

  - Parts (c) and (d) of Theorem 4.6;
  - a relation between *finitely generated* admissible sequences and the condition $p_m < P_m$.

PROPOSITION.

(a) If d is finitely generated, then $m_d < \mu_d < M_d$, and $p_m < P_m$ for all $m_d < m < M_d$. Furthermore, $p_m \geq 2$ for all $m \in (1, \infty)$ with equality if and only if $d = \left(1^{k-1}0\right)^\omega$ and $m = 2^k$ for some positive integer k.

(b) In the other cases we have $m_d = \mu_d = M_d$ and $p_m \geq p_{\mu_d} = P_{\mu_d} > 2$ for all $m \in (1, \infty)$.

Characterization of critical bases. We previously defined p_m starting from an arbitrary admissible sequence. We now assume that $p_m \leq P_m$, namely $m \in [m_d, M_d]$ for an appropriate admissible sequence d. The existence of such a sequence for any given m is the statement of the following result.

LEMMA. *Given a real number $m \geq 2$ there exists a lexicographically largest admissible sequence $d = (d_i)$ such that*

$$\sum_{i=1}^{\infty} \frac{\delta_i}{P_m^i} \leq m - 1.$$

Furthermore, we have $d \neq 1^\omega$ and $m \in I_d$.

Once we univoquely determined the admissible sequence d, we proved that if d is periodic then $p_m \leq P_m$ is the critical base of any alphabet $A = \{a_1, a_2, a_3\}$ whose ratios between the gaps satisfy $\max \left\{ \frac{a_3 - a_1}{a_2 - a_1}, \frac{a_3 - a_1}{a_3 - a_2} \right\} = m$. To this end we first proved this interesting property of bases smaller than P_m:

PROPOSITION. *If (x_i) is a nontrivial univoque sequence in some base $1 < q \leq P_m$, then (x_i) contains at most finitely many zero digits.*

In the next stage we constructively proved Part (a) of Theorem 4.6, by showing that an appropriate suffix δ' of the (periodic) m-admissible sequence associated to p_m is univoque for every base larger than p_m.

PROPOSITION. *Suppose that $m \in I_d$ with d periodic.*
(a) If $q > p_m$, then δ' is a nontrivial univoque sequence in base q.
(b) There are no nontrivial univoque sequences in any base $1 < q < p_m$.

We finally completed the proof of Theorem 4.6 and we showed some additional properties on the periodicity of univoque sequences:

PROPOSITION.
(a) If $d < \tilde{d} < 1^\omega$ are admissible sequences, then $M_d \leq m_{\tilde{d}}$ with equality if and only if $d = S(N, 1)^\omega$ is infinitely generated and $\tilde{d} = S(N-1, 1)S(N, 1)^\omega$.

(b) The sets I_d, where d runs over all admissible sequences $d \neq 1^\omega$, form a partition of the interval $[\frac{1+\sqrt{5}}{2}, \infty)$.

(c) The set C of numbers $m > \frac{1+\sqrt{5}}{2}$ satisfying $p_m = P_m$ is a Cantor set, i.e., a nonempty closed set having neither interior, nor isolated points. Its smallest element is $1 + x \approx 2.3247$ where x is the first Pisot number, i.e., the positive root of the equation $x^3 = x + 1$.

Conclusions and further developments. In this chapter we first considered arbitrary alphabets and we proved the existence of a sharp critical value between the non-existence and the existence of univoque sequences. This notion extends the analogous classical property of the Golden Mean for binary alphabets. The characterization in the case of ternary alphabets revealed the critical base to be an algebraic number like as the Golden Mean for (at least) almost every possible ratio between the gaps, m. Our construction of admissible sequences incidentally yields a new characterization of sequences of the form $1s$ where s is a sturmian word (with alphabet $\{0, 1\}$).

The author wishes the constructions in this chapter to be useful for a generalization to alphabets with more than three digits.

We finally remark that the construction of some univoque sequences has been here functional to the characterization of the critical base for ternary alphabets. We deeper investigate these sequences in Chapter 5.

Univoque sequences for ternary alphabets

The aim of this chapter is to show that the unique expansions presented in Chapter 4 are the only ones for all the sufficiently small bases. This, beside the explicit characterization of a large class of unique expansions, will imply that U_q is a denumerable, regular set for every q smaller than an explicitly determined value depending on the alphabet.

1. Introduction

For an overview on uniqueness and arbitrary alphabets we refer to the introduction of Chapter 4.

Organization of the chapter. Section 2 is dedicated to some recalls on the expansions with digits in the so-called *normal alphabets*, namely the ternary alphabets in the form $A_m = \{0, 1, m\}$. In Section 3 a connection between the theory of sturmian words and the main result of Chapter 4, i.e. the characterization of the generalized Golden Mean for ternary alphabets, is established. In Section 4 we show some characterizing properties of unique expansions. Section 5 contains a lexicographical result on sturmian words which is applied in Section 6, where our main result on the characterization of unique expansions is proved.

2. Some recalls on normal ternary alphabets

2.1. Univoqueness conditions for normal ternary alphabets. In Section 2.3 we recalled the Pedicini's characterization for quasi-greedy, quasi-lazy and unique expansions. We then adapted the non-lexicographic conditions to the case of a normal ternary alphabet $A_m = \{0, 1, m\}$ in Lemma 4.9. Since troughout this chapter we need both lexicographical and "value-oriented" characterization of univoque sequences, we propose an integrated and adapted version of Theorem 4.2 and Theorem 4.3.

REMARK 5.1. *Recall the notation for the shift: $\sigma^n \mathbf{x} = x_{n+1} x_{n+2} \cdots$.*

PROPOSITION 5.1. *Fix a base q and consider the alphabet A_m, with $m \geq 2$ and such that $m - 1 \leq \frac{m}{q-1}$. Let $\mathbf{x}$ be an expansion in base q and alphabet A_m and for every $n \in \mathbb{N}$, consider the conditions:*

$$(154) \qquad \text{whenever } x_n = 0: \quad \pi_q(\sigma^n \mathbf{x}) < 1 \qquad\qquad \text{or} \quad \sigma^n \mathbf{x} <_{lex} \tilde{\gamma}_q(1)$$

$$(155) \qquad \text{whenever } x_n = 1: \quad \pi_q(\sigma^n \mathbf{x}) < m - 1 \qquad \text{or} \quad \sigma^n \mathbf{x} <_{lex} \tilde{\gamma}_q(m-1)$$

$$(156) \qquad \text{whenever } x_n = 1: \quad \pi_q(\sigma^n \mathbf{x}) > \frac{m}{q-1} - 1 \qquad \text{or} \quad \sigma^n \mathbf{x} >_{lex} \tilde{\lambda}_q\left(\frac{m}{q-1} - 1\right)$$

$$(157) \qquad \text{whenever } x_n = m: \quad \pi_q(\sigma^n \mathbf{x}) > \frac{m}{q-1} - (m-1) \quad \text{or} \quad \sigma^n \mathbf{x} >_{lex} \tilde{\lambda}_q\left(\frac{m}{q-1} - (m-1)\right)$$

Then $\mathbf{x}$ is univoque if and only if (154), (155),(156) and (157) are satisfied. Moreover

(a) **x** *is quasi-greedy if and only if*

$$(158) \qquad \qquad \text{whenever } x_n = 0: \quad \sigma^n \mathbf{x} \leq_{lex} \tilde{\gamma}_q(1)$$

$$(159) \qquad \qquad \text{whenever } x_n = 1: \quad \sigma^n \mathbf{x} \leq_{lex} \tilde{\gamma}_q(m-1)$$

(b) **x** *is quasi-lazy if and only*

$$(160) \qquad \qquad \text{whenever } x_n = 1: \quad \sigma^n \mathbf{x} \geq_{lex} \tilde{\lambda}_q\left(\frac{m}{q-1} - 1\right)$$

$$(161) \qquad \qquad \text{whenever } x_n = m: \quad \sigma^n \mathbf{x} \geq_{lex} \tilde{\lambda}_q\left(\frac{m}{q-1} - (m-1)\right)$$

As a consequence of the above proposition, we prove a technical result on quasi-greedy and quasi-lazy expansions of the gaps of the alphabet.

LEMMA 5.1. *Fix a base q and consider the alphabet A_m, with $m \geq 2$ and such that $m - 1 \leq \frac{m}{q-1}$:*
- *if $(\tilde{\gamma}_i)_{i \geq 1} := \tilde{\gamma}_q(m-1) >_{lex} (w1)^\omega$ for some $w \in A_m^*$, then $\tilde{\gamma}_1 \cdots \tilde{\gamma}_{|w|+1} >_{lex} w1$;*
- *if $(\tilde{\lambda}_i)_{i \geq 1} := \tilde{\lambda}_q(\frac{m}{q-1} - 1) <_{lex} (w1)^\omega$ for some $w \in A_m^*$, then $\tilde{\lambda}_1 \cdots \tilde{\lambda}_{|w|+1} <_{lex} w1$.*

PROOF. Suppose $\tilde{\gamma}_q(m-1) >_{lex} (w1)^\omega$ for some $w \in A_m^*$. It follows by condition (155) of Proposition 5.1 that $\tilde{\gamma}_n = 1$ implies $\sigma^n \tilde{\gamma}_q(m-1) \leq_{lex} \tilde{\gamma}_q(m-1)$.

In order to find a contradiction, suppose that $\tilde{\gamma}_1 \cdots \tilde{\gamma}_{|w|+1} = w1$. Then

$$(w1)^\omega = \sigma^{|w|+1}((w1)^\omega) <_{lex} \sigma^{|w|+1}\tilde{\gamma}_q(m-1) <_{lex} \tilde{\gamma}_q(m-1) = w1 \cdots$$

and, consequenlty, $\sigma^{|w|+1}\tilde{\gamma} = w1 \cdots$. By induction, $\tilde{\gamma} = (w1)^\omega$ and this is the required contradiction. The case $\tilde{\lambda}_q(\frac{m}{q-1} - 1) <_{lex} (w1)^\omega$ is similar. $\qquad \square$

2.2. Critical bases of ternary alphabets. In previous chapter we characterized the critical base G_{A_m} by mean of the so called *m-admissible sequences*, namely sequences $\delta = (\delta_i) \in \{1, m\}^{\mathbb{N}}$ satisfying

$$(162) \qquad \qquad 1\delta_2\delta_3 \cdots \leq (\delta_{n+i}) \leq \delta_1\delta_2 \cdots$$

for every $n = 0, 1, \ldots$. At the beginning of Section 7 we also denoted $\delta' := \min\{\sigma^n \delta | \delta_n = 1; n \geq 1\}$.

The relation between *m-admissible* sequences and critical bases becomes evident in the following restatement of Theorem 4.5, Lemma 4.5 and Lemma 4.11.

THEOREM 5.1. *For every $m \geq 2$ there exists an unique m-admissible sequence $\delta_m \in \{1, m\}^\omega$ such that:*

$$(163) \qquad \qquad \text{if } p'_m > 1 \text{ and } \pi_{p'_m}(\delta_m) = m - 1, \text{ then } p'_m \leq 1 + \sqrt{\frac{m}{m-1}};$$

and

$$(164) \qquad \qquad \text{if } p''_m > 1 \text{ and } \pi_{p''_m}(\delta'_m) = \frac{m}{p''_m - 1} - 1, \text{ then } p''_m \leq 1 + \sqrt{\frac{m}{m-1}}.$$

Moreover the critical base G_{A_m} has the following properties:
(a) $G_{A_m} = \max\{p'_m, p''_m\}$;
(b) $G_{A_m} \in [2, 1 + \sqrt{\frac{m}{m-1}}]$;
(c) $G_{A_m} = 1 + \sqrt{\frac{m}{m-1}}$ *if and only if δ_m is not purely periodic;*
(d) for every $q > G_{A_m}$ the sequence δ_m is univoque in base q.

REMARK 5.2. *The bases p'_m and p''_m are well defined by (163) and (164) because the function π_q is monotone with respect to q's greater than 1. In particular this implies that for every $\mathbf{x} \in A_m^{\mathbb{N}}$ and for every $x \in \mathbb{R}$, the equation $\pi_q(\mathbf{x}) = x$ admits at most one solution.*

3. A relation between m-admissible and sturmian sequences

As a consequence of Theorem 5.1 we have the following result.

THEOREM 5.2. *Let $m \geq 2$ be such that $G_{A_m} < 1 + \sqrt{\frac{m}{m-1}}$ and consider the (purely periodic) associated m-admissible sequence $\delta_m = \delta_1 \delta_2 \cdots$. Then the sequence $\mathbf{s}_m := \delta_2 \delta_3 \cdots$ is a sturmian sequence and it is uniquely determined by the conditions:*

$$(165) \qquad \text{if } p'_m > 1 \text{ and } \pi_{p'_m}(\max \mathbf{s}_m) = m - 1, \text{ then } p'_m \leq 1 + \sqrt{\frac{m}{m-1}};$$

and

$$(166) \qquad \text{if } p''_m > 1 \text{ and } \pi_{p''_m}(\min \mathbf{s}_m) = q(\frac{m}{p''_m - 1} - 1) - 1, \text{ then } p''_m \leq 1 + \sqrt{\frac{m}{m-1}}.$$

Moreover for every $q > G_{A_m}$ the sequence $\mathbf{s}_m$ is univoque in base q.

PROOF. The definition of m-admissible sequences given in (162), together with Theorem 1.4, implies that $\mathbf{s}_m := \delta_2 \delta_3 \cdots$ is a sturmian sequence. The condition $G_{A_m} < P_m$ and Theorem 5.1 imply that $\mathbf{s}_m$ is a purely periodic sturmian sequence and, in particular, $\max \mathbf{s}_m = \delta_m$ and $\min s_m = 1\delta'_m$. Hence we may deduce that $\mathbf{s}_m$ is the unique sturmian sequence satisfying (165) and (166) by the equivalent fact that δ_m is uniquely determined by (163) and (164). The univoqueness of $\mathbf{s}_m$ for bases larger than G_{A_m} straightforward follows by Theorem 5.1 as well. $\square$

REMARK 5.3. *Theorem 1.4 also implies that there exists a finite word w satisfying $\mathbf{s}_m = (wm1)^\omega$.*

DEFINITION 5.1. *The sequence $\mathbf{s}_m$ in Theorem 5.2 is defined as the* sturmian sequence associated to A_m.

EXAMPLE 5.1. *Consider $A_3 = \{0, 1, 3\}$ and the sturmian sequence $\mathbf{s} = (13)^\infty$. By solving the equations:*

$$\pi_{p'}((31)^\infty) = 2$$
$$\pi_{p''}((13)^\infty) = \frac{3}{p'' - 1} - 1.$$

which are respectively equivalent to:

$$\left(\frac{3}{p'} + \frac{1}{p'^2}\right) \frac{p'^2}{p'^2 - 1} = 2$$
$$\left(\frac{1}{p''} + \frac{3}{p''^2}\right) \frac{p''^2}{p''^2 - 1} = \frac{3}{p'' - 1} - 1,$$

we get $p'_3 \simeq 2.18614$ and $p''_3 \simeq 1.73205$. Since $p'_3, p''_3 < 1 + \sqrt{\frac{3}{2}}$, $\mathbf{s}_3 = \mathbf{s} = (13)^\omega$. Hence it follows by Theorem 5.1 that $G_{A_3} = p'_3 \simeq 2.18614$ and that $(13)^\omega$ and $(31)^\omega$ are univoque in every base greater than G_{A_3}.

EXAMPLE 5.2. *The sturmian sequences of the alphabets $\{0, 1, 2\}$, $\{0, 1, 4\}$ and $\{0, 1, 8\}$ are respectively:*

$$\mathbf{s}_2 = (1)^\omega \quad \mathbf{s}_4 = (41)^\omega \quad \mathbf{s}_8 = (881)^\omega$$

and $G_{A_2} = G_{A_4} = G_{A_8} = 2$.

4. Univoque sequences in small base

In this section we investigate more closely the properties of expansions in base smaller than $P_m := 1 + \sqrt{\frac{m}{m-1}}$. In fact, as we show further, these properties allow a complete and almost explicit characterization of univoque sequences in small base.

REMARK 5.4. *As a first property of P_m we recall that in Lemma 4.6 we proved that if $\mathbf{x}$ is an univoque sequence in base $q \leq P_m$ and if $x_1 \neq 0$ then $\mathbf{x} \in \{1, m\}^\omega$.*

LEMMA 5.2. *Let $q \leq P_m$.*
If $\mathbf{x} = 1\mathbf{x}'$ is univoque in base q, then $m\mathbf{x}' >_{lex} \tilde{\gamma}_q(1)$.
If $\mathbf{x} = 1m\mathbf{x}''$ is univoque in base q, then $\mathbf{x}'' <_{lex} \tilde{\lambda}_q(\frac{m}{q-1} - 1)$.

PROOF. Since $\mathbf{x}$ is supposed to be univoque, then it follows by conditions (155) and (156) in Proposition 5.1 that

$$\pi_q(\mathbf{x}') > \frac{m}{q-1} - 1 \quad \text{and} \quad \pi_q(m\mathbf{x}'') < 1.$$

Since $q \leq 1 + \sqrt{\frac{m}{m-1}}$, the inequalities above respectively imply:

(167) $\qquad \pi_q(m\mathbf{x}') = \frac{m}{q} + \frac{1}{q}\pi_q(\mathbf{x}') > 1 \quad \text{and} \quad \pi_q(\mathbf{x}'') = q\pi_q(m\mathbf{x}'') - m < \frac{m}{q-1} - 1.$

The sequences $\mathbf{x}'$ and $\mathbf{x}''$ are quasi-greedy and quasi-lazy because they are suffixes of an univoque sequence, which is always quasi-greedy and quasi-lazy. In view of Remark 4.1, we have that $m\mathbf{x}'$ is quasi-greedy, too. Thus, by Proposition 4.1, the conditions $m\mathbf{x}' >_{lex} \tilde{\gamma}_q(1)$ and $\mathbf{x}'' <_{lex} \tilde{\lambda}_q(\frac{m}{q-1} - 1)$ are equivalent to (167) and the proof is complete. $\qquad \square$

We now characterize the univoque expansions in "small" base by mean of a lexicographic comparison with the sturmian associated to the alphabet $\mathbf{s}_m = (wm1)^\omega$ (for the last equality see Remark 5.3).

PROPOSITION 5.2. *Let $m \geq 2$ be such that $G_{A_m} < 1 + \sqrt{\frac{m}{m-1}}$. Let $\mathbf{s}_m = (wm1)^\infty$ be the sturmian word associated to the alphabet A_m and let $\mathbf{x}$ be a sequence with $x_1 = 1$. Then $\mathbf{x}$ is univoque in base $q \in (G_{A_m}, P_m]$ if and only if for every $n \geq 0$:*

(168) $\qquad\qquad\qquad\qquad \min \mathbf{s}_m \leq \sigma^n \mathbf{x} \leq \max \mathbf{s}_m.$

PROOF. First of all we apply the last part of Theorem 1.4 to $\mathbf{s}_m$ and we denote w the finite word satisfying $\max \mathbf{s}_m = (mw1)^\infty$ and $\min \mathbf{s}_m = (1wm)^\infty$. We then divide the proof in several parts.

Part 1. If $\mathbf{x} = 1\mathbf{x}'$ is univoque in base q, then $m\mathbf{x}' >_{lex} \tilde{\gamma}_q(m-1)$.

Since $\mathbf{x}$ is supposed to be univoque, then it follows by the condition (156) of Proposition 5.1 that: $\pi_q(\mathbf{x}') > \frac{m}{q-1} - 1$. This, together with $q \leq 1 + \sqrt{\frac{m}{m-1}}$, implies:

$$\pi_q(m\mathbf{x}') = \frac{m}{q} + \frac{1}{q}\pi_q(\mathbf{x}') > m-1.$$

As $\mathbf{x}'$ is univoque, it is in particular quasi-greedy as well as $m\mathbf{x}'$ — see Remark 4.1. Hence, by the monotonicity of the quasi-greedy expansions, the inequality above implies $m\mathbf{x}' >_{lex} \tilde{\gamma}_q(m-1)$.

Part 2. If $\mathbf{x} = 1m\mathbf{x}''$ is univoque in base q, then $\mathbf{x}'' <_{lex} \tilde{\lambda}_q(\frac{m}{q-1} - 1)$.

Since $\mathbf{x}$ is supposed to be univoque, then it follows by the condition (155) of Proposition 5.1 that: $\pi_q(m\mathbf{x}'') < m - 1$. This, together with $q \le 1 + \sqrt{\frac{m}{m-1}}$, implies:

$$\pi_q(\mathbf{x}'') = q(\pi_q(m\mathbf{x}'') - \frac{m}{q}) < \frac{m}{q-1} - 1.$$

As $\mathbf{x}''$ is univoque, it is in particular quasi-lazy. By the monotonicity of the quasi-lazy expansions, we may conclude $\mathbf{x}'' <_{lex} \tilde{\lambda}_q(\frac{m}{q-1} - 1)$.

Part 4. Approximation of the boundary sequences: $\tilde{\gamma}_q(m-1) = mwm0\cdots$ *and* $\tilde{\lambda}_q(\frac{m}{q-1} - 1) = wm0\cdots$.

Since $q > G_{A_m}$ then it follows by Theorem 5.1 that $\max \mathbf{s}_m = (mw1)^{\infty}$ and $\min \mathbf{s}_m = (1wm)^{\infty}$ are both univoque in base q. Hence, by Theorem 4.9 we have:

$$(169) \qquad (mw1)^{\infty} < \tilde{\gamma}_q(m-1) \quad \text{and} \quad \tilde{\lambda}_q(\frac{m}{q-1} - 1) < (wm1)^{\infty}.$$

Since $q \le P_m$, it follows by (169) and by Lemma 5.2 that:

$$(170) \qquad (mw1)^{\infty} < \tilde{\gamma}_q(m-1) < m(wm1)^{\infty}.$$

Since $\tilde{\gamma}_q(m-1) \ne wm1\cdots$ (Lemma 5.1), it follows by (170) that $\tilde{\gamma}_q(m-1) = mwma\tilde{\gamma}'$ with $a \in \{0, 1\}$ and $\tilde{\gamma}' \in A_m^{\omega}$. We also deduce by $q \le P_m$ and by Lemma 5.2 that $wma\tilde{\gamma}' < \tilde{\lambda}_q(\frac{m}{q-1} - 1) < (wm1)^{\infty}$. Again by Lemma 5.1 we have $\tilde{\lambda}_q(\frac{m}{q-1} - 1) \ne wm1\cdots$: thus $\tilde{\lambda}_q(\frac{m}{q-1} - 1) = wm0\cdots$, $a = 0$ and $\tilde{\gamma}_q(m-1) = mwm0\tilde{\gamma}'$.

Part 5. If part.

Suppose $\mathbf{x}$ univoque and fix n. We distinguish the cases $x_n = 1$ and $x_n = m$.

- If $x_n = 1$, it follows by Proposition 5.1 and Part 4 that for every $n \ge 1$ such that $x_n = 1$

$$wm0\cdots = \tilde{\lambda}_q(\frac{m}{q-1} - 1) < \sigma^n\mathbf{x} < \tilde{\gamma}_q(m-1) = mwm0\cdots$$

By Proposition 5.1 we have that $\mathbf{x} \in \{1, m\}^{\omega}$; thus the inequalities above imply:

$$wm1 \le x_{n+1} \cdots x_{n+|w|+2} \le mw1$$

and, by the arbitrariety of n, we get $(wm1)^{\omega} \le \sigma^n\mathbf{x} \le (mw1)^{\omega}$. Hence, by the last part of Theorem 1.4:

$$(171) \qquad \min \mathbf{s}_m < \sigma^n\mathbf{x} \le \max \mathbf{s}_m.$$

- If $x_n = m$ there exists $0 < k < n$ such that $x_{n-k} = 1$ and $\sigma^{n-k}\mathbf{x} = m^k\sigma^n\mathbf{x}$. Hence it follows by the previous case and by the last part of Theorem 1.4 that:

$$(172) \qquad \min \mathbf{s}_m = 1(wm1)^{\omega} \le \sigma^n\mathbf{x} < m^k\sigma^n\mathbf{x} = \sigma^{n-k}\mathbf{x} \le \max \mathbf{s}_m.$$

Part 6. Only if part.

Suppose now $\mathbf{x}$ satisfying (168). The left inequality implies $x_n \ne 0$ for every $n \ge 1$. Thus we may just verify the conditions of Proposition 5.1 for the cases $x_n = 1$ and $x_n = m$.

- If $x_n = 1$, then

$$\sigma^n\mathbf{x} \overset{(168)}{\le} \max \mathbf{s}_m = (mw1)^{\omega} \overset{\text{Part 4}}{<} \tilde{\gamma}_q(m-1)$$

and

$$\sigma^{n-1}\mathbf{x} = 1\sigma^n\mathbf{x} \overset{(168)}{\ge} \min \mathbf{s}_m = 1(wm1)^{\omega} \overset{\text{Part 4}}{>} 1\tilde{\lambda}_q\left(\frac{m}{q-1} - 1\right).$$

- The right inequality in (168) implies that if $x_n = m$ then there exists $k \geq 1$ such that $x_{n+k} = 1$. Thus

$$\sigma^n \mathbf{x} = m^{k-1} 1 \sigma^{n+k+1} \mathbf{x} > 1 \sigma^{n+k+1} \mathbf{x}$$

(173)

$$\overset{(168)}{\geq} (1wm)^\omega$$

$$\overset{\text{Part 4}}{>} \tilde{\lambda}_q \left(\frac{m}{q-1} - 1 \right)$$

$$\overset{m \geq 2}{>} \tilde{\lambda}_q \left(\frac{m}{q-1} - (m-1) \right).$$

$\square$

5. A lexicographical property of periodic sturmian sequences

This section is devoted to the proof of the following result.

PROPOSITION 5.3. *Let* $\mathbf{s} = (wab)^\omega$ *be a periodic sturmian word with digits in the alphabet* $\{a, b\}$. *If* $\mathbf{x} \in \{a, b\}^\omega$ *satisfies:*

(174) $\min \mathbf{s} \leq \sigma^n \mathbf{x} \leq \max \mathbf{s}.$

for every $n \geq 0$, *then* $\mathbf{x} \in Orb(\mathbf{s})$.

REMARK 5.5. *Proposition 5.3 is a general result on binary alphabets that may seem a bit far from our study of univoque sequences. Nevertheless by comparing the statements of Proposition 5.2 and of Proposition 5.3 we can see how Proposition 5.3 actually implies a further restriction on the univoque sequences in small base. This reasoning is used in the following section, where we explicitly characterize the univoque expansions in small base.*

LEMMA 5.3. *Let* $\mathbf{x}$ *be a purely periodic sequence. Suppose* $\max \mathbf{x} = \overline{v}^\omega$ *(resp.* $\min \mathbf{x} = \underline{v}^\omega$*) and let* u *be a prefix of* $\overline{v}$ *(resp. of* $\underline{v}$*). Then* $u^\omega > \overline{v}^\omega$ *(resp.* $u^\omega < \underline{v}^\omega$*).*

PROOF. By definition u is a prefix of $\overline{v}$ (resp. $\underline{v}$) then there exists a non empty word y such that $\overline{v} = uy$ (resp. $\underline{v} = uy$). Since $\overline{v}^\omega = \max \mathbf{x}$ (resp. $\underline{v}^\omega = \min \mathbf{x}$) then $\overline{v}^\omega > y\overline{v}^\omega$ (resp. $\underline{v} < y\underline{v}^\omega$) Thus for every $n \geq 2$:

$$\overline{v}^\omega = uy\overline{v}^\omega < uuy\overline{v}^\omega < \cdots < u^n y\overline{v}^\omega < u^\omega$$

$$(\text{resp.} \underline{v}^\omega = uy\underline{v}^\omega > uuy\overline{v}^\omega > \cdots > u^n y\overline{v}^\omega > u^\omega).$$

$\square$

PROOF OF PROPOSITION 5.3. Let $\mathbf{x}$ be a sequence satifying (174). Since $\mathbf{s}$ is a periodic sturmian word, there exists a word w such that $\min s = (awb)^\omega$ and $\max s = (bwa)^\omega$ so that (174) can be rewritten as follows.

(175) $(awb)^\omega \leq \sigma^n \mathbf{x} \leq (bwa)^\omega.$

Want to characterize the left special factors of a sequence $\mathbf{x}$ satifying (175). Let y be a left special factor of $\mathbf{x}$. By definition, there exist $k_1, k_2 \geq 0$ and $\mathbf{x}', \mathbf{x}'' \in \{a, b\}^\omega$ such that $\sigma^{k_1} \mathbf{x} = ay\mathbf{x}'$ and $\sigma^{k_2} \mathbf{x} = by\mathbf{x}''$. Thus by (175) we get:

(176) $ay\mathbf{x}' \geq (awb)^\omega \quad \text{and} \quad by\mathbf{x}'' \leq (bwa)^\omega.$

The lexicographic inequalities above are compatible if and only if y is a prefix of w and, consequently, the length of the left special factors is bounded by $| w |$. Hence we may deduce by Proposition 1.1 that $\mathbf{x}$ is purely periodic.

Call y the longest left special factor. Since $\mathbf{x}$ as exactly one left special factor of length $k \leq\mid y\mid$, $F_{|y|+1} =\mid y\mid +2$ and the period of $\mathbf{x}$ is at least $\mid y\mid +2$. By the maximality of y we have $F_{|y|+2} =\mid y\mid +2$, thus the period of $\mathbf{x}$ is equal to $\mid y\mid +2$ and

$$\sigma^{k_1}\mathbf{x} = (ayt')^\omega \quad \text{and} \quad \sigma^{k_2}\mathbf{x} = (byt'')^\omega$$

with $t', t'' \in \{a, b\}$ and, because of the maximality of $\mid y\mid$, with $t' \neq t''$.

In order to complete the proof it suffices to show that $y = w$. Suppose on the contrary that y is a proper prefix of w: then either yt' or yt'' is a prefix of w. By denoting $u := w_1 \cdots w_{|y|+1}$ such a prefix, we get:

$$(aw)^\omega \leq (au)^\omega \quad \text{or} \quad (bu)^\omega \leq (bu)^\omega$$

and, by Lemma 5.3, the required contradiction.

$\square$

REMARK 5.6. *Proposition 5.3 is a reformulation of well known results in the theory of sturmian words, see [AG09] for a survey. We presented a new, independent proof with the purpose of making the reasoning along this chapter as self-contained as possible.*

6. Characterization of univoque expansions in small base

In this section we prove the main result of the chapter. Fix a normal ternary alphabet A_m and define

$$U_q := \{\mathbf{x} \in A_m^\omega \mid \mathbf{x} \text{ is univoque in base } q\}.$$

THEOREM 5.3. *Let m be such that $G_{A_m} < 1 + \sqrt{\frac{m}{m-1}}$. Let U_q be the set of unique expansions in base $q \in (G_{A_m}, 1 + \sqrt{\frac{m}{m-1}}]$ and consider the sturmian word associated to the alphabet, say $\mathbf{s}_m$. Then*

(177)
$$U_q = \{m^t\mathbf{x} \mid \mathbf{x} \in Orb(\mathbf{s}_m), \mathbf{x} = 1\mathbf{x}', \mathbf{x}' \in A_m^\omega; t \geq 0\}$$
$$\cup \{0^t\mathbf{x} \mid \mathbf{x} \in Orb(\mathbf{s}_m), \mathbf{x} = 1\mathbf{x}', \mathbf{x}' \in A_m^\omega, \pi_q(\mathbf{x}) < 1; t \geq 0\}.$$

PROOF. First of all note that if a sequence $\mathbf{x}$ is univoque in base $q > G_{A_m} \geq 2$ for every $m \geq 2$ then for every $t \geq 0$ $\mathbf{y} := m^t\mathbf{x}$ is univoque, too. In fact the univoqueness of $\mathbf{x}$ implies that we need to check the conditions of Proposition 5.1 only for indexes $n \leq t$. If $n \leq t$ then $\sigma^n\mathbf{y} = m^{t-n}\mathbf{x}$ and we distinguish the cases $x_1 = 1$ and $x_1 = m$. The case $x_1 = 0$ cannot occour because of Proposition 5.1.

Now, if $x_1 = 1$ then by the univoqueness condition (156) of Proposition 5.1:

$$\pi_q(\sigma^n\mathbf{y}) = \pi_q(m^{t-n}\mathbf{x}) \geq \pi_q(\mathbf{x}) = \pi_q(x_1\sigma\mathbf{x})$$
$$= \frac{1}{q} + \frac{1}{q}\pi_q(\sigma\mathbf{x}) > \frac{1}{q} + \frac{1}{q}\left(\frac{m}{q-1} - 1\right) \geq \frac{m}{q-1} - (m-1).$$

Since the condition (157) of Proposition 5.1 is satisfied, we deduce the univoqueness of $\mathbf{y}$.

If $x_1 = m$ then by the univoqueness condition (157) of Proposition 5.1:

$$\pi_q(\sigma^n\mathbf{y}) = \pi_q(m^{t-n}\mathbf{x}) \geq \pi_q(\mathbf{x}) = \pi_q(x_1\sigma\mathbf{x})$$
$$= \frac{m}{q} + \frac{1}{q}\pi_q(\sigma\mathbf{x}) > \frac{m}{q} + \frac{1}{q}\left(\frac{m}{q-1} - (m-1)\right) > \frac{m}{q-1} - (m-1)$$

and this implies the univoqueness of $\mathbf{y}$, as well.

By a similar argument we can convince ourselves that the conditions $\mathbf{x} \in U_q$ and $\pi_q(\mathbf{x}) < 1$ are necessary and sufficient for the univoqueness of $0^t\mathbf{x}$ for every $t \geq 0$.

Moreover if $0^t\mathbf{x}$ is univoque only if $x_1 \neq m$. In fact by supposing $y_1 = m$ and by applying the univoqueness condition (157) of Proposition 5.1 we get:

$$\pi_q(\mathbf{y}) = \pi_q(m\sigma\mathbf{y}) = \frac{m}{q} + \frac{1}{q}\left(\frac{m}{q-1} - (m-1)\right) \geq 1.$$

and, consequently, a contradiction with the hypothesis $\pi_q(\mathbf{y}) < 1$.

In view of the previous reasonings we can consider without loss of generality the set $U'_q :=$ $U_q \cap 1A_m^\omega = \{\mathbf{x} \mid \mathbf{x} \text{ is univoque in base } q, \mathbf{x} = 1\mathbf{x}', \mathbf{x}' \in A_m^\omega\}$.

By applying Proposition 5.2 and Proposition 5.3 we get

$$U'_q = \{\mathbf{x} \in A_m^\omega \mid \mathbf{x} = 1\mathbf{x}', \mathbf{x}' \in A_m^\omega, \mathbf{s}_m \leq \sigma^n\mathbf{x} \leq \max \mathbf{s}_m\}$$
$$= \{\mathbf{x} \in A_m^\omega \mid \mathbf{x} = 1\mathbf{x}', \mathbf{x}' \in A_m^\omega, \mathbf{x} \in Orb(\mathbf{s}_m)\}$$

and hence the thesis.   □

COROLLARY 5.1. *If* $1 + \sqrt{\frac{m}{m-1}} \leq \frac{1+\sqrt{m+4}}{2}$, *namely* $m \geq 7.26637$, *for every* $q \in (G_{A_m}, 1 + \sqrt{\frac{m}{m-1}}]$ $U_q = \{m^t\mathbf{x} \mid \mathbf{x} \in Orb(\mathbf{s}_m)\}$.

PROOF. It suffices to show that if $\mathbf{x} \in U_q$ then $\mathbf{x} \neq 0^t 1\mathbf{x}'$. Suppose on the contrary $\mathbf{x} = 0^t 1\mathbf{x}'$ for some $t > 0$. Then by univoqueness conditions (154) and (156) of Proposition 5.1 we deduce:

$$\pi_q(1\mathbf{x}') < 1 \quad \text{and} \quad \pi_q(\mathbf{x}') > \frac{m}{q-1} - 1.$$

The inequalities above are compatible only if $q > \frac{1+\sqrt{m+4}}{2}$ but it is impossible because q is supposed to be smaller than $1 + \sqrt{\frac{m}{m-1}}$.   □

EXAMPLE 5.3. *Consider the alphabet* A_3. *The sturmian sequence associated to* A_3 *is* $\mathbf{s}_3 = (13)^\omega$ *(see Example 5.1) and for every* $q \in (G_{A_3}, 1 + \sqrt{\frac{3}{2}}]$:

$$U_q = \{3^t(13)^\omega \mid t \geq 0\}$$

EXAMPLE 5.4. *Consider the alphabets* A_2, A_4 *and* A_8 *and recall that the respective sturmian sequences are* $(1)^\omega$, $(41)^\omega$ *and* $(881)^\omega$ *while the generalized Golden Mean is always equal to 2 (see Example 5.2).*

Since $m = 2$ *does not satisfy the condition of Corollary 5.1, for every* $q \in (2, 1 + \sqrt{2}]$:

$$U_q = \{2^t(1)^\omega \mid t \geq 0\} \cup \{0^t(1)^\omega \mid t \geq 0\}$$

Conversely $m = 4, 8$ *satisfy the condition of Corollary 5.1, thus for every suitable* q:

$$U_4 = \{4^t(14)^\omega \mid t \geq 0\} \quad \text{and} \quad U_8 = \{(8^t(188)^\omega \mid t \geq 0\}.$$

EXAMPLE 5.5. *The sturmian sequence associated to alphabet* A_5 *is* $\mathbf{s}_5 = (51551551)^\omega$. *Thus* $\mathbf{s}_5$ *has integer digits and more than one occurrence of 1 in its period. Numerical evidence suggests that this is a rare property, in fact considering* $m \in \mathbb{N}$ *and* $2 \leq m \leq 2^{16}$ *this property is true in only 7 cases.*

Nevetheless, in order to enlight the structure of U_q, *it is useful to show how Theorem 5.3 can be applied to this alphabet. For every* $q \in (G_{A_5}, 1 + \sqrt{\frac{5}{4}}]$:

$$U_5 = \{5^t(15515515)^\omega, 5^t(15515155)^\omega, 5^t(15155155)^\omega \mid t \geq 0\}.$$

As a consequence of Proposition 4.3, we have the following extension of Theorem 5.3 to arbitrary alphabets.

COROLLARY 5.2. *Let A an arbitrary alphabet such that $N(A) = A_m$, with $G_{A_m} < 1 + \sqrt{\frac{m}{m-1}}$, and let ϕ be the digit-wise normalizing map. Then for every $q \in (G_{A_m}, 1 + \sqrt{\frac{m}{m-1}}]$, the set of univoque sequences with digit in A and base q, say $U_{A,q}$, satisfies:*

$$U_{A,q} = \{\phi_A(m^t\mathbf{x}) \mid \mathbf{x} \in Orb(\mathbf{s}_m), \mathbf{x} = 1\mathbf{x}', \mathbf{x}' \in A_m^\omega;\ t \geq 0\}$$
$$\cup \{\phi_A(0^t\mathbf{x}) \mid \mathbf{x} \in Orb(\mathbf{s}_m), \mathbf{x} = 1\mathbf{x}', \mathbf{x}' \in A_m^\omega, \pi_q(\mathbf{x}) < 1;\ t \geq 0\}.$$

EXAMPLE 5.6. *Let $A = \{a_1, a_2, a_3\}$ be an alphabet with constant gaps, i.e. $a_2 - a_1 = a_3 - a_2$. Since $N(A) = \{0, 1, 2\}$, $G_A = 2$ and for every $q \in (2, 1 + \sqrt{2}]$:*

$$U_{A,q} = \{a_3^t(a_2)^\omega \mid t \geq 0\} \cup \{a_1^t(a_2)^\omega \mid t \geq 0\}.$$

7. Overview of original contributions, conclusions and further developments

A relation between m-admissible and sturmian words. We recall that m-admissible sequences have been defined in Section 7 as sequences $\delta = (\delta_i) \in \{1, m\}^{\mathbb{N}}$ satisfying

$$1\delta_2\delta_3 \cdots \leq (\delta_{n+i}) \leq \delta_1\delta_2 \cdots$$

for every $n = 0, 1, \ldots$. In Chapter 4 we associated to every m an appropriate m-admissible sequence, here denoted δ_m. The sequence δ_m defines a system of equations whose greatest solution is the critical base of the alphabet A_m.

THEOREM. *Let m be such that m-admissible sequence $\delta_m = \delta_1\delta_2 \cdots$ associated to m is purely periodic, namely $G_{A_m} < 1 + \sqrt{\frac{m}{m-1}}$. Then the sequence $\mathbf{s}_m := \delta_2\delta_3 \cdots$ is a sturmian sequence and it is uniquely determined by the conditions:*

$$\text{if } p_m' > 1 \text{ and } \pi_{p_m'}(\max \mathbf{s}_m) = m - 1, \text{ then } p_m' \leq 1 + \sqrt{\frac{m}{m-1}};$$

and

$$\text{if } p_m'' > 1 \text{ and } \pi_{p_m''}(\min \mathbf{s}_m) = q(\frac{m}{p_m'' - 1} - 1) - 1, \text{ then } p_m'' \leq 1 + \sqrt{\frac{m}{m-1}}.$$

Moreover for every $q > G_{A_m}$ the sequence $\mathbf{s}_m$ is univoque in base q.

Under the assumption that δ_m is purely periodic, we defined the sequence $\mathbf{s}_m$ the *sturmian sequence associated to the alphabet A_m*. By a well known result on periodic sturmian words, we have that $\mathbf{s}_m = (wm1)^\omega$ for some palindrome word $w \in \{1, m\}^*$.

Univoque sequences in small bases. When the q is sufficiently small, namely $q \leq P_m = 1 + \sqrt{\frac{m}{m-1}}$, then the univoque sequences satisfy particular characterizing properties.

PROPOSITION. *Let $m \geq 2$ be such that $G_{A_m} < 1 + \sqrt{\frac{m}{m-1}}$ and let $\mathbf{s}_m = (wm1)^\infty$ be the sturmian word associated to the alphabet A_m. If $\mathbf{x}$ is a sequence with $x_1 = 1$, then $\mathbf{x}$ is univoque in base $q \in (G_{A_m}, P_m]$ if and only if for every $n \geq 0$:*

$$\min \mathbf{s}_m \leq \sigma^n\mathbf{x} \leq \max \mathbf{s}_m.$$

REMARK. *The univoque sequences in small base belong to $\{1, m\}^{\mathbb{N}}$: this result has been proved in Chapter 4 and it is confirmed by the result above.*

Characterization of univoque sequences in small base. Fix a normal ternary alphabet A_m and define

$$U_q := \{\mathbf{x} \in A_m^\omega \mid \mathbf{x} \text{ is univoque in base } q\}.$$

THEOREM. *Let m be such that $G_{A_m} < 1 + \sqrt{\frac{m}{m-1}}$ and let U_q be the set of unique expansions in base $q \in (G_{A_m}, 1 + \sqrt{\frac{m}{m-1}}]$. Consider the sturmian word associated to the alphabet, say $\mathbf{s}_m$. Then*

$$U_q = \{m^t \mathbf{x} \mid \mathbf{x} \in Orb(\mathbf{s}_m), \mathbf{x} = 1\mathbf{x}', \mathbf{x}' \in A_m^\omega; t \geq 0\}$$

$$\cup \{0^t \mathbf{x} \mid \mathbf{x} \in Orb(\mathbf{s}_m), \mathbf{x} = 1\mathbf{x}', \mathbf{x}' \in A_m^\omega, \pi_q(\mathbf{x}) < 1; t \geq 0\}.$$

The result is extended to the general ternary alphabets.

COROLLARY. *Let A an arbitrary alphabet such that $N(A) = A_m$, with $G_{A_m} < 1 + \sqrt{\frac{m}{m-1}}$, and let ϕ be the digit-wise normalizing map. Then for every $q \in (G_{A_m}, 1 + \sqrt{\frac{m}{m-1}}]$, the set of univoque sequences with digit in A and base q, say $U_{A,q}$, satisfies:*

$$U_{A,q} = \{\phi_A(m^t \mathbf{x}) \mid \mathbf{x} \in Orb(\mathbf{s}_m), \mathbf{x} = 1\mathbf{x}', \mathbf{x}' \in A_m^\omega; t \geq 0\}$$

$$\cup \{\phi_A(0^t \mathbf{x}) \mid \mathbf{x} \in Orb(\mathbf{s}_m), \mathbf{x} = 1\mathbf{x}', \mathbf{x}' \in A_m^\omega, \pi_q(\mathbf{x}) < 1; t \geq 0\}.$$

Conclusions and further developments. In this chapter we focused on the minimal univoque sequences, i.e. the univoque sequences that first appear when choosing a base larger than the critical base. Our characterization concerns the set of ternary alphabets with ratio between gaps m and periodic m-admissible sequence. This is a very large set of alphabets, because in Chapter 4 we proved that the set of ratios with aperiodic m-admissible sequence is a Cantor set. We proved that when the m-admissible sequence is periodic, the set of univoque sequences U_q in base $q \leq P_m$ is uniquely composed by eventually periodic sequences with constant anti-period. This implies that U_q is recognizable by a finite automaton. The periodic suffix of minimal univoque sequences is proved to be a periodic sturmian sequence.

After the appearence of [KLP] on the archive for electronic preprints ArXiv.org, Jean-Paul Allouche remarked the relation between aperiodic admissible sequences and proper sturmian words (see Remark 4.5). This stimulated the study of the connection between univoque and periodic sturmian sequences, here established by a common characterizing property. A deeper investigation of this relation could be useful for a generalization to arbitrary alphabets.

Acknowledgements

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