SVOLGIMENTO PROVA SCRITTA di ANALISI 1 del 12/7/2013

1) $z \neq 0$

$$\left| \frac{z \cdot \bar{z} + 3}{\bar{z}} \right| \geq 2$$

$$\left| |z|^2 + 3 \right| \geq 2 |\bar{z}|$$

$$|a^2+b^2+3| \geq 2\sqrt{a^2+b^2} \quad ; \quad a^2+b^2+3 \text{ è reale positivo}$$

$$\cancel{a^2+b^2} \quad a^2+b^2+3 \geq 2\sqrt{a^2+b^2}$$

$$a^2+b^2 = \rho^2 \quad \Rightarrow \quad \rho^2 + 3 - 2\rho \geq 0$$

$\rho_{1,2} = 1 \pm \sqrt{1-3}$ $\Rightarrow$ la disequazione è verificata $\forall \rho > 0$ $(z \neq 0)$.

Quindi la disequazione è verificata $\forall z \neq 0$

2) f è definita su tutto $\mathbb{R}$.

f è sempre non negativa.

$$\lim_{x\to 0^-} f(x) = f(0) = 1 = \lim_{x\to 0^+} f(x)$$

$f \in C^0(\mathbb{R})$.

$$f'(x) = \begin{cases} \dfrac{x}{\sqrt{x^2+1}}\, e^{\sqrt{x^2+1}-1} & \text{se } x<0 \\ \left[e^{x\log(x^2+1)}\right]' = \underbrace{(x^2+1)^x}_{\to 0}\left[\underbrace{\log(x^2+1)}_{\to 0} + \dfrac{2x^2}{\underbrace{(x^2+1)}_{\to 0}}\right] & \text{se } x>0 \end{cases}$$

Per $x<0$ $\quad f'(x)>0 \iff x>0$

$\Rightarrow f'(x)<0 \quad \forall x<0$

Per $x>0$ $\quad f'(x)\geq 0$

$$\lim_{x\to 0^-} f'(x) = 0 = \lim_{x\to 0^+} f'(x)$$

$\Rightarrow f$ è derivabile in $\mathbb{R}$

$f'(0)=0$.

f decresce in $(-\infty, 0)$ e cresce in $(0, +\infty)$

In $x=0$ ~~MAX~~. ASS. : $f(0)=1$.

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Poichè $\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = +\infty$

$\Rightarrow \nexists$ MAX. ASS.

3)
$$f(x) = \frac{\cancel{x^2} - \frac{x^6}{6} + o(x^6) \cancel{-1} + \frac{x^6}{2} + o(x^6) \cancel{+1} \cancel{- x^2} + \frac{x^4}{2} + o(x^4)}{x^2 \cdot x^2}$$

$$= \frac{\frac{x^4}{2}}{x^4}$$

$\Rightarrow$ Il numeratore ha ordine di infinitesimo $\alpha = 4$.

$$\lim_{x \to 0} f(x) = \frac{1}{2}.$$

4)
$$\log(n^n + e^n) = \log(n^n) + \log\left(1 + \frac{e^n}{n^n}\right)$$

$$\sim \log(n^n) = n \log n \qquad 4$$

$$\Rightarrow \quad a_n \sim \frac{n \log n}{n^4} \cancel{\xrightarrow[n \to]{} 0}$$

$$= \frac{\log n}{n^3} \xrightarrow[n \to +\infty]{} 0$$

Poiché $\dfrac{\log n}{n^3} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$

e $\sum \dfrac{1}{n^2}$ converge, per il criterio del confronto converge anche la serie assegnata.

5)
$$\frac{2x+1}{x^3+3x^2+3x+1} = \frac{(2x+1)}{(x+1)^3}$$

$$= \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$= \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3}$$

$$= \frac{Ax^2 + (2A+B)x + A + B + C}{(x+1)^3}$$

$$\Rightarrow \begin{cases} A = 0 \\ B = 2 \\ C = 1 - 2 = -1 \end{cases}$$

$$\int f(x)\,dx = \int \left[\frac{2}{(x+1)^2} - \frac{1}{(x+1)^3} \right] dx$$

Altrimenti:

$$\boxed{\frac{2x+1}{(x+1)^3} = \frac{2(x+1) - 1}{(x+1)^3} = \frac{2}{(x+1)^2} - \frac{1}{(x+1)^3}}$$

$$\int \left[\frac{2}{(x+1)^2} - \frac{1}{(x+1)^3} \right] dx = \frac{-2}{(x+1)} + \frac{1}{2(x+1)^2} + C$$

$$\Rightarrow \int_0^{+\infty} f(x)\,dx = -\lim_{x \to +\infty} \left[\frac{2}{(x+1)} - \frac{1}{2(x+1)^2} \right] + \left[2 - \frac{1}{2} \right] = \frac{3}{2}.$$

5_{bis}) $y' = z$ (6)

$$\Rightarrow \begin{cases} x z' + z = \log x & \forall x > 0 \\ z(1) = 0 \end{cases}$$

Devidendo per x :

$$z' + \frac{1}{x} z = \frac{\log x}{x}$$

$$\Rightarrow z(x) = e^{-\int_1^x \frac{1}{t} dt} \left[\int_1^x e^{\int_1^t \frac{1}{s} ds} \frac{\log t}{t} dt \right]$$

$$= e^{-\log t|_1^x} \left[\int_1^x e^{\log s|_1^t} \frac{\log t}{t} dt \right]$$

$$= e^{-\log x} \left[\int_1^x e^{\log t} \frac{\log t}{t} dt \right]$$

$$= \frac{1}{x} \left[\int_1^x \cancel{t} \frac{\log t}{\cancel{t}} dt \right] = \frac{1}{x} \left[t \log t - t \right]_1^x$$

$$= \frac{1}{x} \left[x \log x - x + 1 \right] = \log x - 1 + \frac{1}{x} .$$

$$y(x) = \int_1^x z'(t)\,dt =$$

$$\int_1^x \left[\log t - 1 + \frac{1}{t}\right] dt$$

$$= \left[t\log t - t - t + \log t\right]_1^x$$

$$= x\log x - 2x + \log x + 2$$

$$= (x+1)\log x + 2(1-x)$$